Advanced Complexity

TD n°3

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Exercise 1 : Space hierarchy theorem

Using a diagonal argument, prove that for two space-constructible functions f and g such that $f(n) = o(g(n))$ (and as always $f, g \geq log$) we have $\mathsf{SPACE}(f(n)) \subsetneq \mathsf{SPACE}(g(n))$.

Exercise 2 : Polylogarithmic space

Let $\mathsf{polyL} = \cup_{k\in\mathbb{N}} \mathsf{SPACE}(log^k(n))$. Show that $\mathsf{polyL} \neq \mathsf{P}$.

Exercise 3 : Padding argument

1. Show that if $\mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$ for some $c > 0$, then $\mathsf{PSPACE} \subseteq \mathsf{NP}$.
2. Deduce that $\mathsf{DSPACE}(n^c) \neq \mathsf{NP}$.

Exercise 4 : My very first PSPACE-complete problem

Show that the following problem is PSPACE-complete (not assuming anything about QBF) :

— INPUT : a Turing Machine M and a word w and a number t written in unary

— QUESTION : does M accepts w within space t ?

Exercise 5 : PSPACE and games

The Geography game is played as follow :

— The game starts with a given name of a city, for instance *Cachan* ;

— the first player gives the name of a city whose first letter coincides with the last letter of the previous city, for instance *Nice* ;

— the second player gives then another city name, always starting with the last letter of the previous city, for instance *Evry* ;

— the first player plays again, and so on – with the restriction that no player is allowed to give the name of a city already used in the game ;

— the loser is the first player who does not find a new city name to continue.

This game can be described using a directed graph whose vertices represent cities and where an edge (X, Y) means that the last letter of the city X is the same as the first letter of the city Y. This graph has also a vertex marked as the initial vertex of the game (the initial city). Each player choses a vertex of the graph, the first player choses first, and the two players alternate their moves. At each move, the sequence of vertices chosen by the two players must form a simple path in the graph, starting from the distinguished initial vertex.

Player 1 wins the game if, after some number of moves, Player 2 has no valid move (that is no move that forms a simple path with the sequence of previous moves).

`GEOGRAPHY` is the following problem :

— INPUT : a directed graph G and an initial vertex s.

— QUESTION : is the player 1 sure to win the game on G starting at s ?

Show that `GEOGRAPHY` is PSPACE-complete by :

1. Showing that `GEOGRAPHY` is PSPACE
2. That the satisfiability of a QSAT formula of the form $\exists x_1 \forall x_2 \ldots \exists x_n \bigwedge(C_i)$ where C_i is a clause can be expressed as a `GEOGRAPHY` instance.

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