Rewriting Techniques, 2: termination, interpretation

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Exercise 1:
Show that the strict order $>$ defined by
$$ s > t \text{ iff } |s| > |t| \text{ and, for all } x \in V, |s|_x > |t|_x $$
is a reduction order, where $|s|$ is the size of $s$ and $|s|_x$ is the number of occurrences of $x$ in $s$.

A matrix interpretation on integers is the following:
- a positive integer $d$;
- for every symbol $f$ of arity $n$, $n$ matrices $M_{f,1}, \ldots, M_{f,n} \in \mathbb{N}^{d \times d}$;
- for every symbol of arity $n$, a vector $V_f \in \mathbb{N}^d$;
- a non-empty set $I \subseteq \{1, \ldots, d\}$ satisfying that for every symbol $f$ of arity $n$ the map
$$ L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \text{ defined as } L_f(X_1, \ldots, X_n) = V_f + \sum_{i=1}^n M_{f,i}X_i $$
is monotonic with respect to $>_I$ where $X >_I Y$ holds if and only if for every $i \in \{1, \ldots, d\}$, $X[i] \geq Y[i]$ and there is $j \in I$ such that $X[j] > Y[j]$.

Then $(\mathbb{N}^d, (L_f)_f, >_I)$ is a well-founded monotone algebra.

Exercise 2:
Consider the TRS $\{ \text{s(a)} \to \text{s(p(a))}, \text{p(b)} \to \text{p(s(b))} \}$.

1. Prove that its termination cannot be proved by a polynomial interpretation on integers;
2. Use the following matrix interpretation to prove termination w.r.t. $>_\{1,2\}$.

$$ L_a(X) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} X \quad \quad L_p(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X \quad \quad L_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \quad L_b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} $$

3. Why does it fail if we take $>_\{1\}$ instead? Is there another matrix interpretation that works with this ordering?

Exercise 3:
Let $A \subseteq \mathbb{N}$ and $P_f$ be respectively the domain and the interpretation, for each function symbol $f$, of a polynomial interpretation of integers for a TRS (note: the TRS is therefore terminating). Take $a \in A \setminus \{0\}$.

1. Define $\pi_a : T(F, X) \to A$ as the function which maps every variable $x$ to $a$ and every term of the form $f(t_1, \ldots, t_n)$ to $P_f(\pi_a(t_1), \ldots, \pi_a(t_n))$. Prove that $\pi_a(t)$ is greater or equal to the length of every reduction starting from $t$.
2. Show that there exists $d$ and $k$ positive integers such that for every $f \in F$ of arity $n$ and every $a_1, \ldots, a_n \in A \setminus \{0\}$ it holds $P_f(a_1, \ldots, a_n) \leq d \prod_{i=1}^n a_i^k$.
3. From the previous point, pick $d$ to be also greater or equal than $a$ and fix $c \geq k + \log_2(d)$. Prove that $\pi_a(t) \leq 2^{2^{|t|}}$.

Consider now any finite TRS and a function symbol $f$. Prove that there exists an integer $k$ such that if $s \to t$ then $|t|_f \leq k(|s|_f + 1)$, where $|.|_f$ is the number of $f$.

Deduce that the TRS
$$ \{ a(0,y) \to s(y), \; a(s(x),0) \to a(x,s(0)), \; a(s(x),s(y)) \to a(x,a(s(x),y)) \}, $$
simulating the Ackermann’s function, cannot be proved terminating using a polynomial interpretation over integers.
Exercise 4:
Consider exercise on semantic labeling of TD1, where the following rewriting system was defined

\[
\begin{align*}
  f(sX) &\rightarrow f(p(sX)) \circ (sX) \\
p(sX) &\rightarrow z
\end{align*}
\]

1. Prove that RPO cannot prove the termination of the system.
2. Prove that a labeled system can be proved terminating with RPO.

Exercise 5:
We consider the Ackermann’s function

\[
\begin{align*}
  \text{Ack} &\ {0} \ {y} \ = \ {y} + \ {1} \\
  \text{Ack} &\ {x} \ {0} \ = \ \text{Ack} \ {(x-1)} \ {1} \\
  \text{Ack} &\ {x} \ {y} \ = \ \text{Ack} \ {(x-1)} \ (\text{Ack} \ {x} \ {(y-1)})
\end{align*}
\]

(a) Prove its termination by induction.
(b) The following rewrite system simulates Ack

\[
\begin{align*}
  a(0, y) &\rightarrow s(y) \\
a(s(x), 0) &\rightarrow a(x, s(0)) \\
a(s(x), s(y)) &\rightarrow a(x, a(s(x), y))
\end{align*}
\]

Prove its termination using a LPO.
(c) Consider the well-founded domain \(\text{Mult}(\mathbb{N} \times \mathbb{N}), (\succ_{\text{lex}})_{\text{mal}}\). Prove the termination of Ack using the following abstraction:

\[
\phi : T(\{a, s\}, X) \rightarrow \text{Mult}(\mathbb{N} \times \mathbb{N})
\]

\[
t \rightarrow \{ (|u|, |v|) \mid t|_{p\in\text{Pos}(t)} = a(u, v) \}
\]

where \(|0| = 1, |a(x, y)| = |x| + |y| + 1\) and \(|s(x)| = |x| + 1\).

A weight function for a signature \(\Sigma\) is a pair \((w, w_0)\) consisting of a mapping \(w : \Sigma \rightarrow \mathbb{N}\) and a constant \(w_0 > 0\) such that \(w(c) \geq w_0\) for all constant \(c \in \Sigma\). Let \((w, w_0)\) be a weight function. The weight of a term \(t\) is defined as follows:

\[
w(t) = \begin{cases} 
  w_0 & \text{if } t \text{ is a variable} \\
  w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n)
\end{cases}
\]

We denote \(|s|_x\) for \(x\) a variable the number of times that \(x\) occurs in \(s\). Let \(>\) be a precedence and \((w, w_0)\) a weight function. We define the Knuth-Bendix order (KBO) \(\succ\) on terms inductively as follows: \(s >_{\text{KBO}} t\) if \(|s|_x \geq |t|_x\) for all variables \(x\) and either

1. \(w(s) > w(t)\), or
2. \(w(s) = w(t)\) and one of the following alternatives holds
   (a) \(t\) is a variable and \(s = f^n(t)\) for some unary function symbol \(f\) and \(n > 0\),
   (b) \(s = f(s_1, \ldots, s_n), t = f(t_1, \ldots, t_n)\) and there is \(i \in \{1, \ldots, n\}\) such that \(s_j = t_j\) for all \(1 \leq j < i\) and \(s_i >_{\text{KBO}} t_i\), or
   (c) \(s = f(s_1, \ldots, s_n), t = g(t_1, \ldots, t_m)\) and \(f > g\).

Exercise 6:
Using a KBO, prove the termination of:

1. \(\{ l(x) + (y + z) \rightarrow x + l(l(y) + z), l(x) + (y + (z + w)) \rightarrow x + (z + (y + w)) \}\)
2. \(\{ r^n(l^k(x)) \rightarrow l^k(r^m(x)) \}\), where \(n, k > 0\) and \(m \geq 0\).