Exercise 1: But first, a bit of homomorphism, with nutts.

A bottom-up tree transducer (NUTT) is a tuple \( U = (Q, F, F', Q_f, \Delta) \) where \( Q \) is a finite set (of states), \( F \) and \( F' \) are finite ranked sets (of input and output), \( Q_f \subseteq Q \) (final states) and \( \Delta \) is a finite set of rules of the form:

\[
\begin{align*}
&\bullet \ f(q_1(x_1), ..., q_n(x_n)) \rightarrow q(u) \text{ where } f \in F \text{ and } u \in T(F', \{x_1, ..., x_n\}) \\
&\bullet \ q(x_1) \rightarrow q'(u) \text{ where } u \in T(F', \{x_1\})
\end{align*}
\]

We say that \( U \) is linear when the right side of the rules of \( \Delta \) are. This defines a rewrite system \( \rightarrow_U \) on \( T(F \cup F' \cup Q) \). The relation induced by \( U \) is then \( R(U) = \{(t, t') \mid t \in T(F), t' \in T(F'), t \rightarrow_U q(t'), q \in Q_f\} \).

1) Prove that tree morphisms are a special case of NUTT that is if \( \mu : T(F) \rightarrow T(F') \) is a morphism, then there exists a NUTT \( U_\mu \) such that \( R(U_\mu) = \{(t, \mu(t)) \mid t \in T(F)\} \). Be sure that if \( \mu \) is linear then \( U_\mu \) is too.

2) Prove that the domain of a NUTT \( U \), that is \( \{t \in T(F) \mid \exists t' \in T(F'), (t, t') \in U\} \), is recognizable.

3) Prove that the image of a recognizable tree language \( L \) by a linear NUTT \( U \), that is \( \{t' \in T(F') \mid \exists t \in L, (t, t') \in U\} \), is recognizable.

Exercise 2: And a bit of minimization

**Definition 1** An equivalence relation \( \equiv \) on \( T \) is a congruence on \( T(F) \) if for every \( f \in F_n \):

\[
(\forall i, 1 \leq i \leq n, u_i \equiv v_i) \Rightarrow f(u_1, ..., u_n) \equiv f(v_1, ..., v_n)
\]

For a given tree language \( L \), let us define the congruence \( \equiv_L \) on \( T(F) \) by : \( u \equiv_L v \) if for all contexts \( C \in C(F) : C[u] \in L \iff C[v] \in L \)

Prove that the following are equivalent:

1. \( L \) is a recognizable tree language
2. \( L \) is the union of some equivalence classes of a congruence of finite index
3. the relation \( \equiv_L \) is a congruence of finite index.

Then, show how to obtain the minimal automaton of a language.

Exercise 3: MSO on finite trees

We consider trees with maximum arity 2. Give MSO formulae which express the following:

1. \( X \) is closed under predecessors
2. \( x \subseteq y \) (with \( \subseteq \) the prefix relation on positions)
3. 'a' occurs twice on the same path
4. 'a' occurs twice not on the same path
5. There exists a sub tree with only a's
6. The frontier word contains the chain 'ab'
Exercise 4: From formulae to automaton

Give tree automata recognizing the languages on trees of maximum arity 2 defined by the formulae:

1. \((x \in S \land (x \downarrow_1 y \Rightarrow y \in S)) \land (z \in S \Rightarrow P_f(z))\)
2. \(\exists S. (x \in S \land (x \downarrow_1 y \Rightarrow y \in S)) \land (z \in S \Rightarrow P_f(z))\)