A matrix interpretation on integers is the following:

- a positive integer \( d \);
- for every symbol \( f \) of arity \( n \), \( n \) matrices \( M_{f,1}, \ldots, M_{f,n} \in \mathbb{N}^{d \times d} \);
- for every symbol of arity \( n \), a vector \( V_f \in \mathbb{N}^d \);
- a non-empty set \( I \subseteq \{1, \ldots, d\} \) satisfying that for every symbol \( f \) of arity \( n \) the map

\[
L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \text{ defined as } L_f(X_1, \ldots, X_n) = V_f + \sum_{i=1}^n M_{f,i}X_i
\]

is monotonic with respect to \( >_I \) if \( X >_I Y \) holds if and only if for every \( i \in \{1, \ldots, d\} \), \( X[i] \geq Y[i] \) and there is \( j \in I \) such that \( X[j] > Y[j] \).

Then \((\mathbb{N}^d,(L_f)_f,>_I)\) is a well-founded monotone algebra.

**Exercise 1:**
Consider the TRS \{ \text{s(a)} \to \text{s(p(a))}, \text{p(b)} \to \text{p(s(b))} \}.

1. Prove that its termination cannot be proved by a polynomial interpretation on integers;
2. Use the following matrix interpretation to prove termination w.r.t. \( >_{(1,2)} \).

\[
L_a(X) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} X \quad L_p(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X \quad L_a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad L_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

3. Why does it fail if we take \( >_{(1)} \) instead? Is there another matrix interpretation that works with this ordering?

**Solution:**
(1) From the first rule, the degree of \( P \) must be one. The same holds for \( P_a \), thanks to the second rule. This implies that \( P_{s(a)} \) of the form \( s_1a + s_0 \) whereas \( P_{s(p(a))} \) of the form \( s_1p_1a + s_1p_0 + s_0 \) which, since \( p_1 \geq 1 \), is sufficient to conclude that the termination of this TRS cannot be proved by a polynomial interpretation on integers.

(2) It holds that:

\[
L_a(L_a) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} >_{(1,2)} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = L_a(L_p(L_a))
\]
\[
L_p(L_b) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} >_{(1,2)} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = L_p(L_a(L_b))
\]

(3) From the second rule, \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} >_{(1)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) does not hold. No, let \( L_a(X) = M_aX + V_a \) and \( L_p(X) = M_pX + V_p \). For the first rule it will hold

\[
L_a(L_a) = M_aL_a + V_a
\]
\[
L_a(L_p(L_a)) = M_aM_pL_a + M_aV_p + V_a
\]

To make \( >_{(1)} \) it must therefore hold

\[
(M_a)_{1,1}(L_a)_{1,1} + \cdots + (M_a)_{1,d}(L_a)_{1,d} > (M_p)_{1,1}((M_a)_{1,1}(L_a)_{1,1} + \cdots + (M_a)_{1,d}(L_a)_{1,d}) + \cdots
\]

which implies \( (M_p)_{1,1} = 0 \). Similarly, from the second rule, \( (M_a)_{1,1} = 0 \). This implies that no polynomial interpretation with the ordering \( >_{(1)} \) can be defined for this TRS, since for any \( m > n, (m_0,0,\ldots,0) >_{(1)} (n_0,0,\ldots,0) \) but \( L_a(m_0,0,\ldots,0) =_{(1)} L_a(n_0,0,\ldots,0) \).
Exercise 2:  
Let $A \subseteq \mathbb{N}$ and $P_f$ be respectively the domain and the interpretation, for each function symbol $f$, of a polynomial interpretation of integers for a TRS (note: the TRS is therefore terminating). Take $a \in A \setminus \{0\}$. 

1. Define $\pi_a : T(F, X) \rightarrow A$ as the function which maps every variable $x$ to $a$ and every term of the form $f(t_1, \ldots, t_n)$ to $P_f(\pi_a(t_1), \ldots, \pi_a(t_n))$. Prove that $\pi_a(t)$ is greater or equal to the length of every reduction starting from $t$.

2. Show that there exists $d$ and $k$ positive integers such that for every $f \in F$ of arity $n$ and every $a_1, \ldots, a_n \in A \setminus \{0\}$ it holds $P_f(a_1, \ldots, a_n) \leq d \prod_{i=1}^{n} a_i^k$.

3. From the previous point, pick $d$ to be also greater or equal than $a$ and fix $c \geq k + \log_2(d)$.

Prove that $\pi_a(t) \leq 2^{2^{c|t|}}$.

Consider now any finite TRS and a function symbol $f$. Prove that there exists an integer $k$ such that if $s \rightarrow t$ then $|t|_f \leq k(|s|_f + 1)$, where $|.|_f$ is the number of $f$.

Deduce that the TRS 

$$
\{ a(0,0) \rightarrow a(y), \ a(s(x),0) \rightarrow a(x,a(0)), \ a(s(x),s(y)) \rightarrow a(x,a(s(x),y)) \}
$$

simulating the Ackermann’s function, cannot be proved terminating using a polynomial interpretation over integers.

Solution:

- The proof is by induction on the $\rightarrow$ relation. Let $t$ be irreducible. Then the length of all its reductions is 0 and $\pi_a(t) \geq 0$ by definition. For the inductive step, suppose $t \rightarrow t'$ s.t. $t \rightarrow t' \rightarrow \ldots$ is the maximal reduction from $t$. There exists a context $C$, a valuation $\sigma$ and a rewriting rule $l \rightarrow r$ such that $t = C[\sigma] \rightarrow C[\sigma] = t'$. W.l.o.g. we can consider just terms of the form $f(t_1, \ldots, t_n)$. By inductive hypothesis, $\pi_a(\sigma(X)) = P_f(\pi_a(\sigma(X_1)), \ldots, \pi_a(\sigma(X_n)))$ is greater or equal to the length of every reduction starting from $r \sigma$. It follow that $\pi_a(\sigma) = P_f(\pi_a(\sigma(X_1)), \ldots, \pi_a(\sigma(X_n))) \geq \pi_a(r \sigma) + 1$ and therefore $\pi_a(\sigma)$ is greater or equal to the length of every reduction starting from $l \sigma$.

- Let $\{ s_0, \ldots, s_m \}$ be the coefficient of the polynomial $P_f$, let $d \geq \sum_{i=0}^{n} a_i$ (so $d \geq 1$) and let $k \geq 1$ be also greater or equal to the degree of $P_f$. The thesis can be rewritten as $P_f(a_1, \ldots, a_n) \leq (\sum_{i=0}^{m} s_i) \prod_{i=1}^{n} a_i^j = \sum_{i=1}^{m} (s_i \prod_{j=1}^{n} a_i^j)$. Moreover there exists $k_1, k_2, \ldots, k_m \leq k$ such that $P_f(a_1, \ldots, a_n) = \sum_{i=1}^{m} (s_i \prod_{j=1}^{n} a_i^j)$ for all $i \in [1, m]$ and all $i \in [1, m]$. Moreover $a_1, \ldots, a_n \in A \setminus \{0\}$, and therefore the thesis trivially holds since for all $i \in [1, m] s_i \prod_{j=1}^{n} a_i^j \leq s_i \prod_{j=1}^{n} a_j^j$.

By induction of $t$. If $t$ is a variable, then $|t| = 1$ and $\pi_a(t) = a \leq 2^{a} \leq 2^{2^{k|\sigma|}} \leq 2^{2^{c|t|}}$. If $t$ is of the form $f(t_1, \ldots, t_n)$ then $\pi_a(t) = P_f(\pi_a(t_1), \ldots, \pi_a(t_n))$. By inductive hypothesis, since $P_f$ is monotone, $\pi_a(t) \leq P_f(2^{2^{c|t_1|}}, \ldots, 2^{2^{c|t_n|}})$. From (2) it follows that $P_f(2^{2^{c|t_1|}}, \ldots, 2^{2^{c|t_n|}}) \leq d \prod_{i=1}^{n} (2^{2^{c|t_i|}})^k = d 2^{\sum_{i=1}^{n} k (2^{c|t_i|})} = 2^{\log_2(d) \sum_{i=1}^{n} k (2^{c|t_i|})} \leq 2^{\log_2(d) + k \sum_{i=1}^{n} (2^{c|t_i|})}$. Since $d \geq a \geq 1$ and $k \geq 1$ it holds that $c \geq 1$ and therefore $2^{(\log_2(d) + k \sum_{i=1}^{n} (2^{c|t_i|})} \leq 2^{2^{c|t|}} \leq 2^{2^{c|t|}}$.

W.l.o.g. consider $s = l \sigma$ and $t = r \sigma$ for a rewriting rule $l \rightarrow r$ and a valuation $\sigma$. The number of occurrences of $f$ in $l \sigma$ is $|l|_f + \sum_{p \in \{ l|l|_p \in X \}} |\sigma(l_p)|_f$ where $|l|_f$ only depends on the left side of the rewriting rule. Similarly, $|r \sigma|_f = |r|_f + \sum_{p \in \{ r|l|_p \in X \}} |\sigma(r_p)|_f$ where $|r|_f$ depends only on the right side of the rewriting rule. Let $V$ the number of variables in $r$ (i.e. $\{ p|l|_p \in X \}$). It holds that $|r \sigma|_f \leq |r|_f + V \sum_{p \in \{ l|l|_p \in X \}} |\sigma(l_p)|_f$. Since every variable of $r$ also occurs in $l$ it must hold that $|r \sigma|_f \leq |r|_f + V \max_{p \in \{ l|l|_p \in X \}} |\sigma(l_p)|_f$. Moreover $\max_{p \in \{ l|l|_p \in X \}} |\sigma(l_p)|_f$ is trivially less or equal that all the occurrences of $f$. 

Page 2
We consider the Ackermann’s function

\[ \text{Ack} \]

Consider exercise on semantic labeling of TD1, where the following rewriting system was defined

- \( f (s \ X) \rightarrow f (p (s \ X)) \circ (s \ X) \)
- \( p (s \ X) \rightarrow s (p (s \ X)) \)

1. Prove that RPO cannot prove the termination of the system.
2. Prove that a labeled system can be proved terminating with RPO.

**Solution:**

1. Considering the first rewrite rule, \( f > \circ \) is needed. Then we must check that
   - (a) \( f (s \ X) >_{rpo} f (p (s \ X)) \) and
   - (b) \( f (s \ X) >_{rpo} (s \ X) \).

   Second condition is immediate since \( s \ X < f (s \ X) \).

   First condition needs \( s \ X >_{rpo} p (s \ X) \) which is impossible since \( s \ X <_{rpo} p (s \ X) \) since \( s \ X < p (s \ X) \).

2. With the following interpretation on \( \mathbb{N} \): \( z = 0, s X = X + 1, p X = \max(0, X - 1), \)

   \( \text{lab}(R) \) is the following (infinite) system, for all \( i \in \mathbb{N} \)

   \[
   \begin{align*}
   f_{i+1} (s_i \ X) &\rightarrow f_i (p_{i+1} (s_i \ X)) \circ (s_i \ X) & p (s \ z) &\rightarrow z \\
   p_{i+2} (s (s \ X)) &\rightarrow s_i (p_{i+1} (s_i \ X))
   \end{align*}
   \]

   If we take \( f_i > \circ, f_i > f_{i+1}, f_i > p_{i+1} \), the first rule is ensured decreasing. The second rule is trivially decreasing. Taking \( p_{i+2} > s_i \) and \( p_{i+2} > p_{i+1} \) makes the last rule decreasing.

**Exercise 4:**

We consider the Ackermann’s function

\[ \text{Ack} \]

1. Prove its termination via well-founded induction.
2. The following rewrite system simulates \( \text{Ack} \):
\[
\begin{align*}
    a(0, y) &\rightarrow s(y) \\
    a(s(x), 0) &\rightarrow a(x, a(0)) \\
    a(s(x), s(y)) &\rightarrow a(x, a(s(x), y))
\end{align*}
\]
Prove its termination using a LPO.

3. Consider the well-founded domain \((\text{Mult}(N \times N), (>_{\text{lex}})_{\text{mult}})\). Prove the termination of \(\text{Ack}\) using the following abstraction:
\[
\phi : T(\{a, s\}, X) \rightarrow \text{Mult}(N \times N) \\
t \rightarrow \{(\{u\}, \{v\}) \mid t|_{p \in \text{Pos}(t)} = a(u, v)\}
\]
where \(|0| = 1\), \(|a(x, y)| = |x| + |y| + 1\) and \(|s(x)| = |x| + 1\).

**Solution:**

1. Induction on \((N \times N, >_{\text{lex}})\). We prove that the calculus of \(\text{Ack} u v\) terminates by induction \((u, v)\) ordered lexicographically on integers.
   - Base cases: \(\text{Ack}\) terminates for \((0, n), n \in N\);
   - We need to show that \(\text{Ack}\) terminates for \((n, m), n > 0\). Induction hypothesis: \(\text{Ack}\) terminates for all \((j, k)\) such that \(j < n\) or \((j = n\) and \(k < m\)). If \(m = 0\) then by induction hypothesis the function terminates since \((n - 1, 1) < (n, m)\). Instead, if \(m > 0\), by induction hypothesis the function terminates on input \((n, m - 1)\) with output \(r\) and terminates on input \((n - 1, r)\).

2. Let \(>\) be such that \(a > s\) and let \(\text{status}(a) = \text{status}(a) = \text{lex}\). It holds:
   - \(a(0, t) >_{\text{rpo}} s(t)\), since \(a > s\) and \(a(0, t) >_{\text{rpo}} t\);
   - \(a(s(t), 0) >_{\text{rpo}} a(t, a(0))\) since \(a(s(t), 0) > t\), \(a(s(t), 0) > s(0)\) and \((a(t), 0) >_{\text{rpo}} \text{lex}(t, s(0))\);
   - \(a(s(t), s(t')) >_{\text{rpo}} a(t, a(s(t), t'))\) since \(a(s(t), s(t')) >_{\text{rpo}} t\), \(a(s(t), s(t')) >_{\text{rpo}} a(s(t), t'), (a(t), s(t')) >_{\text{rpo}} \text{lex}(t, a(s(t), t'))\)  

3. We can show that whenever \(s \rightarrow t\), then \(\phi(s) < \phi(t)\).

A weight function for a signature \(\Sigma\) is a pair \((w, w_0)\) consisting of a mapping \(w : \Sigma \rightarrow N\) and a constant \(w_0 > 0\) such that \(w(c) \geq w_0\) for all constant \(c \in \Sigma\). Let \((w, w_0)\) be a weight function. The weight of a term \(t\) is defined as follows
\[
    w(t) = \begin{cases} 
    w_0 & \text{if } t \text{ is a variable} \\
    w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n)
    \end{cases}
\]
We denote \(|s|_x\) for \(x\) a variable the number of times that \(x\) occurs in \(s\). Let \(>\) be a precedence and \((w, w_0)\) a weight function. We define the Knuth-Bendix order \((\text{KBO}) >_{\text{kbo}}\) on terms inductively as follows: \(s >_{\text{kbo}} t\) if \(|s|_x \geq |t|_x\) for all variables \(x\) and either

1. \(w(s) > w(t)\), or
2. \(w(s) = w(t)\) and one of the following alternatives holds
   - \(t\) is a variable and \(s = f^n(t)\) for some unary function symbol \(f\) and \(n > 0\),
   - \(s = f(s_1, \ldots, s_n), t = f(t_1, \ldots, t_n)\) and there is \(i \in \{1, \ldots, n\}\) such that \(s_j = t_j\) for all \(1 \leq j < i\) and \(s_i >_{\text{kbo}} t_i\), or
   - \(s = f(s_1, \ldots, s_n), t = g(t_1, \ldots, t_m)\) and \(f > g\).

**Exercise 5:**
Using a KBO, prove the termination of:
1. \{ l(x) + (y + z) \rightarrow x + (l(1(y)) + z), l(x) + (y + (z + w)) \rightarrow x + (z + (y + w)) \}\n
2. \{ r^n(1^k(x)) \rightarrow 1^k(r^m(x)) \}, \text{ where } n, k > 0 \text{ and } m \geq 0.

\textbf{Solution:}

1. Let $1 > +$, $w(1) = 0$, $w(+) = w(x) > 0$ for each variable $x$. It holds that $w$ is admissible. For both rules, it holds that the weight does not change after the rewrite step. To prove that $l(x) + (y + z) >_{kbo} x + (l(1(y)) + z)$ we therefore need to prove the third condition, which holds since $1(x) >_{kbo} x$. Similarly, to prove $l(x) + (y + (z + w)) >_{kbo} x + (z + (y + w))$, it is sufficient to show that $1(x) >_{kbo} x$. Notice that it does not holds $(y + (z + w) >_{kbo} z + (y + w)$ because of the ordering of the variables.

2. Let $r > 1$, $w(r) = 0$ and $w(1) = 1$. It holds that $w$ is admissible. Let $s \rightarrow t$ with $s = C[r^n(1^k(t'))]$ and $t = C[1^k(r^m(t'))]$. From the definition of $w$, it holds that $w(s) = w(t)$ since the number of occurrences of the function symbol $1$ is the same in $s$ and $t$. By applying the definition of $>_{kbo}$, we get that we need to show $r^n(1^k(t')) >_{kbo} 1^k(r^m(t'))$, which holds since $r > 1$. 