From Büchi Automata to Cyclic and Infinite Proofs

Internship at ITU Copenhagen

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Purpose

Encode Büchi automata as formulas in a proof-theoretical framework with (co)-induction.

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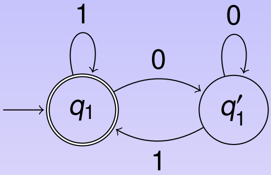
$\rightsquigarrow$

Logics dealing with infinite proofs, cyclic proofs; mixing inductive and co-inductive formulas; strongly related; well describe Büchi Automata.

Big Picture



$$\downarrow$$

- Common and used: explicit (co)-induction

$$\mu\mathrm{LK} \subseteq \mu\mathrm{LK}^{\omega} \subseteq \mu\mathrm{LK}^{\infty}$$

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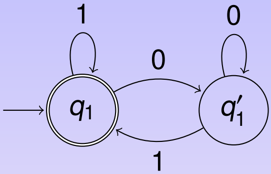

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- Infinite proofs: satisfies adequacy, impractical
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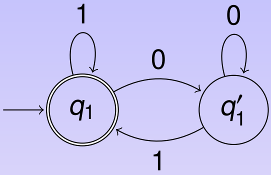

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$$\mu\mathrm{LK} \subseteq \mu\mathrm{LK}^{\omega} \subseteq \mu\mathrm{LK}^{\infty}$$

- Cyclic proofs

Outline

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1. Introduction
2. Büchi Automata
3. μLK
4. μLK$^\omega$ and μLK$^\infty$
5. Conclusion

Büchi Automata

Definition (Büchi Automata)

A *Büchi automaton* is a quintuple $\mathcal{A} = (Q, \Sigma, \delta, Q_I, Q_F)$, where

- Q is a finite set (the states);
- Σ is an alphabet;
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ the nondeterministic transition function;
- $Q_I \subseteq Q$ the initial states and $Q_F \subseteq Q$ the final states.

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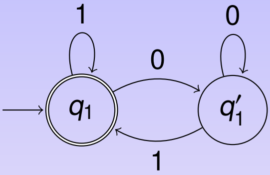
Definition (Acceptance condition)

- A run α on a word is accepting by an automaton $\iff$ α visits a final state infinitely often;
- A word is recognized by an automaton $\iff$ there exists an accepting run on it.

An Example of a Büchi Automaton



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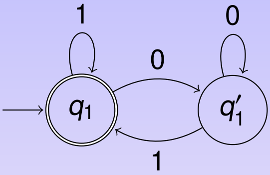

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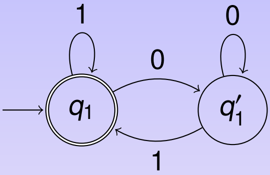

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"Proof"

By reading the word 1^ω, I can build *step by step* an accepting run in $\mathcal{A}$:

"From state q_1, I read 1 and jump to q_1 and so on so forth."

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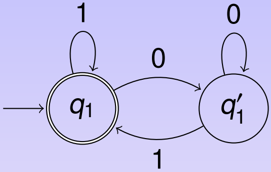

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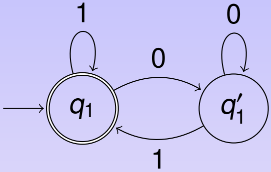

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$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash_? [q_1]\ 1^\omega}\ ?}{\vdash_? \exists tl\ \ (1 :: 1^\omega = 1 :: tl \wedge [q_1]\ tl) \vee \left(1 :: 1^\omega = 0 :: tl \wedge [q_1']\ tl\right)}\ \exists R, \vee R_1}{\vdash_? [q_1]\ 1 :: 1^\omega}\ ?$$

Goals

- Adequacy: $w \in \mathcal{L}(\mathcal{A}) \iff \vdash \llbracket \mathcal{A} \rrbracket \lfloor w \rfloor$

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$$\llbracket A_1 \rrbracket x \vdash \llbracket A_2 \rrbracket x \iff \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2).$$

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  Proof of inclusion $\leadsto$ inclusion and a certificate;
- Usable and generic logic: properties over automata are used in a wider context.

Outline

μLK

Definition (Formula of μLK)

$$
\begin{array}{lcll}
P & ::= & \top \mid \bot & \\
 & \mid & \exists x.\ P \mid \forall x.\ P & x \in \mathcal{V} \\
 & \mid & P \wedge P \mid P \vee P \mid P \Rightarrow P & \\
 & \mid & s = t & t,s \text{ some terms}
\end{array}
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 & \mid & p & p \in \mathcal{V}_f \\
 & \mid & \mu(\lambda p.\lambda x_1.\dots.\lambda x_n.\ P)\ t_1 \dots t_n & p \in \mathcal{V}_f,\ t_i \text{ a term} \\
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\begin{array}{lll}
N = & \mu B_{\mathrm{nat}} & = \mu\,(\lambda p_n\ .\lambda x.\ x = 0 \vee (\exists y\ x = s(y) \wedge p_n\ y)) \\
S = & \nu B_{\mathrm{stream}} & = \nu\,(\lambda p_s\ .\lambda w.\ \exists w' \exists n\ w = n{:}w' \wedge N\ n \wedge p_s\ w')
\end{array}
$$

Rules of μLK

Sequent calculus:

- identity group: Ax, cut, $=R$, $=L$;
- logical group: $\top$, $\bot$, $\wedge L_i$, $\wedge R$, $\vee L$, $\vee R_i$, $\Rightarrow L$, $\Rightarrow R$, $\forall L$, $\forall R$, $\exists L$, $\exists R$;
- structural group: WL,WR (weak), Cl,CR (contraction).

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- structural group: WL,WR (weak), Cl,CR (contraction).

$+$ explicit (co)-induction:

$$\frac{\Gamma \vdash B(\mu B)\,\mathbf{t}}{\Gamma \vdash \mu B\,\mathbf{t}}\ \mu R \qquad \frac{\Gamma,\ S\,\mathbf{t} \vdash P \quad BS\,\mathbf{x} \vdash S\,\mathbf{x}}{\Gamma,\ \mu B\,\mathbf{t} \vdash P}\ \mu L$$

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$$\cfrac{\cfrac{\Phi_0 \text{ or } \Phi_n}{\Gamma \vdash t = 0 \vee \exists y\ t = s(y) \wedge \mu B_{\mathrm{nat}}\ y}}{\Gamma \vdash \mu B_{\mathrm{nat}}\ t}\ \mu R$$

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$$\cfrac{\cfrac{\Pi}{S\ t \vdash P} \qquad \cfrac{\cfrac{\Psi_0}{\vdash S\ 0} \quad \cfrac{\Psi_n}{S\ x \vdash S\ (s(x))}}{x = 0 \vee \exists y\ x = s(y) \wedge S\ y \vdash S\ x}\ \vee L, (\exists L), = L}{\mu B_{\mathrm{nat}}\ t \vdash P}\ \mu L$$

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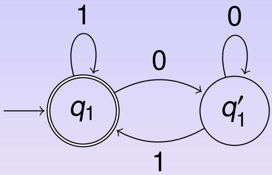
μLK vs. Büchi automata



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\begin{aligned}
[q_1] = \nu(\lambda q_1.\lambda w.\ \exists w'&\\
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μLK vs. Büchi automata

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Explicit (co)-induction rules $\leadsto$ replaced by some cycles or infinite branches.

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$$\dfrac{\dfrac{}{t = 0 \vdash \mu B_{\text{nat}}\ t} \qquad \dfrac{\dfrac{}{\mu B_{\text{even}}\ t' \vdash \mu B_{\text{nat}}\ (s(s(t')))}}{t = s(s(t')) \wedge \mu B_{\text{even}}\ t' \vdash \mu B_{\text{nat}}\ t}}{\mu B_{\text{even}}\ t \vdash \mu B_{\text{nat}}\ t}\ \mu L'$$

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Guard Condition

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$$\begin{aligned} P &= \mu(\lambda p.\ \nu(\lambda q.\ p)) \\ Q &= \nu(\lambda q.\ P) \end{aligned}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\uparrow \alpha}{\vdash Q}}{\vdash P}\,\mu R}{\vdash Q \ :\alpha}\,\nu R}{\vdash P}\,\mu R \qquad \dfrac{\dfrac{\dfrac{\uparrow \tau}{P \vdash \bot}}{Q \vdash \bot}\,\nu L}{P \vdash \bot \ :\tau}\,\mu L}{\vdash \bot}\,\mathit{cut}$$

Guard Condition - interleaved (co)-inductive formulas

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Fix

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“Definition”: table of (co)-induction

$$(Q,\ \epsilon,\ \trianglerighteq,\ <)$$

- Q: names of (co)-inductive formulas (defined atoms);
- $\epsilon : Q \rightarrow \{\mu; \nu\}$;
- $P \trianglerighteq A$: A is the unfolding of $P \in Q$;
- $<$: Who is on the top of who?

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Second bug

$$\begin{array}{rcl} \mathrm{Nat} & \trianglerighteq & B_{\mathrm{nat}}\ \mathrm{Nat} \qquad\qquad \epsilon(\mathrm{Nat}) = \mu \\ B_{\mathrm{nat}} & = & \lambda p_n.\ \lambda n.\ n = 0 \vee \exists n'\ n = s(n') \wedge p_n\ n' \end{array}$$

$$\frac{}{\mathrm{Nat}\ t \vdash \bot\ \ : \alpha}\ \mu L$$

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- $<$: Who is on the top of who?

Second bug

$$\begin{array}{rcl} \mathrm{Nat} & \trianglerighteq & B_{\mathrm{nat}}\ \mathrm{Nat} \qquad\qquad \epsilon(\mathrm{Nat}) = \mu \\ B_{\mathrm{nat}} & = & \lambda p_n.\ \lambda n.\ n = 0 \vee \exists n'\ n = s(n') \wedge p_n\ n' \end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash B_{\mathrm{nat}}\ \mathrm{Nat}\ t}\ Ax}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash \mathrm{Nat}\ t}\ \mu R \qquad \dfrac{\uparrow \alpha}{\mathrm{Nat}\ t \vdash \bot}}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash \bot}\ cut}{\mathrm{Nat}\ t \vdash \bot\ \ : \alpha}\ \mu L$$

Guard Condition - observation

$$\dfrac{\dfrac{}{t = 0 \vdash \mathrm{Nat}\ t} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\uparrow \alpha}{s_5 : \mathrm{Even}\ t' \vdash \mathrm{Nat}\ t'}}{s_4 : \mathrm{Even}\ t' \vdash \mathrm{Nat}\ (s(t'))}}{s_3 : \mathrm{Even}\ t' \vdash \mathrm{Nat}\ (s(s(t')))}}{s_2 : t = s(s(t')) \wedge \mathrm{Even}\ t' \vdash \mathrm{Nat}\ t}}{s_1 : (A =)\mathrm{Even}\ t \vdash \mathrm{Nat}\ t : \ \alpha}\ \mu L$$

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$$\mathcal{O}_A(\alpha) = (\mu L, \mathrm{Even})$$

Guard Condition - observation

$$
\dfrac{\dfrac{}{t = 0 \vdash \mathrm{Nat}\ t} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\uparrow \alpha}{s_5 : \mathrm{Even}\ t' \vdash \mathrm{Nat}\ t'}}{s_4 : \mathrm{Even}\ t' \vdash \mathrm{Nat}\ (s(t'))}}{s_3 : \mathrm{Even}\ t' \vdash \mathrm{Nat}\ (s(s(t')))}}{s_2 : t = s(s(t')) \wedge \mathrm{Even}\ t' \vdash \mathrm{Nat}\ t}}{s_1 : (A =)\mathrm{Even}\ t \vdash \mathrm{Nat}\ t : \ \alpha}\ \mu L
$$

$$
O_A(\alpha) = (\mu L, \mathrm{Even})
$$

“Definition”: Observations

- The trace of $A \in s_0$ in the branch $s_0, s_1, \ldots$ is a serie of formulas $A_0, A_1, \ldots$ such that:
  - $A_i \in s_i$ (on the same side);
  - if A_i is active in the conclusion s_i then A_{i+1} is active in the premise of s_{i+1}.
- The observation of a formula in a branch is the serie of (r, A) where r is a (co)-inductive rules applied to A appearing in the trace.

$$\dfrac{\dfrac{\dfrac{}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash B_{\mathrm{nat}}\ \mathrm{Nat}\ t}\ Ax}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash \mathrm{Nat}\ t}\ \mu R \qquad \dfrac{}{\mathrm{Nat}\ t \vdash \mathrm{Nat}\ t}}{\dfrac{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash \bot}{\mathrm{Nat}\ t \vdash \bot \quad : \alpha}\ \mu L}\ cut$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash B_{\mathrm{nat}}\ \mathrm{Nat}\ t}\,Ax}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash \mathrm{Nat}\ t}\,\mu R \qquad \dfrac{\cancel{\uparrow \alpha}}{\mathrm{Nat}\ t \vdash \mathrm{Nat}\ t}}{B_{\mathrm{nat}}\ \mathrm{Nat}\ t \vdash \bot}\,cut}{\mathrm{Nat}\ t \vdash \bot \ : \alpha}\,\mu L$$

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“Definition”: Refinement of Guard Condition

A proof is valid $\iff$ each infinite brach is either inductive or co-inductive.

- inductive branch: there is an observation o on the left such that that $\epsilon\left(\max_{(r,n)\in\mathcal{I}nf(o)} \{n\}\right) = \mu$;

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A proof is valid $\iff$ each infinite brach is either inductive or co-inductive.

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- co-inductive branch: there is an observation o on the right such that that $\epsilon\left(\max_{(r,n)\in Inf(o)}\{n\}\right) = \nu$.

μLK$^\infty$

- Bijection between observations of $[q]\ t$ and runs starting with q;

μLK$^{\infty}$

- Bijection between observations of $[q]\ t$ and runs starting with q;
- completeness and soundness of acceptance;

μLK$^{\infty}$

- Bijection between observations of $[q]\ t$ and runs starting with q;
- completeness and soundness of acceptance;
- completeness and soundness of inclusion.

"Definition": Refinement of Guard Condition

A proof is valid $\iff$ each cycle is either inductive or co-inductive.

- inductive cycle: there is an observation o on the left such that that $\epsilon\left(\max_{(r,n)\in o}\{n\}\right) = \mu$;

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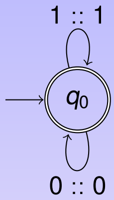
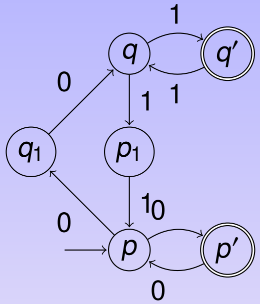
Unexpected bug

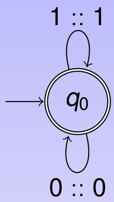


$\not\subseteq$

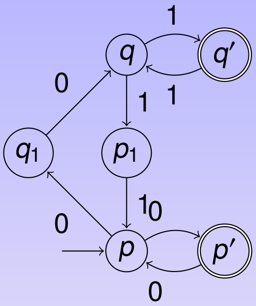


$\mathcal{L}(L) \not\subseteq \mathcal{L}(A)$ $\qquad (0 :: 0 :: 1 :: 1)^\omega \notin \mathcal{L}(A)$

Unexpected bug



$\not\subseteq$

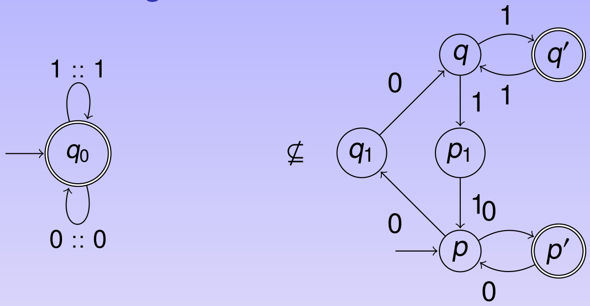


$\mathcal{L}(L) \not\subseteq \mathcal{L}(A) \qquad (0 :: 0 :: 1 :: 1)^\omega \notin \mathcal{L}(A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\looparrowright \alpha}{L\ y \vdash p\ y, q\ y}}{L\ y \vdash \mathbf{p'}\ (0 : y), q_1\ (0 : y)}\ \nu R}{L\ y \vdash p\ (0 : 0 : y), \cancel{q}\ (0 : 0 : y)} \qquad \dfrac{\dfrac{\dfrac{\looparrowright \alpha}{L\ y \vdash p\ y, q\ y}}{L\ y \vdash p_1\ (1 : y), \mathbf{q'}\ (1 : y)}\ \nu R}{L\ y \vdash \cancel{p}\ (1 : 1 : y), q\ (1 : 1 : y)}}{L\ x \vdash p\ x, q\ x \ : \alpha}$$

Validity condition holds but does not respect the Büchi automata semantics.

Unexpected bug



$$\mathcal{L}(L) \not\subseteq \mathcal{L}(A) \qquad (0 :: 0 :: 1 :: 1)^\omega \notin \mathcal{L}(A)$$

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Validity condition holds but does not respect the Büchi automata semantics. Traces are broken.

A few counter-examples later

"Definition": Refinement[2] of Guard Condition

A proof is valid $\iff$ each cycle is either inductive or co-inductive.

- inductive cycle: there is an observation $o = o_1, o_2, \ldots o_p$ on the left such that:
  - $o_1 = o_p$;
  - $\max_{(r,n)\in o} \{n\} = n_1$ and $\epsilon(n_1) = \mu$;
- co-inductive cycle: there is an observation $o = o_1, o_2, \ldots o_p$ on the right such that:
  - $o_1 = o_p$;
  - $\max_{(r,n)\in o} \{n\} = n_1$ and $\epsilon(n_1) = \nu$.

Outline

$$\mu LK = \mu LK^{\omega} \subseteq \mu LK^{\infty}$$

1. New logic $\mu\mathrm{LK}^{\omega}$ and $\mu\mathrm{LK}^{\infty}$ supporting mutual inductive and coinductive definitions with implicit (co)induction;

$$\mu LK = \mu LK^{\omega} \subseteq \mu LK^{\infty}$$

1. New logic μLK$^\omega$ and μLK$^\infty$ supporting mutual inductive and coinductive definitions with implicit (co)induction;
2. they are strongly related with μLK: we can translate back and forth between μLK proofs and cyclic proofs;

$$\mu LK = \mu LK^{\omega} \subseteq \mu LK^{\infty}$$

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4. soundness and completeness of Büchi acceptance and inclusion;

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4. soundness and completeness of Büchi acceptance and inclusion;
5. first result for cut-elimination of infinite proofs.

The End

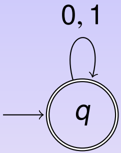
Thanks for listening !

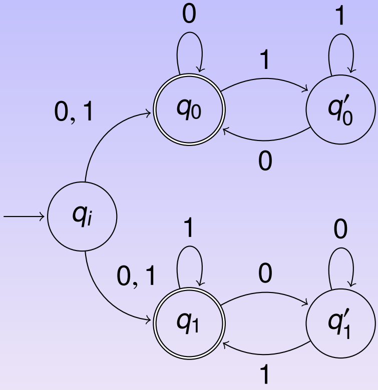
References I

[Bro06] James Brotherston. *Sequent Calculus Proof Systems for Inductive Definitions*. PhD thesis, University of Edinburgh, November 2006.

[San02] Luigi Santocanale. A calculus of circular proofs and its categorical semantics. In *Proceedings of Foundations of Software Science and Computation Structures (FOSSACS02)*, number 2303 in LNCS, pages 357–371. Springer, January 2002.

Nondeterministic





$$\mathcal{L}(\mathcal{A}_1) = (0|1)^\omega \qquad \subseteq \qquad \mathcal{L}(\mathcal{A}_2) = (0|1)^\omega$$

Encoding

Encoding of $\mathcal{A} = (Q, \Sigma, \delta, Q_I, Q_F)$:

$$[\mathcal{A}] = \lambda w. \bigvee_{q \in Q_I} [q]^{\emptyset}\ w$$

$$[q]^{\gamma} = \begin{cases} q & \text{if } q \in \gamma \\ \mu\left(\lambda q.\lambda w.\exists w' \bigvee_{q' \in \delta(q,\alpha),\ \alpha \in \Sigma} w = \alpha \cdot w' \wedge [q']^{\gamma \cup \{q\}}\ w'\right) & \text{if } q \in Q_F \\ \nu\left(\lambda q.\lambda w.\exists w' \bigvee_{q' \in \delta(q,\alpha),\ \alpha \in \Sigma} w = \alpha \cdot w' \wedge [q']^{\gamma \cup \{q\}}\ w'\right) & \text{else} \end{cases}$$

Adequacy

- $w \in \mathcal{L}(\mathcal{A}) \iff \vdash \llbracket \mathcal{A} \rrbracket \lfloor w \rfloor$: The proof tries all the possible runs in parallel.

Adequacy

- $w \in \mathcal{L}(\mathcal{A}) \iff \vdash \llbracket \mathcal{A} \rrbracket \lfloor w \rfloor$: The proof tries all the possible runs in parallel.
  $w \in \mathcal{L}(\mathcal{A}) \iff$ there is at least one accepted run $\iff$ there is at least on valid observation $\iff$ $\llbracket \mathcal{A} \rrbracket \lfloor w \rfloor$ is provable;
- There is a bijection between the runs and the observations of the proof $\vdash \llbracket \mathcal{A} \rrbracket \lfloor w \rfloor$.

$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2) \Rightarrow [\![A_1]\!]x \vdash [\![A_2]\!]x$:

- We prove the inclusion in μLK$^\infty$:

$[\![A_1]\!]x \vdash [\![A_2]\!]x \Rightarrow \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$:

$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2) \Rightarrow [\![A_1]\!]x \vdash [\![A_2]\!]x$:

- We prove the inclusion in μLK$^\infty$:
  - on the right we test all the runs in $\mathcal{A}_2$ in parallel;

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- We prove the inclusion in μLK$^\infty$:
  - on the right we test all the runs in $\mathcal{A}_2$ in parallel;
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    - if the run on the left is valid, then the word is in $\mathcal{L}(\mathcal{A}_1)$ so is in $\mathcal{L}(\mathcal{A}_2)$. Then one of the run on the right is valid;

$\llbracket A_1 \rrbracket x \vdash \llbracket A_2 \rrbracket x \Rightarrow \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$:

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    - else the observation of $\llbracket \mathcal{A}_1 \rrbracket$ in this branch is valid.

$\llbracket A_1 \rrbracket x \vdash \llbracket A_2 \rrbracket x \Rightarrow \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$:

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    - else the observation of $\llbracket \mathcal{A}_1 \rrbracket$ in this branch is valid.
- then these proofs are regular so are in $\mu\mathrm{LK}^\omega$;

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- then we can build a proof in μLK.

$[\![A_1]\!]x \vdash [\![A_2]\!]x \Rightarrow \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$:

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$\llbracket A_1 \rrbracket x \vdash \llbracket A_2 \rrbracket x \Rightarrow \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$:

- If we prove the inclusion in one of the logics we can prove it in μLK$^\infty$;

$\mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2) \Rightarrow \llbracket A_1 \rrbracket x \vdash \llbracket A_2 \rrbracket x$:

- We prove the inclusion in $\mu\mathsf{LK}^\infty$:
    - on the right we test all the runs in $\mathcal{A}_2$ in parallel;
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    - for each branch of the proof:
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        - else the observation of $\llbracket \mathcal{A}_1 \rrbracket$ in this branch is valid.
- then these proofs are regular so are in $\mu\mathsf{LK}^\omega$;
- then we can build a proof in $\mu\mathsf{LK}$.

$\llbracket A_1 \rrbracket x \vdash \llbracket A_2 \rrbracket x \Rightarrow \mathcal{L}(\mathcal{A}_1) \subseteq \mathcal{L}(\mathcal{A}_2)$:

- If we prove the inclusion in one of the logics we can prove it in $\mu\mathsf{LK}^\infty$;
- if $w \in \mathcal{L}(\mathcal{A}_1)$ then $\Pi_1 : \vdash \llbracket \mathcal{A}_1 \rrbracket \lfloor w \rfloor$ and:

$$\dfrac{\dfrac{\Pi_1}{\vdash \llbracket \mathcal{A}_1 \rrbracket \lfloor w \rfloor} \qquad \dfrac{\Pi_2}{\llbracket \mathcal{A}_1 \rrbracket \lfloor w \rfloor \vdash \llbracket \mathcal{A}_2 \rrbracket \lfloor w \rfloor}}{\vdash \llbracket \mathcal{A}_2 \rrbracket \lfloor w \rfloor}\ \mathit{cut}$$

Outline

6. Results
7. Mental Repository

Inclusions of the Logics

Theorem 1

$$\mu LK \subseteq \mu LK^\omega \subseteq \mu LK^\infty$$

- $LK^\omega \subseteq \mu LK^\infty$:

Inclusions of the Logics

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- $LK^{\omega} \subseteq \mu LK^{\infty}$: unfold the cycles infinitly often.

$$\dfrac{\dfrac{\Psi \qquad \dfrac{\hookrightarrow \alpha}{\Gamma \vdash P}}{\Pi}}{\Gamma \vdash P \;:\; \alpha} \quad \Longrightarrow \quad \dfrac{\dfrac{\Psi \qquad \dfrac{\dfrac{\Psi \quad \vdots}{\Pi}}{\Gamma \vdash P \diamond}}{\Pi}}{\Gamma \vdash P \diamond}$$

Inclusions of the Logics

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- $\mu LK \subseteq \mu LK^\omega$: not the same language. We need a translation and a table of (co)-induction.

Table of (co)-induction:

- $Q = \left\{\widehat{\varepsilon B} \mid B \text{ closed operator of } \mu LK, \varepsilon \in \{\mu; \nu\}\right\}; \quad \epsilon(\widehat{\varepsilon B}) = \varepsilon;$

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$$\langle \_ \rangle : \mu LK \text{ formula } \rightarrow \mu LK^\omega \text{ formula}$$

$$\begin{array}{ll} \langle P \boxdot Q \rangle = \langle P \rangle \boxdot \langle Q \rangle & \boxdot \in \{\wedge; \vee; \Rightarrow\} \\ \langle \circledast x\ B \rangle = \circledast x\ \langle B \rangle & \circledast \in \{\forall; \exists\} \\ \langle \varepsilon B \rangle = \widehat{\varepsilon B} & \varepsilon \in \{\mu; \nu\} \\ \langle a \rangle = a & \end{array}$$

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- $Q = \left\{\widehat{\varepsilon B} \mid B \text{ closed operator of } \mu LK, \varepsilon \in \{\mu; \nu\}\right\}; \quad \epsilon(\widehat{\varepsilon B}) = \varepsilon;$

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$$\langle \circledast x\ B\rangle = \circledast x\ \langle B\rangle \qquad \circledast \in \{\forall; \exists\}$$

$$\langle \varepsilon B\rangle = \widehat{\varepsilon B} \qquad \varepsilon \in \{\mu; \nu\}$$

$$\langle a\rangle = a$$

- $\widehat{\varepsilon B} \trianglerighteq \langle B\ \varepsilon B\rangle;$

Table of (co)-induction:

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$$\langle P \boxdot Q\rangle = \langle P\rangle \boxdot \langle Q\rangle \qquad \boxdot \in \{\wedge;\vee;\Rightarrow\}$$

$$\langle \circledast x\ B\rangle = \circledast x\ \langle B\rangle \qquad \circledast \in \{\forall;\exists\}$$

$$\langle \varepsilon B\rangle = \widehat{\varepsilon B} \qquad \varepsilon \in \{\mu;\nu\}$$

$$\langle a\rangle = a$$

- $\widehat{\varepsilon B} \trianglerighteq \langle B\ \varepsilon B\rangle$;
- $\widehat{\varepsilon B} < \widehat{\varepsilon' B'} \iff \varepsilon' B'$ sub-formula of B

$\mu LK \subseteq \mu LK^\omega$

Lemma: Functoriality in μLK$^\omega$

If B is monotonic (i.e. the p_i appears only in positive positions in B) then for all predicates $P_1, P_2, \ldots P_n$ this rule is admissible in μLK$^\omega$: Let B a predicate operator: $B = \lambda p.\lambda \mathbf{x}.\ A$ and P and Q some predicates then this rule is admissible in μLK$^\omega$:

$$\frac{\langle P \rangle\ \mathbf{x} \vdash \langle Q \rangle\ \mathbf{x}}{\langle B\ P \rangle\ \mathbf{t} \vdash \langle B\ Q \rangle\ \mathbf{t}}\ \mathit{functo}$$

and all the observations involve names n such that for all names m appearing in $\langle P \rangle$ or $\langle Q \rangle$, $n < m$.

Theorem 1

$$\Gamma \vdash_{\mu LK} \Delta \quad \Rightarrow \quad \langle\Gamma\rangle \vdash_{\mu LK^{\omega}} \langle\Delta\rangle$$

By induction on the size of the proof then case analysis on the first rule.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, S\ t} \qquad \dfrac{\Pi_2}{S\ x \vdash BS\ x}}{\Gamma \vdash \Delta, \nu B\ t}\ \nu R$$

$$\downarrow$$

$$\dfrac{\dfrac{\Pi_1^*}{\langle\Gamma\rangle \vdash \langle\Delta\rangle, \langle S\rangle\ t} \qquad \dfrac{\dfrac{\dfrac{\Pi_2^*}{\langle S\rangle\ x \vdash \langle BS\rangle\ x} \qquad \dfrac{\dfrac{\dfrac{\hookrightarrow \alpha}{\langle S\rangle\ x \vdash \widehat{\nu B}\ x}}{\langle S\rangle\ x, \langle S\rangle\ x \vdash \widehat{\nu B}\ x}\ WL}{\langle S\rangle\ x, \langle BS\rangle\ x \vdash \langle B(\nu B)\rangle\ x}\ functo}{\langle S\rangle\ x \vdash \langle B(\nu B)\rangle\ x}\ cut}{\langle S\rangle\ x \vdash \widehat{\nu B}\ x\ :\ \alpha}\ \nu R}{\langle\Gamma\rangle \vdash \langle\Delta\rangle, \widehat{\nu B}\ t}\ \text{cut, cut, } \forall R, \Rightarrow R, \ldots$$

Theorem 2

$$\mu LK^\omega \subseteq \mu LK$$

Soon a complete proof.

Cut Elimination

Proof of normalisation of μLK$^\infty$.

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Proof of normalisation of μLK^∞. We must show:

1. normalisation: the reduction rules provides a limit proof;

$$d(\Pi, \Pi') = \frac{1}{1 + \text{ minimum depth of two different nodes}}$$

2. validity: the limit proof is also valid.

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Exploration of the reduction

The sub-part of the proof which is explored by the reduction.

Strategy of reduction

Always reduce the first cut rule which is no followed by another cut rule.

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Lemma 3

For a cut rule: $\dfrac{\Pi_1 \quad \Pi_2}{s}\,cut$. If there is an infinite observation of the cut formula in Π_i contained in the exploration, then there is a dual observation in Π_{1-i} in the exploration.

Lemma 4

There exists an infinite branch in the exploration which has a valid observation of a formula in the root.

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Lemma 4 $+$ Lemma 5 $\rightsquigarrow$ Normalisation $+$ Validity !

Results

- Infinite proofs: cut-elimination, regular proofs $= \mu\mathsf{LK}^\omega$

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- μLK: cut-elimination; as expressive as $\mu\mathsf{LK}^{\omega}$

$$\mu\mathsf{LK} \; = \; \mu\mathsf{LK}^{\omega} \; \subseteq \; \mu\mathsf{LK}^{\infty}$$

- Cyclic proofs: consistent, as expressive as μLK

Outline

Encoding

Encoding from the Büchi automata to formulas of a logic so as to reason over the automata within the logic.

We must trust the encoding (and the logic) for working within the logic instead of manipulating automata directly.