# Automates d'arbre

## TD n°3 : Trees and Logic

**Exercise 1 : But first, a bit of homomorphism, with nutts.**

A bottom-up tree transducer (NUTT) is a tuple $U = (Q, \mathcal{F}, \mathcal{F}', Q_f, \Delta)$ where $Q$ is a finite set (of states), $\mathcal{F}$ and $\mathcal{F}'$ are finite ranked sets (of input and output), $Q_f \subseteq Q$ (final states) and $\Delta$ is a finite set of rules of the form :

- $f(q_1(x_1), ..., q_n(x_n)) \to q(u)$ where $f \in \mathcal{F}$ and $u \in T(\mathcal{F}', \{x_1, ..., x_n\})$
- $q(x_1) \to q'(u)$ where $u \in T(\mathcal{F}', \{x_1\})$.

We say that $U$ is linear when the right side of the rules of $\Delta$ are. This defines a rewrite system $\to_U$ on $T(\mathcal{F} \cup \mathcal{F}' \cup Q)$. The relation induced by $U$ is then $\mathcal{R}(U) = \big\{(t, t') \mid t \in T(\mathcal{F}),\, t' \in T(\mathcal{F}'),\, t \to_U^* q(t'),\, q \in Q_f\big\}$.

1) Prove that tree morphisms are a special case of NUTT that is if $\mu : T(\mathcal{F}) \longrightarrow T(\mathcal{F}')$ is a morphism, then there exists a NUTT $U_\mu$ such that $\mathcal{R}(U_\mu) = \big\{(t, \mu(t)) \mid t \in T(\mathcal{F})\big\}$. Be sure that if $\mu$ is linear then $U_\mu$ is too.
2) Prove that the domain of a NUTT $U$, that is $\big\{t \in T(\mathcal{F}) \mid \exists t' \in T(\mathcal{F}'),\, (t, t') \in U\big\}$, is recognizable.
3) Prove that the image of a recognizable tree language $L$ by a linear NUTT $U$, that is $\big\{t' \in T(\mathcal{F}') \mid \exists t \in L,\, (t, t') \in U\big\}$, is recognizable.

**Exercise 2 : And a bit of minimization**

**Definition 1** *An equivalence relation $\equiv$ on $T$ is a congruence on $T(\mathcal{F})$ if for every $f \in \mathcal{F}_n$ :*

$$(\forall i, 1 \leq i \leq n, u_i \equiv v_i) \Rightarrow f(u_1, ..., u_n) \equiv f(v_1, ..., v_n)$$

*For a given tree language $L$, let us define the congruence $\equiv_L$ on $T(\mathcal{F})$ by : $u \equiv_L v$ if for all contexts $C \in C(\mathcal{F})$ :*

$$C[u] \in L \Leftrightarrow C[v] \in L$$

Prove that the following are equivalent :

1. L is a recognizable tree language
2. L is the union of some equivalence classes of a congruence of finite index
3. the relation $\equiv_L$ is a congruence of finite index.

   Then, show how to obtain the minimal automaton of a language.

**Exercise 3 : MSO on finite trees**

We consider trees with maximum arity 2. Give MSO formulae which express the following :

1. X is closed under predecessors
2. $x \subseteq y$ (with $\subseteq$ the prefix relation on positions)
3. 'a' occurs twice on the same path
4. 'a' occurs twice not on the same path
5. There exists a sub tree with only a's
6. The frontier word contains the chain 'ab'

**Exercise 4 : From formulaes to automaton**

Give tree automatons recognizing the languages on trees of maximum arity 2 defined by the formulae :

1. $(x \in S \wedge (x \downarrow_1 y \Rightarrow y \in S)) \wedge (z \in S \Rightarrow P_f(z))$
2. $\exists S.(x \in S \wedge (x \downarrow_1 y \Rightarrow y \in S)) \wedge (z \in S \Rightarrow P_f(z))$