Probabilistic Aspects of Computer Science: Exercise 1

Exercise 1 *Let $Q = \{q_1, \dots q_k\}$ be a finite set and $M = (Q, \mu^{(0)}, \delta)$ be an irreducible Markov chain. Show that if there is an i such that $\delta_{i,i} > 0$ then M is aperiodic also.*

Exercise 2 *Show that if a Markov Chain has two stationary distributions, then it has an infinite number of stationary distributions.*

Exercise 3 *Given a Markov chain $M = \{Q, \mu^{(0)}, \delta\}$ on the state space $Q = \{q_1, q_2, \dots, q_k\}$, a distribution μ on Q is said to be reversible if for each $1 \leq i, j \leq k$ $\mu_i \delta_{i,j} = \mu_j \delta_{i,j}$. A Markov chain M is said to be reversible if M has a reversible distribution.*

Show that if μ is a reversible distribution for the Markov chain M, then μ is a stationary distribution.

We have the following properties of the distance.

Exercise 4 *Let Q be a set of k elements. For every distribution μ, ν, ρ on Q we have the following.*

1. $d(\nu, \mu) = d(\mu, \nu) \geq 0$ *and* $(\mu, \nu) = 0$ *iff* $\mu = \nu$.
2. $d(\mu, \nu) \leq d(\mu, \rho) + d(\rho, \mu)$.
3. $d(\mu, \nu) \leq 1$.
4. *If a $k \times k$ matrix δ is a transition matrix, i.e., for each $1 \leq i \leq k$, $\sum_{j=1}^{k} \delta(q_i, q_j) = 1$ then*
$$d(\mu\delta, \nu\delta) \leq d(\mu, \nu).$$

Exercise 5 *Let X_0 be a random variable in $\mathbb{Z}$ and Z_n a sequence of random variables in $\{-1, 1\}$ independent identically distributed, independent of X_0, such that $P(Z_n = 1) = p$ for all n, where $0 < p < 1$. Define $\{X_n\}_{n \geq 0}$ by $X_{n+1} = X_n + Z_{n+1}$.*

1. *Show that $\{X_n\}$ is a Markov chain, denoted as 'random walk on $\mathbb{Z}$', and give its transition matrix.*

2. *Is this chain aperiodic?*

3. *Suppose $X_0 = 0$. What is the probability to come back to 0 after n steps?*

Exercise 6 *Let $\{X_n\}_{n\geq 0}$ be a Markov chain and $i_0, \cdots, i_{n-1}, i, j_1, \cdots, j_k$ states. Show that* $P(X_{n+1} = j_1, \cdots, X_{n+k} = j_k \mid X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0)$ $= P(X_{n+1} = j_1, \cdots, X_{n+k} = j_k \mid X_n = i)$

Exercise 7 *Three gangsters A, B and C meet at a crossroads in Washington D.C. and naturally begin shooting each other. Every surviving man shoots another surviving man of his choice every 10 seconds. The probability to hit the target for A, B and C is α, β and γ. A is the most hated among the three so that as long as he lives, B and C ignore each other and shoot him. A cannot stand B, so that as long as B lives, A shoots him. C is shooted by another one only if A or B dies. What is the probability that A, B or C survives?*

Exercise 8 *Let $\{X_n\}_{n\geq 0}$ be a Markov chain on the state space E et transition matrix δ. For $L \geqslant 1$, we define $Y_n = (X_n, X_{n+1}, \cdots, X_{n+L})$.*

1. *The process $\{Y_n\}_{n\geq 0}$ takes its value in $F = E^{L+1}$. Show that it is a Markov chain and give its transition matrix.*

2. *Show that $X_{nn\geq 0}$ is irreductible. Show that it is the case for $Y_{nn\geq 0}$ if we restrict its state space to $F = \{(i_0, \cdots, i_L \in E^{L+1} \mid \delta_{i_0 i_1}, \cdots, \delta_{i_{L-1} i_L} \geq 0)\}$.*

3. *Show that if $X_{nn\geq 0}$ has a stationary distribution μ then $Y_{nn\geq 0}$ has also a stationary distribution $\mu^\star$. Explicit $\mu^\star$.*