Exercise 1: Language theory

Show that the following problems are PSPACE-complete:

1. NFA Universality
   - INPUT: a non-deterministic automaton $A$ over alphabet $\Sigma$
   - QUESTION: $L(A) = \Sigma^*$?
   
   Bonus: what is the complexity of this problem for a DFA?

2. NFA Equivalence
   - INPUT: two non-deterministic automata $A_1$ and $A_2$ over the same alphabet $\Sigma$
   - QUESTION: $L(A_1) = L(A_2)$
   
   Bonus: what is the complexity of this problem for a DFA?

3. DFA Intersection Vacuity
   - INPUT: deterministic automata $A_1, \ldots, A_m$ for some $m$
   - QUESTION: $\bigcap_{i=1}^{m} L(A_i) = \emptyset$?

Solution:

1. The converse problem is in NP, and PSPACE is stable by complement (remark that there will be a word not accepted of polynomial size in the size of the automaton if there is a word not accepted). For the hardness, suppose $M$ is a TM with polynomial space bound $p(n)$, and $w$ is an input to $M$ of length $n$. We will show how to take $M$ and $w$, and write down, in polynomial time, a regular expression $E$ that is $\Sigma^*$ if and only if $M$ does not accept $w$ (PSPACE is closed under complementation).

Considering that a run of $M$ can be written by writing the sequence of configurations (of length $p(n)$ with the blank symbols) separated by a #. Then, we construct $E = F + G + H$ where $F, G$ and $H$ respectively define the runs that do not start right, move right, or finish right:

- $H$: finishes wrong. $M$ fails to accept if the sequence has no accepting state. Thus, let $H = (\Sigma - Q_f)^*$, where $Q_f$ is the set of accepting states of $M$.

- $F$: Starts wrong. Any string in which the first $p(n) + 2$ symbols are not #, $q_0$ (the start state), $w$, and $p(n) - n$ blanks, is not the beginning of an accepting computation, and so should be in $L(E)$. We can write $F$ as the sum of the terms:
  - $(\Sigma - \{\#\})\Sigma^*$, i.e., all strings that do not begin with #.
  - $\Sigma(\Sigma - q_0)\Sigma^*$, i.e., all strings that do not have $q_0$ as their second symbol.
  - $\Sigma^{i+1}(\Sigma - a_i)\Sigma^*$, where $a_i$ is the $i$th position of $w$.
  - $\Sigma^i(\Sigma - B)\Sigma^*$, for all $n + 3 \leq i \leq p(n) + 1$.
  - $(\Sigma + \epsilon)\Sigma^{p(n)+1}$. This term covers all strings that are shorter than $p(n) + 2$ symbols, and therefore cannot have an initial state, regardless of the symbols found there.

- $G$: moves wrong. We need to capture all strings that have some point at which symbols separated by distance roughly $p(n)$ do not reflect a move of $M$. The idea is similar to that used in Cook’s theorem. Each position of a configuration is determined by the symbol at that position in the previous configuration and the
two neighboring positions. Thus, $G$ is the sum of terms $(\Sigma^*)UVW(\Sigma^p(n))X(\Sigma^*)$, where $U, V, W, X$ are four symbols of $\Sigma$ such that if $UVW$ were three consecutive symbols of a configuration of $M$, then $X$ would not be the symbol in the same position as $V$ in the next ID. For example, if none of $U, V, W$ are a state, then $X$ could be any symbol but $V$.

We can indeed compute those formulae in polynomial time, and if $M$ accepts, $E \neq \Sigma^*$, and if $M$ rejects, there is no accepting run and $E = \Sigma^*$.

Bonus: For a DFA, we check if a non acceptable state can be reached, it is NL.

2. We can guess a word (letter by letter, because its size is bounded by $2^{|Q_1|+|Q_2|}$) which is accepted by one and not the other, we use the power set construction on the fly for the second one, and so the co problem is NP and the problem is PSPACE. We can also build the two minimal automata by exhaustive search and check if they are indeed equal, which is also PSPACE. For the hardness, we are given an automaton $A$. We simply construct $B$ such that $L(B) = \Sigma^*$, and then:

$$L(A) = \Sigma^* \iff L(A) = L(B)$$

Bonus: For a DFA, the minimization is in $O(n \times \log(\log(n)))$ where $n$ is the number of state with the Hopcroft algorithm.

3. As for the previous one we guess a word which invalidate the problem. For the hardness we are given $A$. For the hardness, we can use the same proof as in the first one, but instead of making a global $E$, we build all the corresponding deterministic $A_i$ and check the universality of the union. Beware, for the moves wrong regexp, we can easily produce a NFA detecting a wrong formation, but with determinism, we need $p(n)$ DFA, one for each position where the mistake might be.

**Exercise 2: Did you get padding?**

Show that if $P = \text{PSPACE}$, then $\text{EXPTIME} = \text{EXPSPACE}$.

**Solution:**

We assume that $\text{PSPACE} \subseteq P$. Let $L_1$ be accepted by $M_1 \in \text{EXPSPACE}$, with $M_1$ deciding in $p(2^n)$ space, where $p$ is a polynomial function. We define

$$L_2 = \{(x, 1^{2|x|}) | M_1 \text{ accepts } x \text{ in space } p(|x| + 2|x|)\}$$

$L_2$ is accepted by $M_2$ which simulates $M_1$ and rejects if it uses too much space. Clearly, $M_2$ is in $\text{PSPACE} \subseteq P$. So we have $M_3$ which accepts $L_2$ in polynomial time. We can then build $M_4$ which on input $x$ produces $(x, 1^{2|x|})$ and simulates $M_3$. $M_4$ recognizes $L_1$, and $M_4 \in \text{EXPTIME}$. We do have $\text{EXPSPACE} \subseteq \text{EXPSPACE}$.

**Exercise 3: Too fast!**

Show that $\text{ATIME}(\log n) \neq L$.

**Solution:**

When considering $\text{ATIME}(\log n)$, we do not even have the time to read the full input. So any language which is in $L$ and needs for the input to be completely read will yield the result. For instance, one may use the palindromes language, or $0^n$ on a two letter alphabet, or $0^{2^k}$ on a one letter alphabet.

**Exercise 4: Direct application**

Show that $\text{EXPSPACE} = \text{AEXPTIME}$.

**Hint:** You may use that if $f$ is space-constructible, then :

$$\text{SPACE}(\text{poly}(f(n)) = \text{ATIME}(\text{poly}(f(n)))$$
Solution:
We set \( f(n) = 2^n \), then:

\[
\text{SPACE}(2^p(n)) \subseteq \text{SPACE}(\text{poly}(2^p(n))) = \text{ATIME}(\text{poly}(2^p(n))) \subseteq \text{AEXPTIME}
\]

As \( p \) is arbitrary, then \( \text{EXPSPACE} \subseteq \text{AEXPTIME} \). And conversely:

\[
\text{ATIME}(2^p(n)) \subseteq \text{ATIME}(\text{poly}(2^p(n))) = \text{SPACE}(\text{poly}(2^p(n))) \subseteq \text{EXPSPACE}
\]

Exercise 5: Closure under morphisms
Given a finite alphabet \( \Sigma \), a function \( f : \Sigma^* \rightarrow \Sigma^* \) is a morphism if \( f(\Sigma) \subseteq \Sigma \) and for all \( a = a_1 \cdots a_n \in \Sigma^* \), \( f(a) = f(a_1) \cdots f(a_n) \) (\( f \) is uniquely determined by the value it takes on \( \Sigma \)).

1. Show that \( \text{NP} \) is closed under morphisms, that is: for any language \( L \in \text{NP} \), and any morphism \( f \) on the alphabet of \( L \), \( f(L) \in \text{NP} \).

2. Show that if \( \text{P} \) is closed under morphisms, then \( \text{P} = \text{NP} \).

Solution:
1. If we are given \( L \) and \( M \) a NTM recognizing \( L \), then we can build \( M' \) which on inputs \( f(a) \) guesses \( a_1, \ldots, a_n \in \Sigma^n \) such that \( f(a) = f(a_1) \cdots f(a_n) \) and finally simulates \( M \) on \( a_1 \cdots a_n \). \( M' \) is a NTM in polynomial time which recognizes \( f(L) \). Thus \( f(L) \in \text{NP} \).

2. SAT is in \( \text{NP} \), and in particular, with the certificate definition, we know that

\[
L = \{ (\phi, u) | u \text{ is a correct valuation of } \phi \}
\]

is in \( \text{P} \). We may consider an alphabet such that the symbols for \( \phi \) and \( u \) are disjoints, and then, we may consider the morphism \( f \) such that \( f(\phi) = \phi \) and \( f(u) = 0^n \). We have \( f(L) = \text{SAT} \), \( L \in \text{P} \), and by hypothesis, \( \text{P} \) is closed under morphism. Thus, \( \text{SAT} \in \text{P} \), and with the Cook-Levin theorem, \( \text{P} = \text{NP} \).