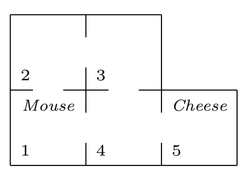
Probabilistic Aspects of Computer Science: Exercise 2

Exercise 1 *Exercise 6 and exercise 8 from last week's exercises.*

Exercise 2 *Let* $M = \{Q, \mu^{(0)}, \delta\}$ *be a Markov chain on states* $Q = \{q_1, q_2, \ldots, q_k\}$. *Recall that if the state* q_{i_0} *of* M *is absorbing, a state* $q_j \in \mathtt{Bad}_{i_0}$ *iff* q_{i_0} *is not reachable from* q_{i_0} *and* $q_j \in \mathtt{VeryGood}_{i_0}$ *iff no state in* $\mathtt{Bad}_{i_0}$ *is reachable from* q_j. *Recall also that if* $q_j \in \mathtt{Bad}_{i_0}$ *then* $\lim_{n\to\infty}(\delta^n)_{j,i_0} = 0$.

Show that if $q_j \in \mathtt{VeryGood}_{i_0}$, *then we have that* $\lim_{n\to\infty}(\delta^n)_{j,i_0} = 1$.

Exercise 3 *A mouse is wandering in a maze partioned into* 5 *rooms . We suppose that it is changing rooms in each time period* $t = 0, 1, 2, 3, \ldots$ *and that, if on time* n, *it is in a room with* k *doors, then it chooses one of them with probability* $1/k$ *and is on time* $n + 1$ *in the next room. All its choices are supposed to be independent.*



The path of the mouse can be described as a Markov chain whose states are the rooms and the transition matrix is the matrix of probabilities of going from one room to another one.

1. *Give the transition matrix of* M *and compute the probability of the path defined by event* $(X_0 = 1, X_1 = 2, X_2 = 1, X_3 = 3, X_4 = 5)$.

2. *Suppose that the initial state is room* 1. *Compute the probability that the mouse reaches room* 5 *in* $2, 3$ *and* 4 *steps.*

3. *Suppose that initial distribution is such that the probability of the mouse being in room* i *is proportional to the number of doors in room* i. *What is the corresponding initial distribution?*

4. *Does the resulting Markov chain admit a unique stationary distribution? Explain why? If yes, compute the distribution.*

5. *Now, assume that when the mouse reaches room* 5, *it eats the cheese. What is the probability that the mouse eats the cheese?*

Exercise 4 *We consider a Markov chain* $M = \{Q, \mu^{(0)}, \delta\}$ *on* 4 *states whose transition matrix is given by:*

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. *Give the transition graph? Which states are absorbing? Do the other states converge? For each state* q *of Markov chain, what is* $\lim_{n\to\infty} \mu^{(0)}\delta^n$*?*

2. *Compute the probability of the two following events:* $(X_0 = 1, \forall n \geq 1 \quad X_n = 3)$ *and* $(X_n = 1,$ *if* n *is even and* $X_n = 2$ *otherwise.)*

3. *Show that any distribution of the form* $\pi_0 = (0, 0, r_0, s_0)$ *with* $r_0 + s_0 = 1$ *is a stationary distribution. Is there any other stationary distribution?*

Exercise 5 *We now discuss a synchronous consensus protocol. Assume that there are* N *processes trying to agree upon a leader. Let the processes be* $p_1, p_2, \ldots, p_N$*. The protocol proceeds in rounds. In each round, each process* p_i *tosses a* N*-faced fair dice and if the result is* j*, then* p_i *declares* j *to be its choice for the leader. If all processes agree on the choice then the agreed choice is declared to be the leader. If they disagree, then start again by tossing the coins. Let* $X^{(n)}$ *be the random variable that denotes the leader at beginning of round* n *(if there is no leader then* $X^{(n)}$ *takes the value* 0.*) Note that* $X^{(0)}, X^{(1)}, \ldots$ *is a Markov chain. Let us call this Markov chain* SM*.*

1. *Give the transition matrix for the Markov chain* SM*.*

2. *If* δ *is the transition matrix of* SM*, does* $\lim_{n\to\infty} \delta^n$ *exists? If yes, please compute it. Does* $\lim_{n\to\infty} X^{(n)}$ *exists? If yes, please compute it.*

3. *What are stationary distributions of* SM?

4. *What is the probability that a leader is elected?*