# Automates d'arbre

## TD n°2 : Decision problems & tree homomorphisms

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**Exercise 1 : Back from TD1 : an abstract language.**

1) Let $\mathcal{E}$ be a finite set of linear terms on $T(\mathcal{F}, \mathcal{X})$. Prove that $Red(\mathcal{E}) = \{C[t\sigma] \mid C \in \mathcal{C}(\mathcal{F}), t \in \mathcal{E}, \sigma \text{ ground substitution}\}$ is recognizable.
2) Prove that if $\mathcal{E}$ contains only ground terms, then one can construct a DFTA recognizing $Red(\mathcal{E})$ whose number of states is at most $n+2$, where $n$ is the number of nodes of $\mathcal{E}$.

**Exercise 2 : Back from TD1 : Commutative closure**

Let $\mathcal{F} = \{f(2), a(0), b(0)\}$.

1) Let $L_1$ be the smallest set such that :
   - $f(a,b) \in L_1$
   - $t \in L_1 \Rightarrow f(a, f(t,b)) \in L_1$

   Prove that $L_1$ is recognizable.
2) Prove that $L_2 = \{t \in T(\mathcal{F}) \mid |t|_a = |t|_b\}$ is not recognizable.
3) Let $L$ be recognizable on $\mathcal{F}$ and $C(L)$ be the closure of $L$ by the congruence generated by the equation $f(x,y) = f(y,x)$. Prove that $C(L)$ is recognizable.
4) Let $L$ be recognizable on $\mathcal{F}$ and $AC(L)$ be the closure of $L$ by the congruence generated by the equations $f(x,y) = f(y,x)$ and $f(x, f(y,z)) = f(f(x,y), z)$. Prove that $AC(L)$ is not recognizable in general.

**Exercise 3 : Decisions problems**

We consider the **(GII)** problem (ground instance intersection) :

**Instance** : $t$ a term in $T(\mathcal{F}, \mathcal{X})$ and $\mathcal{A}$ a NFTA

**Question** : Is there at least one ground instance of $t$ accepted by $\mathcal{A}$ ?

1) Suppose that $t$ is linear. Prove that **(GII)** is P-complete.
2) Suppose that $\mathcal{A}$ is deterministic. Prove that **(GII)** is NP-complete.
3) Prove that **(GII)** is EXPTIME-complete.
   *hint : for the hardness, reduce the intersection non-emptiness problem (admitted to be EXPTIME-complete).*
4) Deduce that the complement problem :
   **Instance** : $t$ a term in $T(\mathcal{F}, \mathcal{X})$ and linear terms $t_1$, ..., $t_n$
   **Question** : Is there a ground instance of $t$ which is not an instance of any $t_i$ ?
   is decidable.

**Exercise 4 : Stability vs Recognizability**

We can see the set of runs of an NFTA $\mathcal{A} = (Q, \mathcal{F}, Q_f, \Delta)$ as a tree language on $\mathcal{F} \times Q = \{(f,q)(n) \mid f(n) \in \mathcal{F},\ q \in Q\}$ as the smallest set $Run(\mathcal{A})$ included in $T(\mathcal{F} \times Q)$ such that :

- if $a \to q \in \Delta$, then $(a,q) \in Run(\mathcal{A})$
- if $f(q_1, ..., q_n) \to q \in \Delta$ and $t_1, ..., t_n \in Run(\mathcal{A})$ with $t_i(\epsilon) = (\_, q_i)$ then $(f,q)(t_1, ..., t_n) \in Run(\mathcal{A})$.

Then the set of accepting runs can be seen as $Acc(\mathcal{A}) = \{t \in Run(\mathcal{A}) \mid t(\epsilon) = (\_, q),\ q \in Q_f\}$.

1) Prove that $Acc(\mathcal{A})$ is in the smallest class **Stab** of sets which contains all the $T(\mathcal{F})$ for any finite ranked set $\mathcal{F}$ and which is stable by image of linear morphisms and inverse image of

morphisms. For example, you should be able to prove that $Acc(\mathcal{A}) = \beta^{-1}(\gamma(\delta^{-1}(T(\mathcal{F}'))))$ where $\gamma$ is linear.

2) Deduce that **Stab** = **Rec**.