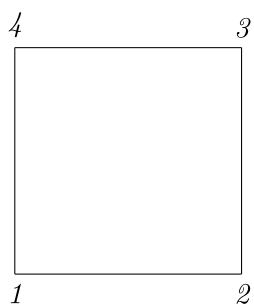
Probabilistic Aspects of Computer Science: Exercise 1

Exercise 1 *Show that if* $(\Omega, \mathcal{F}, \mu)$ *is a probability space and* $A, B \subseteq \Omega$ *then* $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$ *and* $\mu(A \setminus B) = \mu(A) - \mu(A \cap B)$.

Exercise 2 *Consider the case of tossing* 4 *fair coins. How can we model this situation as a probability space? What is the probability of getting at least* 2 *heads? What is the probability of getting exactly* 2 *tails?*

Exercise 3 *Consider a person walking on the following square.*



The person starts at intersection 1*. Then the person tosses* **two** *fair coins. If the first coin turns up heads then the person decides to stay at* 1*; otherwise the person tosses another coin. If the second coin turns up heads then the person moves anti-clockwise to intersection* 2*, otherwise the person moves clockwise to intersection* 4.

At intersection 2*, the person again tosses two coins and if the first coin turns up heads then the person decides to stay at intersection* 2*; otherwise the person tosses another coin. As before, if the second coin toss comes up heads then the person moves anti-clockwise to* 3 *otherwise the person moves clockwise to* 1.

The same process is repeated at 3 *and* 4*. At each intersection, the person tosses two coins. If the first coin toss is head then the person stays at the intersection; otherwise it moves to one of its neighbors depending on the result of the second coin toss. Let the resulting Markov chain be* M_{sq2}.

1. *What is the transition matrix of* M_{sq2}*? Also draw the transition graph of* M_{sq2}*. What is the initial distribution of* M_{sq2}*?*
2. *Is* M_{sq2} *aperiodic? Irreducible?*
3. *Show by induction that for all time instants* $n > 0$*, the distribution* $\mu^{(n)}$ *of* M_{sq2} *is* $[\frac{1}{4} + \frac{1}{2^{n+1}}, \frac{1}{4}, \frac{1}{4} - \frac{1}{2^{n+1}}, \frac{1}{4}]$.
4. *Does* $\lim_{n\to\infty} \mu^{(n)}$*exist? If yes, what is* $\lim_{n\to\infty} \mu^{(n)}$*exist?*

Exercise 4 *If $\mu^{(n)}$ are the distributions at time n of a Markov chain $M = (Q, \mu^{(0)}, \delta)$ such that $\lim_{n\to\infty} \mu^{(n)}$ exists then show that the distribution $\mu_{\mathsf{stat}} = \lim_{n\to\infty} \mu^{(n)}$ satisfies the equation $\mu_{\mathsf{stat}} = \mu_{\mathsf{stat}}\delta$.*

Exercise 5 *We sketch the proof of following theorem–*

> *Let $Q = \{q_1, \ldots q_k\}$ be a finite set and let $M = (Q, \mu^{(0)}, \delta)$ be a Markov chain. If M is irreducible and aperiodic, then there is a N such that*
>
> $$\forall n \geq N, \forall 1 \leq i, j \leq k.\ (\delta^n)_{i,j} > 0.$$

We use the following fact from elementary number theory.

Fact: *A set $S = \{n_1, n_2, \ldots,\}$ of natural numbers is said to be closed under addition if $\ell + k \in S$ whenever ℓ and k are in S. If $S = \{n_1, n_2, \ldots,\}$ is closed under addition and $g.c.d(n_1, n_2, \ldots) = 1$, then there is a M such that for all $n \geq M$, $n \in S$.*

1. *Show using the above fact that if δ is the transition matrix of an aperiodic Markov chain on $Q = \{q_1, \ldots q_k\}$, then there is a natural number M such that for all $n \geq M$ and for all $1 \leq i \leq k$ $(\delta^n)_{i,i} > 0$.*
2. *Using Part 1 of the exercise, prove the Theorem.*

Exercise 6 *Let $Q = \{q_1, \ldots q_k\}$ be a finite set and $M = (Q, \mu^{(0)}, \delta)$ be an irreducible Markov chain. Show that if there is an i such that $\delta_{i,i} > 0$ then M is aperiodic also.*

Exercise 7 *We give a very simplified version of the internet assumed by the PageRank algorithm employed by search algorithms (for example, by Google). The algorithm assumes that the internet consists of some webpages which have hyperlinks to other webpages. The person browsing the internet decides (probabilistically) whether to click on one of these links or visit a new page by entering it in the address bar.*

Assuming that there are N webpages in the world, the PageRank algorithm creates a Markov chain M with N states follows. Let the webpages be named $p_1, p_2, \ldots p_N$. Then the set $\{p_1, p_2 \ldots, p_N\}$ are the states of M. Now the transition function δ of M is defined as follows. If the webpage p_i has links to every other page then $\delta_{i,j} = \frac{1}{N-1}$. If p_i has links to $N' < N - 1$ webpages then $\delta_{i,j} = \frac{0.85}{N'}$ if p_i has a link to page p_j and $\delta_{i,j} = \frac{0.15}{N-N'-1}$ if p_i does not have a link to page p_j. The initial distribution of M assigns probability $\frac{1}{N}$ to each of the states.

1. *Is the Markov chain aperiodic?*
2. *Is the Markov Chain irreducible?*