Rewriting Techniques, 4: dependency pairs, argument filtering

05–12–2019

Exercise 1:

**Definition 1** (Weakly monotone algebra). A *weakly monotone* \( F \)-algebra \((A, >, \geq)\) consists of a non-empty \( F \)-algebra \( A \) together with a proper order \( > \) and a preorder \( \geq \) on the carrier \( A \) of \( A \) such that \( > \cdot \geq \subseteq > \) or \( \geq \cdot > \subseteq > \), and every algebra operation is monotone with respect to \( \geq \) in all coordinates, i.e., if \( f \in F \) has arity \( n \geq 1 \), for all \( a_1, \ldots, a_n, b \in A \) and \( i \in [1, n] \) with \( a_i \geq b \) then

\[
R(A(a_1, \ldots, a_i, \ldots, a_n)) \geq R(A(a_1, \ldots, b, \ldots, a_n))
\]

A monotone algebra \((A, >, \geq)\) is said *well-founded* if \( > \) is so.

Prove that if \((A, >, \geq)\) is a well-founded weakly monotone algebra then \((A, >, \geq)\) is a reduction pair.

**Exercise 2:**

(a) Let \( R \) be a rewrite system and such that each defined symbol has positive arity. Prove that if every cycle \( C \) of the dependency graph of \( R \) has a simple projection \( \pi \) such that \( \pi(C) \subseteq \geq \) and \( \pi(C) \cap \triangleright \neq \emptyset \), where \( \pi(C) = \left\{(\pi(s), \pi(t)) \mid (s, t) \in C \right\} \) and \( \geq \) is the subterm relation, then \( R \) terminates.

Consider the following rewriting system:

\[
\begin{align*}
m(1) & \rightarrow 1 \\
m(a(x, y)) & \rightarrow a(s(x), m(y)) \\
q(0, 0) & \rightarrow a(0, 1) \\
q(s(x), s(y)) & \rightarrow m(q(x, y)) \\
q(0, 0) & \rightarrow 1 \\
q(0, s(y)) & \rightarrow a(0, q(s(0), s(y)))
\end{align*}
\]

(b) Compute the marked dependency pairs and the dependency graph approximation.

(c) Prove the termination of the rewrite system by finding a suitable simple projection that satisfied the constraints in question 1.

An *argument filter* \((AF\) for short) for a signature \( F \) is a mapping \( \pi \) that associates with every \( n \)-ary function symbol an argument position \( i \in [1, n] \) or a (possibly empty) list \([i_1, \ldots, i_m]\) of argument positions with \( 1 \leq i_1 < \cdots < i_m \leq n \).

The signature \( F_\pi \) consists of all function symbols \( f \) such that \( \pi(f) \) is some list \([i_1, \ldots, i_m]\), where in \( F_\pi \) the arity of \( f \) is \( m \). Every argument filter \( \pi \) induces a mapping from \( T(F, \text{Var}) \) to \( T(F_\pi, \text{Var}) \), also denoted by \( \pi \):

\[
\pi(t) = \begin{cases} 
  t & \text{if } t \text{ is a variable} \\
  \pi(t_i) & \text{if } t = f(t_1 \ldots t_n) \text{ and } \pi(f) = i \\
  f(\pi(t_{i_1}) \ldots \pi(t_{i_m})) & \text{if } t = f(t_1 \ldots t_n) \text{ and } \pi(f) = [i_1, \ldots, i_m]
\end{cases}
\]

**Exercise 3:**

Let \( R \) be the following TRS,

\[
\begin{align*}
0 - y & \rightarrow 0 \\
x - 0 & \rightarrow x \\
s \cdot s \cdot y & \rightarrow x - y \\
0 & \rightarrow 0 \\
s \cdot s \cdot y & \rightarrow (x - y) \cdot s \cdot y
\end{align*}
\]
(a) Give dependency pairs of $R$.
(b) Find a weakly monotone polynomial interpretation on $\mathbb{N}$ to prove termination of $R$.
(c) Find an argument filter to prove termination using LPO with empty precedence.

**Exercise 4:**
Consider the rewriting system $R$:

\[
\begin{align*}
0 \leq y & \rightarrow \text{true} & x - 0 & \rightarrow x & \text{gcd } 0 \ y & \rightarrow y \\
(s \ x) \leq 0 & \rightarrow \text{false} & x - (s \ y) & \rightarrow p \ (x-y) & \text{gcd } (s \ x) \ 0 & \rightarrow s \ x \\
s \ x \leq s \ y & \rightarrow x \leq y & p \ (s \ x) & \rightarrow x & \text{gcd } (s \ x) \ (s \ y) & \rightarrow \text{if } (y \leq x) \ (s \ x) \ (s \ y) \\
\text{if true } (s \ x) \ (s \ y) & \rightarrow \text{gcd } (x-y) \ (s \ y) & \text{if false } (s \ x) \ (s \ y) & \rightarrow \text{gcd } (y-x) \ (s \ x)
\end{align*}
\]

(a) Compute the dependency pairs of $R$.
(b) How many different argument filters does $R \cup \text{DP}(R)$ admit?
(c) Prove the termination of $R$.

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**For next week: critical pairs, KB completion**

**Exercise 5:**
Compute the critical pairs of the following rewrite systems. Which one are locally confluent?

(a) $s(p(s(y))) \rightarrow y, s(p(x)) \rightarrow p(s(x))$
(b) $0 + y \rightarrow y, x + 0 \rightarrow x, s(w) + z \rightarrow s(w + z), v + s(k) \rightarrow s(v + k)$
(c) $a(x, x) \rightarrow 0, a(y, p(y)) \rightarrow 1$
(d) $a(a(x, y), z) \rightarrow a(x, a(y, z)), a(w, 1) \rightarrow w$

**Exercise 6:**
Is the TRS consisting of the rewriting rules

\[
\begin{align*}
0 + x & \rightarrow x & \text{gcd } x \ 0 & \rightarrow x & \text{gcd } (x + y) \ x & \rightarrow \text{gcd } x \ y \\
s \ x + y & \rightarrow s \ (x + y) & \text{gcd } 0 \ x & \rightarrow x & \text{gcd } x \ (x + y) & \rightarrow \text{gcd } x \ y
\end{align*}
\]

confluent?

**Exercise 7:**
Complete the ES consisting of the equation $(x \cdot y) \cdot (y \cdot z) \approx y$ (of central groupoids).