Exercise 1: Extensions

Definition 1 (extension encoding)
Let $t$ be an unranked tree on $\Sigma$. Let $F_{\Sigma}^\Sigma = \{@ (2)\} \cup \{a (0) \mid a \in \Sigma\}$. We define the ranked tree $\text{ext}(t)$ by induction on the size of $t$ by:
- for $a \in \Sigma$, $\text{ext}(a) = a$
- if $t = a(t_1, ..., t_n)$ with $n \geq 1$, $\text{ext}(t) = @ (\text{ext}(a(t_1, ..., t_{n-1})), \text{ext}(t_n))$

that is $\text{ext}(a(t_1, ..., t_n))$ is equal to:

Give the extension encoding of:

Exercise 2: The soundess of the extension
Let $L$ be a language of unranked trees. Prove that $L$ is recognizable by a NFHA iff $\text{ext}(L)$ is recognizable by a NFTA.

Exercise 3: Complexity
Show that the emptiness problem for NFHA(NFA) is in PTIME.
Definition 2 If $X$ is a set of propositional variables, let $\mathbb{B}(X)$ be the set of positive propositional formulae on $X$, i.e., formulae generated by the grammar $\phi ::= \bot \mid \top \mid x \in X \mid \phi \lor \phi \mid \phi \land \phi$.

Definition 3 A AWA (Alternating Word Automata) is a tuple $A = (Q, \Sigma, Q_0, Q_f, \delta)$ where $\Sigma$ is a finite set (alphabet), $Q$ is a finite set (of states), $Q_0 \subseteq Q$ (initial states), $Q_f \subseteq Q$ (final states) and $\delta$ is a function from $Q \times \Sigma$ to $\mathbb{B}(Q)$ (transition function). A run of $A = (Q, \Sigma, Q_0, Q_f, \delta)$ on a word $w$ is a tree $t$ labelled by $Q \times \mathbb{N}$ such that:
- if $w = \varepsilon$, then $t = (q_0, 0)$ with $q_0 \in Q_0$.
- if $w = a.w'$, then $t = (q_0, k)(t_1, \ldots, t_n)$ where $k$ is the length of $w$, $q_0 \in Q_0$ and such that for all $i$, $t_i$ is a run of $w'$ on $(Q, \Sigma, \{q_i\}, Q_f, \delta)$ for some $q_i$ satisfying $\{q_1, \ldots, q_n\} \models \delta(q_0, a)$.

Definition 4 We say that a run is accepting if every leaf of the form $(q, 0)$ satisfies that $q \in Q_f$. Notice that a run may have leaves of the form $(q, i)$ with $i > 0$ if $\emptyset \models \delta(q_0, a)$. Those leaves are considered as 'success' leaves in this semantic. The language of a AWA is the set of words which have an accepting run.

1. Let $\Sigma = \{0, 1\}$ and $A = (Q, \Sigma, Q_0, Q_f, \delta)$ the AWA such that $Q = \{q_0, q_1, q_2, q_3, q_4, q_1', q_2'\}$, $Q_f = \{q_0, q_1, q_2, q_3, q_4\}$ and:

$$
\delta = \{
q_00 \rightarrow (q_0 \land q_1) \lor q_1' & q_01 \rightarrow q_0 \\
q_10 \rightarrow q_2 & q_11 \rightarrow \top \\
q_20 \rightarrow q_3 & q_21 \rightarrow q_3 \\
q_30 \rightarrow q_4 & q_31 \rightarrow q_4 \\
q_40 \rightarrow \top & q_41 \rightarrow \top \\
q_1'0 \rightarrow q_1' & q_1'1 \rightarrow q_2' \\
q_2'0 \rightarrow q_2' & q_2'1 \rightarrow q_1'
\}
$$

Give an example of an accepting computation of $A$ on $w = 00101$ and an example of a non accepting computation of $A$ on $w$.

2. Prove that for all AWA, we can compute in exponential time a non-deterministic automaton which accepts the same language.