Homework on rewriting theory

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evaluation criteria: correctness, presentation and precision

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two disjoint signatures (i.e. $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$), $\mathcal{V}$ be a set of variables disjoint from $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{R}_1$ be a set of rewrite rules on $\mathcal{F}_1$ such that $\rightarrow_1 = \rightarrow_{\mathcal{R}_1}$ terminates on $\mathcal{T}_1 = \mathcal{T}(\mathcal{F}_1, \mathcal{V})$, and $\mathcal{R}_2$ be a set of rewrite rules on $\mathcal{F}_2$ such that $\rightarrow_2 = \rightarrow_{\mathcal{R}_2}$ terminates on $\mathcal{T}_2 = \mathcal{T}(\mathcal{F}_2, \mathcal{V})$. Then, let $\rightarrow$ be the rewrite relation on $\mathcal{T} = \mathcal{T}(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{V})$ generated by $\mathcal{R}_1 \cup \mathcal{R}_2$. We are going to study sufficient conditions for the termination of $\rightarrow$.

To this end, we will use the following notions.

A (multi-holes) context is a term of $C = \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{V})$ where $\Box$, the empty context, is a new constant of arity 0. If $C$ is a context and $p_1, \ldots, p_n$ are the positions of the occurrences of $\Box$ in $C$ from left to right, then $C[t_1, \ldots, t_n]$ denotes the term of $\mathcal{T}$ obtained by replacing the $i$-th occurrence of $\Box$ by $t_i$ for every $i$ in $\{1, \ldots, n\}$.

A symbol is of color $k \in \{1, 2\}$ if it belongs to $\mathcal{F}_k$. A non-empty context non-reduced to a variable is of color $k$ if it belongs to $C_k = \mathcal{T}(\mathcal{F}_k \cup \{\Box\}, \mathcal{V})$. The opposite color of $k$, written $\overline{k}$, is 2 if $k = 1$, and 1 if $k = 2$.

Every element of $\mathcal{T}$ is of the form $C[t_1, \ldots, t_n]$ with $C$ a variable or a non-empty context of color $k$ and every $t_i$ headed by a symbol of color $\overline{k}$. $C$ is called the cap of $t$ and is denoted by $\text{cap}(t)$. The terms $t_1, \ldots, t_n$ are called the aliens of $t$. Their multiset is denoted by $\text{aliens}(t)$.

**Exercise 1 (2 points)** Let $\rightarrow_h$ be the restriction of $\rightarrow$ to homogeneous terms, that is, the relation such that $t \rightarrow_h u$ iff $t \rightarrow u$ and both $t$ and $u$ belong to $\mathcal{T}_1 \cup \mathcal{T}_2$. Prove that $\rightarrow_h$ terminates.

**Exercise 2 (3 points)** The rank of a term $t \in \mathcal{T}$, $\text{rk}(t)$, is the maximum number of color layers in $t$: $\text{rk}(t) = 1 + \sup_{a \in \text{aliens}(t)} \text{rk}(a)$. Prove that the rank cannot increase by reduction: if $t \rightarrow u$, then $\text{rk}(t) \geq \text{rk}(u)$.

Hint 1: Look how evolve cap$(t)$ and aliens$(t)$ when $t \rightarrow u$.

Hint 2: Proceed by induction on $\text{rk}(t)$.

**Exercise 3 (4 points)** A rewrite rule $l \rightarrow r$ is collapsing if $r$ is a variable. Prove that $\rightarrow$ terminates if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-collapsing.

Hint: Look how cap$(t)$ and aliens$(t)$ evolve when $t \rightarrow u$, and devise a lexicographic combination of well-founded relations to prove the termination of every term $t \in \mathcal{T}$.

**Exercise 4 (4 points)** Given a term $t$, we define $S(t)$ to be the multiset made of $t$, the aliens of $t$, the aliens of the aliens of $t$, ..., $S(t) = \Sigma_{i \geq 1} S_i(t)$ where $S_1(t) = \{t\}$ and, for all $i \geq 1$, $S_{i+1}(t) = \Sigma_{a \in \text{aliens}(t)} S_i(a)$.
A rewrite rule $l \rightarrow r$ is duplicating if some variable has more occurrences in $r$ than it has in $l$. Prove that $\rightarrow$ terminates if both $R_1$ and $R_2$ are non-duplicating.

Hint: Look how $rk(t)$ and $S(t)$ evolve when $t \rightarrow u$, and devise a lexicographic combination of well-founded relations to prove the termination of every term.

**Exercise 5 (3 points)** Assume that $R_1$ is non-collapsing and non-duplicating.

Let $\| t \| = \begin{cases} 0 & \text{if } t \in V \\ \sum_{a \in \text{aliens}(t)} \| a \| & \text{if } \text{cap}(t) \in \mathcal{T}_1 \\ 1 + \sup_{a \in \text{aliens}(t)} \| a \| & \text{if } \text{cap}(t) \in \mathcal{T}_2 \end{cases}$

Prove that $\| t \| \geq \| u \|$ whenever $t \rightarrow u$.

A reduction $t \rightarrow u$ is destructive at level 1 if it is done in $\text{cap}(t)$ and $t$ and $u$ have different colors. It is destructive at level 2 if it is a destructive reduction at level 1 in some alien of $t$.

Observe that, if the reduction is destructive at level 1 or 2, then $\| t \| > \| u \|$. 

**Exercise 6 (4 points)** Prove that $\rightarrow$ terminates if $R_1$ is non-collapsing and non-duplicating.