Rewriting Techniques, 3: multisets, RPO, WFI

28-11-2019

Exercise 1:
Let $A$ and $B$ be two terminating relations on the same set of terms. Show that if $AB \subseteq BA$ then $A \cup B$ is a terminating relation. Use well-founded induction.

Given a strict order $>$ on a set $A$, we define the corresponding multiset order $>_\text{mul}$ on $\text{Mult}(A)$ as follows: $M >_\text{mul} N$ if and only if there exist $X,Y \in \text{Mult}(A)$ such that

1. $\emptyset \neq X \subseteq M$;
2. $N = (M \setminus X) \cup Y$;
3. $\forall y \in Y \exists x \in X \, x > y$.

Exercise 2:
Order the multisets $\{2\}$, $\{1,3\}$, $\{1,1,2\}$, $\{2,2,2\}$, $\{1,2\}$.

Exercise 3:
Given a strict order $(A, >)$, define the following single-step relation on $\text{Mult}(A)$:

$$M >_\text{mul} N \iff \exists x \in M, Y \in \text{Mult}(A). N = (M - \{x\}) \cup Y \land \forall y \in Y. x > y$$

Show that $>_\text{mul}$ is the same as the transitive closure of $>_1\text{mul}$ (hint: show that each relation is included in the other using appropriate inductions). Conclude that $>_\text{mul}$ is transitive.

The lexicographic order $>_{\text{lex}}$ for the Cartesian product $\times$ of two domains $(A, >_A)$ and $(B, >_B)$ is defined as follows: $(a_1, b_1) >_{\text{lex}} (a_2, b_2)$ if and only if $a_1 > a_2$ or $a_1 = a_2$ and $b_1 > b_2$. This order can be readily extended con Cartesian products of arbitrary length by recursively applying this definition, i.e. by observing that $A \times B \times C = A \times (B \times C)$.

In the following, let $\Sigma$ be a finite signature, $V$ a set of variables and $T(\Sigma, V)$ the terms built from those sets.

For every $f \in \Sigma$ let $\text{status}(f) \in \{\text{mul}, \text{lex}\}$ be its status (status is then called status function) and let $>$ be a strict order on $\Sigma$. The recursive path order $>_{\text{rpo}}$ on $T(\Sigma, V)$ induced by $>$ is defined as follows. $s >_{\text{rpo}} t$ if and only if one of the following holds:

1. $t$ is a variable appearing in $s$ and $s \neq t$, or
2. there exists $i \in [1, m]$ such that $s_i >_{\text{rpo}} t$, or
3. $f > g$ and $s >_{\text{rpo}} t_j$ for all $j \in [1, n]$, or
4. $f = g$, for all $j \in [1, m]$ it holds $s >_{\text{rpo}} t_j$ and $(s_1, \ldots, s_m)(>_{\text{rpo}})\text{status}(f)(t_1, \ldots, t_m)$.

The lexicographic path order is a recursive path order s.t. for all $f \in \Sigma$, $\text{status}(f) = \text{lex}$, whereas the multiset path order is a recursive path order s.t. for all $f \in \Sigma$, $\text{status}(f) = \text{mul}$, where we define $(s_1, \ldots, s_m)(>_{\text{rpo}})\text{mul}(t_1, \ldots, t_m)$ as $\{ s_1, \ldots, s_m \}(>_{\text{rpo}})\text{mul} \{ t_1, \ldots, t_m \}$.

Exercise 4:
Show that it is not possible to prove termination using lexicographic path ordering of the following term rewrite system:

$$\{ a(a(x)) \rightarrow a(x), \ s(a(x)) \rightarrow a(x) \}$$
Exercise 5:
Show that the termination of the following rewriting system cannot be proven with lexicographic path order but can be proven with multiset path order.

\[
\begin{align*}
0 + x & \rightarrow 0 \\
0 \times x & \rightarrow x \\
s(x) + y & \rightarrow s(x + y) \\
s(x) \times y & \rightarrow (y \times x) + y
\end{align*}
\]

For next week: simple termination, dependency pairs

A binary relation \( R \) on terms has the subterm property if \( C[t] R t \) for all non-empty contexts \( C \) and terms \( t \). A simplification order is a rewrite order with the subterm property.

Exercise 6:
Questions 1 is independant from 2, 3 and 4.

1. Show that a transitive relation \( \triangleright \) on terms has the subterm property if and only if

\[
f(t_1, \ldots, t_n) \triangleright t_i
\]

for all function symbols \( f \) of arity \( n \geq 1 \), terms \( t_1, \ldots, t_n \) and \( i \in \{1, \ldots, n\} \).

2. Let \( (w, w_0) \) be an admissible weight function. Assume that \( f \) is of arity 1, \( w(f) = 0 \) and that there is \( g \) such that \( f \not\gg g \). Prove that under this conditions \( \triangleright_{kbo} \) does not satisfy the subterm property.

3. Let \( \triangleright \) be a precedence and \( (w, w_0) \) an admissible weight function, show that \( t \in \text{Var}(s) \) and \( s \neq t \) then \( s \triangleright_{kbo} t \).

4. Let \( \triangleright \) be a precedence and \( (w, w_0) \) an admissible weight function. Show that the Knuth-Bendix order \( \triangleright_{kbo} \) has the subterm property.

Exercise 7:
We consider the following TRS:

\[
\begin{align*}
m(x, 0) & \rightarrow 0 \\
m(s(x), s(y)) & \rightarrow m(x, y) \\
q(0, s(y)) & \rightarrow 0 \\
q(s(x), s(y)) & \rightarrow s(q(m(x, y), s(y))) \\
p(0, y) & \rightarrow y \\
p(x, s(y)) & \rightarrow s(p(x, y)) \\
m(m(x, y), z) & \rightarrow m(x, p(y, z))
\end{align*}
\]

1. Which rule makes the termination of this TRS not provable with KBO or RPO?
2. What are the defined symbols?
3. Compute the marked dependency pairs.
4. Draw the dependency graph approximation.
5. What are the inequalities that are enough to consider? What can instead be ignored?
6. Find a weakly monotonic polynomial interpretation on integers satisfying those inequalities.

Exercise 8:
Consider the following TRS

\[
\begin{align*}
f \; 0 & \rightarrow s \; 0 \\
f \; (s \; 0) & \rightarrow s \; 0 \\
f \; (s \; (s \; x)) & \rightarrow f \; (f \; (s \; x))
\end{align*}
\]

1. Compute the dependency pairs.
2. Prove the termination by constructing a suitable reduction pair based on a weakly monotone interpretation in \( \mathbb{N} \).
3. Is the system simply terminating?