Probabilistic Aspects of Computer Science: Exercise 6

Exercise 1 *Given any PFA $\mathcal{A}$, show that there is another automota $\mathcal{B}$ such that:*

1. $\mathcal{L}_{>\frac{1}{2}}(\mathcal{B}) = \mathcal{L}_{>\frac{1}{\sqrt{2}}}(\mathcal{A})$.
2. *for any x,* $\mathcal{L}_{>x^2}(\mathcal{B}) = \mathcal{L}_{>x}(\mathcal{A})$.

Exercise 2 *Show that any prefix code is uniquely decodable. Give an example of a non-singular code that is not uniquely decodable. Give an example of a uniquely decodable code that is not a prefix code.*

Exercise 3 *Show that if $\mathcal{C}$ is a prefix code, then the code obtained by reversing codewords of $\mathcal{C}$ is uniquely decodable. Deduce a code which is not a prefix-code but that is yet uniquely decodable.*

Exercise 4

1. *Let X be a random variable whose domain is a finite set denoted as $\mathcal{X}$, and $Y = \varphi(X)$, where φ is a bijective and deterministic function. Show that: $H(X) = H(Y)$.*
2. *Let Z be a random variable with domain a finite set $\mathcal{Z}$, and ψ a deterministic function, not necessarly bijective. Show that: $H(Z, \psi(Z)) = H(Z)$.*

Exercise 5 *Let $\mathcal{Q} = (1, 2, \ldots, m)$ be a finite set and k a natural integer. Show that the entropy of the probability distribution $(p_1, \ldots, p_k, p_{k+1}, \ldots, p_m)$ is less than the entropy of the probability distribution $(p_1, \ldots, p_{k-1}, \frac{p_k+p_{k+1}}{2}, \frac{p_k+p_{k+1}}{2}, p_{k+2}, \ldots, p_m)$.*

Exercise 6 *Show that if $H(Y|X) = 0$, then Y is a function of X (i.e. for all x with $p(x) > 0$, there is only one possible value of y with $p(x, y) > 0$.*