Advanced Complexity

TD n°5

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October 11, 2017

Exercise 1 : P-choice

A language L is said P-peek ($L \in \mathsf{P}p$) if there is a function $f : \{0,1\}^* \times 0, 1^* \to \{0,1\}^*$ computable in polynomial time such that $\forall x, y \in \{0,1\}^*$:

— $f(x,y) \in \{x,y\}$

— if $x \in L$ or $y \in L$ then $f(x,y) \in L$

f is called the peeking function for L.

1. Show that $\mathsf{P} \subseteq \mathsf{P}p$
2. Show that $\mathsf{P}p$ is closed under complementary
3. Show that if there exist L NP-hard in $\mathsf{P}p$, then $\mathsf{P} = \mathsf{NP}$
4. Let $r \in [0,1]$ a real number, we define L_r as the set of words $b = b_1...b_n \in \{0;1\}^*$ such that $0, b_1...b_n \leq r$. Show that $L_r \in \mathsf{P}p$
5. Deduce that there exist a non-recursive language in $\mathsf{P}p$

Exercise 2 : Complete problems for levels of PH

Show that the following problem is Σ_k^P-complete (under polynomial time reductions).

$\Sigma_k\mathtt{QBF}$: • INPUT : A quantified boolean formula $\psi := \exists X_1 \forall X_2 \exists ... Q_k X_k \phi(X_1, ..., X_k)$, where $X_1, ..X_k$ are k disjoint sets of variables, Q_k is the quantifier $\forall$ if k is even, and the quantifier $\exists$ if k is odd, ϕ is a boolean formula over variables $X_1 \cup \cdots \cup X_k$;

— QUESTION : is the input formula true ?

Define a similar problem $\Pi_k\mathtt{QBF}$ such that $\Pi_k\mathtt{QBF}$ is Π_k^P-complete.

Exercise 3 : Oracle machines

Let O be a language. A Turing machine with oracle O is a Turing machine with a special additional read/write tape, called the oracle tape, and three special states : $q_{query}, q_{yes}, q_{no}$. Whenever the machine enters the state q_{query}, with some word w written on the oracle tape, it moves **in one step** to the state q_{yes} or q_{no} depending on whether $w \in O$.

We denote by P^O (resp. NP^O) the class of languages decided in polynomial time by a deterministic (resp. non-deterministic) Turing machine with Oracle O. Given a complexity class $\mathcal{C}$, we define $\mathsf{P}^{\mathcal{C}} = \bigcup_{O \in \mathcal{C}} \mathsf{P}^O$ (and similarly for NP).

1. Prove that for any $\mathcal{C}$-complete language L, $\mathsf{P}^{\mathcal{C}} = \mathsf{P}^L$ and $\mathsf{NP}^{\mathcal{C}} = \mathsf{NP}^L$.
2. Show that for any language L, $\mathsf{P}^L = \mathsf{P}^{\bar{L}}$ and $\mathsf{NP}^L = \mathsf{NP}^{\bar{L}}$.
3. Prove that if $\mathsf{NP} = \mathsf{P}^{SAT}$ then $\mathsf{NP} = \mathsf{coNP}$.

Exercise 4 : Collapse of PH

1. Prove that if $\Sigma_k^P = \Sigma_{k+1}^P$ for some $k \geq 0$ then $\mathsf{PH} = \Sigma_k^P$. (Remark that this is implied by $\mathsf{P} = \mathsf{NP}$).
2. Show that if $\Sigma_k^P = \Pi_k^P$ for some k then $\mathsf{PH} = \Sigma_k^P$ (*i.e.* PH collapses).
3. Show that if $\mathsf{PH} = \mathsf{PSPACE}$ then PH collapses.
4. Do you think there is a polynomial time procedure to convert any QBF formula into a QBF formula with at most 10 variables ?

Exercise 5 : Relativization

Show that there is an oracle O such that $\mathsf{P}^O = \mathsf{NP}^O$.