Complexité avancée - TD 3

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Exercise 1: My very first PSPACE-complete problem
Show that the following problem is PSPACE-complete (not assuming anything about QBF):

- INPUT: a Turing Machine M and a word w and a number t written in unary
- QUESTION: does M accepts w within space $|t|$?

Exercise 2: Padding argument
Show that the following problem is PSPACE-complete (not assuming anything about QBF):

- INPUT: a context-sensitive language L and a word w
- QUESTION: does $w \in L$?

Hint: You can find inspiration in the exercise 7 of last week's exercise sheet, and in propositon 3.8 from the lecture notes.

Exercise 3: PSPACE and games
`FORMULA-GAME` is the following game. There are two players, Player 1 and Player 2 which alternatively make moves on a given board. The board is a boolean formula $\phi(x_1, \dots x_{2n})$, and the moves of the players consist in picking truth values for the variables $x_1, \dots x_{2n}$ in this order. Specifically, Player 1 choses the value of x_1, then Player 2 choses the value of x_2, then Player 1 choses the value of x_3, and so on. Player 1 wins the game if ϕ is true under the variable assignment produced in the game. Player 1 has a winning strategy if he has a way of choosing his moves so that he wins the game no matter the moves of Player 2.

Show that the following problem is PSPACE-complete:

- INPUT: a boolean formula ϕ
- QUESTION: does Player 1 have a winning strategy for `FORMULA-GAME` on board ϕ?

Exercise 4: PSPACE and games
The Geography game is played as follow:

- The game starts with a given name of a city, for instance *Cachan*;
- the first player gives the name of a city whose first letter coincides with the last letter of the previous city, for instance *Nice*;
- the second player gives then another city name, always starting with the last letter of the previous city, for instance *Evry*;

- the first player plays again, and so on – with the restriction that no player is allowed to give the name of a city already used in the game;
- the loser is the first player who does not find a new city name to continue.

This game can be described using a graph whose vertices represent cities and where an edge (X, Y) means that the last letter of the city X is the same as the first letter of the city Y. This graph has also a vertex marked as the initial vertex of the game (the initial city). Each player choses a vertex of the graph, the first player choses first, and the two players alternate their moves. At each move, the sequence of vertices chosen by the two players must form a simple path in the graph, starting from the distinguished initial vertex.

Player 1 wins the game if, after some number of moves, Player 2 has no valid move (that is no move that forms a simple path with the sequence of previous moves).

`GEOGRAPHY` is the following problem:

- INPUT: a graph G and an initial vertex s.
- QUESTION: does player 1 have a wining strategy for the game on G starting at s ?

Show that `GEOGRAPHY` is PSPACE-complete.

Exercise 5: Language theory

Show that the following problems are PSPACE-complete:

1. NFA Universality:
   - INPUT: a non-deterministic automaton A over alphabet Σ
   - QUESTION: $\mathcal{L}(A) = \Sigma^*$?

   Bonus: what is the complexity of this problem for a DFA ?

2. NFA Equivalence
   - INPUT: two non-deterministic automata A_1 and A_2 over the same alphabet Σ
   - QUESTION: $L(A_1) = L(A_2)$

   Bonus: what is the complexity of this problem for a DFA ?

3. DFA Intersection Vacuity:
   - INPUT: determinitic automata $A_1, \dots, A_m$ for some m
   - QUESTION: $\bigcap_{i=1}^{m} L(A_i) = \emptyset$?

Exercise 6: A translation result

Show that if $\mathsf{P} = \mathsf{PSPACE}$, then $\mathsf{EXPTIME} = \mathsf{EXPSPACE}$.

Exercise 7: Descriptive complexity

adapted from an exercise by Cristina Sirangelo

1. Let `FO-SAT` be the following problem:
   - INPUT: a first-order formula ϕ
   - QUESTION: is ϕ satisfiable ? That is, does ϕ have a model M (denoted $M \models \phi$) ?

What is the complexity of this problem ?

2. Define FO-Combined Complexity:

    - INPUT: a first-order formula ϕ and a finite structure M (on the same signature)
    - QUESTION: does $M \models \phi$?

    Show that this problem is PSPACE-complete.

3. What if the model is fixed ? That is, for a fixed model M, one want to decide:

    - INPUT: a first-order formula ϕ
    - QUESTION: does $M \models \phi$

4. Given a first-order formula ϕ, Data Complexity is the problem:

    - INPUT: a model M
    - QUESTION: does $M \models \phi$?

    Show that this problem is in L.