Probabilistic Aspects of Computer Science:

DTMC

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MPRI M1

1. Discrete Event Systems
2. Renewal Processes with Arithmetic Distribution
3. Discrete Time Markov Chains (DTMC)
4. Finite DTMC
Plan

1. Discrete Event Systems

Renewal Processes with Arithmetic Distribution

Discrete Time Markov Chains (DTMC)

Finite DTMC
A path of a discrete-event system

Observe that $s_3$ and $s_{12}$ are hidden to the observations $\{X(\tau)\}$. 
Informally

An execution of a discrete-event system is an infinite sequence of events: \( e_1, e_2, \ldots \) occurring after some (possibly null) delay.

More formally

A discrete-event system is defined by two families of random variables: 

\( S_0, S_1, S_2, \ldots \) such that \( S_0 \) is the initial state and \( S_i \) is the state of the system after the occurrence of \( e_i \).

\( T_0, T_1, T_2, \ldots \) such that \( T_0 \) is the elapsed time before the occurrence of \( e_0 \) and \( T_i \) is the elapsed time between the occurrences of \( e_i \) and \( e_{i+1} \).

(Do you need some explanations?)
A DES is non Zeno almost surely iff $\Pr(\sum_{i \in \mathbb{N}} T_i = \infty) = 1$. (cf Achilles and the tortoise)

When a DES is non Zeno:

- $N(\tau) \overset{\text{def}}{=} \min(\{n \mid \sum_{k \leq n} T_k > \tau\})$, the number of events up to time $\tau$, is defined for almost every sample.

- $X(\tau) \overset{\text{def}}{=} S_{N(\tau)}$ is the observable state at time $\tau$.

When $\Pr(S_0 = s) = 1$, one says that the process starts in $s$. 
Two kinds of analysis

- **Transient analysis.** Computation of measures depending on the elapsed time since the initial state:
  
  for instance $\pi_\tau$, the distribution of $X_\tau$.

- **Steady-state analysis.** Computation of measures depending on the long-run behaviour of the system:
  
  for instance $\pi_\infty \overset{\text{def}}{=} \lim_{\tau \to \infty} \pi_\tau$.

  *(requires to establish its existence)*

Analysis of a web server

- What is the probability that a connection is established within 10s?
- What is the mean number of clients on the long run?
Performance indices

Definition

- A *performance index* is a function from states to numerical values.
- The measure of an index \( f \) w.r.t. to a state distribution \( \pi \) is given by:
  \[
  \sum_{s \in S} \pi(s) \cdot f(s)
  \]
- When range of \( f \) is \( \{0, 1\} \) it is an *atomic property* and its measure can be rewritten:
  \[
  \sum_{s \models f} \pi(s)
  \]

Analysis of a web server

- Let \( S' \) be the subset of states such that the server is available. Then the probability that the server is available at time 10 is:
  \[
  \sum_{s \in S'} \pi_{10}(s)
  \]
- Let \( s \) be a state and \( cl(s) \) be the number of clients in \( s \). Then the mean number of clients on the long run is:
  \[
  \sum_{s \in S} \pi_{\infty}(s) \cdot cl(s)
  \]
Renewal process

A renewal process is a very simple case of DES.

- It has a single state.
- The time intervals between events are integers obtained by sampling i.i.d. (independent and identically distributed) random variables.
- *Renewal instants* are the instants corresponding to the occurrence of events.

Using a lamp

- A bulb is used by some lamp.
- The bulb seller provides some information about the quality of the bulb.

for \( n > 0 \), \( f_n \) is the probability that the bulb duration is \( n \) days
A DTMC is a stochastic process which fulfills:

- For all $n$, $T_n$ is the constant 1.
- The process is memoryless ($X_n$ denotes $X(n)$)

$$\Pr(X_{n+1} = s_j \mid X_0 = s_{i_0}, \ldots, X_{n-1} = s_{i_{n-1}}, X_n = s_i) = \Pr(X_{n+1} = s_j \mid X_n = s_i) = P[i, j] \overset{\text{def}}{=} p_{ij} \quad \text{independent of } n$$

The behavior of a DTMC is defined by $X_0$ and $P$.
Two infinite DTMCs

A random walk

A simulation of renewal process
Transient analysis of a DTMC

The transient analysis is easy (and effective in the finite case):

$$\pi_n = \pi_0 \cdot P^n \text{ with } \pi_n \text{ the distribution of } X_n$$

\[
\begin{bmatrix}
0.3 & 0.7 & 0.0 \\
0.0 & 0.0 & 1.0 \\
0.2 & 0.8 & 0.0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\pi_0 \\
1.0 & 0.0 & 0.0 & 0.0 \\
\pi_2 \\
0.09 & 0.21 & 0.70
\end{bmatrix}
\begin{bmatrix}
\pi_1 \\
0.3 & 0.7 & 0.0 \\
\pi_3 \\
0.167 & 0.623 & 0.21
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.7 \\
1 \\
0.8
\end{bmatrix}
\]

\[
\begin{bmatrix}
0.3 \quad 0.7 \quad 0.0 \\
0.0 \quad 0.0 \quad 1.0 \\
0.2 \quad 0.8 \quad 0.0
\end{bmatrix}
\]
Plan

Discrete Event Systems

2. Renewal Processes with Arithmetic Distribution

Discrete Time Markov Chains (DTMC)

Finite DTMC
Analysis of renewal process

Let $u_n$ be the probability that $n$ is a renewal instant.
What can be expected about $u_n$ when $n$ goes to $\infty$?

Using a lamp

- Let $f_2 = 0.4$ and $f_3 = 0.6$
- Then $u_0 = 1$ (initial bulb), $u_1 = 0$
- $u_2 = f_2 = 0.4$, $u_3 = f_3 = 0.6$ (one change)
- $u_4 = u_2 f_2 = 0.16$, $u_5 = u_3 f_2 + u_2 f_3 = 0.48$ (two changes)

The renewal equation

$$u_n = u_0 f_n + \cdots + u_{n-1} f_1 \text{ when } n > 0 \quad (1)$$

Is there a limit?

$u_{10} = 0.3696$, $u_{20} = 0.38867558$, $u_{30} = 0.38423714$, $u_{40} = 0.38463453$, $u_{50} = 0.38461546$

What is its value?
The average renewal time

The average renewal time is: \( \mu \overset{\text{def}}{=} \sum_{i \in \mathbb{N}} i f_i \)
(\( \mu \) may be infinite)

Intuitively, during \( \mu \) time units, there is on average one renewal instant.

So one could expect: \( \lim_{n \to \infty} u_n = \mu^{-1} \overset{\text{def}}{=} \eta \).
(with the convention \( \infty^{-1} \overset{\text{def}}{=} 0 \))

Example (continued)

- \( \mu = 0.4 \cdot 2 + 0.6 \cdot 3 = 2.6 \)
- \( \eta = 0.38461538 \) very close to \( u_{50} = 0.38461546 \)
Is there always a limit?

Let \( f_2 = 1 \), then \( \lim_{n \to \infty} u_{2n} = 1 \) and \( \lim_{n \to \infty} u_{2n+1} = 0 \).

Periodicity

- The *periodicity* of the renewal process is: \( \gcd(\{i \mid f_i > 0\}) \).
- When the periodicity is 1 the renewal process is said *aperiodic*.

The renewal theorem

When the process is aperiodic, \( \lim_{n \to \infty} u_n \) exists and is equal to \( \eta \).
Another renewal equation

Let $\rho_k \overset{\text{def}}{=} \sum_{i>k} f_i$ be the probability that the duration before a new renewal instant is strictly greater than $k$.

$$\mu = \sum_{i \in \mathbb{N}} i f_i = \sum_{i \in \mathbb{N}} \sum_{0 \leq k < i} f_i = \sum_{k \in \mathbb{N}} \sum_{i > k} f_i = \sum_{k \in \mathbb{N}} \rho_k$$

Let $L_{nk}$ (with $k \leq n$) be the event “The last renewal instant in $[0, n]$ is $k$”

Observe that: $\Pr(L_{nk}) = u_k \rho_{n-k}$.

Letting $n$ fixed and summing over $k$, one obtains:

$$\rho_0 u_n + \rho_1 u_{n-1} + \cdots + \rho_n u_0 = 1$$  (2)
A first sufficient condition

If \( \limsup_{n \to \infty} u_n \leq \eta \) then \( \lim_{n \to \infty} u_n \) exists and is equal to \( \eta \).

Sketch of proof (for \( \mu \) finite)
Let \( u_{n_1}, u_{n_2}, \ldots \) be an arbitrary subsequence converging toward \( \eta' \) (\( \leq \eta \)).
Let us select some integer \( r > 0 \) and some \( \varepsilon > 0 \).

There exists \( m \) such that:

\[
\forall n_i \geq m \ |u_{n_i} - \eta'| \leq \varepsilon \land \forall 1 \leq r' \leq r \ u_{n_i - r'} - \eta \leq \varepsilon
\]

Using (2) for \( n_i \geq m \):

\[
\rho_0(\eta' + \varepsilon) + (\eta + \varepsilon) \sum_{r' = 1}^{r} \rho_{r'} + \sum_{r' > r} \rho_{r'} \geq 1
\]

Letting \( \varepsilon \) goes to 0,

\[
\rho_0 \eta' + \eta \sum_{r' = 1}^{r} \rho_{r'} + \sum_{r' > r} \rho_{r'} \geq 1
\]

Letting \( r \) goes to \( \infty \) (recall that \( \rho_0 = 1 \)),

\[
\eta' + \eta(\mu - 1) \geq 1
\]

Which can be rewritten as: \( \eta' - \eta \geq 0 \) implying \( \eta' = \eta \)
Standard results

Let \( a_1, \ldots, a_k \) be natural integers whose gcd is 1.

Then there exists \( n_0 \) such that:

\[
\forall n \geq n_0 \exists \alpha_1, \ldots, \alpha_k \in \mathbb{N} \ n = a_1 \alpha_1 + \cdots + a_k \alpha_k
\]

Sketch of proof

- Using Euclid algorithm, there exists \( y_1, \ldots, y_k \in \mathbb{Z} \) such that:
  \[
  1 = a_1 y_1 + \cdots + a_k y_k
  \]
- Let us note \( s = a_1 + \cdots + a_k \) and \( x = \sup_i |y_i|(s - 1) \).
- Let \( n \geq xs \) and perform the Euclidian division of \( n \) by \( s \).

Let \((x_{n,m})_{n,m \in \mathbb{N}}\) be a bounded family of reals.

Then there exists an infinite sequence of indices \( m_1 < m_2 < \cdots \) such that:

For all \( n \in \mathbb{N} \) the subsequence \((x_{n,m_k})_{k \in \mathbb{N}}\) is convergent.

Sketch of proof

- Build nested subsequences of indices \((m_k^n)_{k \in \mathbb{N}}\) such that:
  \((x_{n,m_k^n})_{k \in \mathbb{N}}\) is convergent.
- Pick the “diagonal” subsequence of indices \( m_k = m_k^k \).
The key lemma

Let $f$ be an aperiodic distribution and $(w_n)_{n \in \mathbb{N}}$ such that $\forall n, w_n \leq w_0$ and:

$$w_n = \sum_{k=1}^{\infty} f_k w_{n+k}$$

(3)

Then for all $n$, $w_n = w_0$.

Sketch of proof

Let $A = \{k \mid f_k > 0\}$, $B = \{k \mid \exists k_1, \ldots, k_n \in A \ \exists \alpha_1, \ldots, \alpha_n \in \mathbb{N} \ k = \sum \alpha_i k_i\}$. There exists $n_0$ such that $[n_0, \infty[ \subseteq B$.

$$w_0 = \sum_{k=1}^{\infty} f_k w_k \leq w_0 \sum_{k=1}^{\infty} f_k = w_0$$

implying $w_k = w_0$ for all $k \in A$.

Let $k \in A$,

$$w_k = \sum_{k'=1}^{\infty} f_{k'} w_{k+k'} \leq w_0 \sum_{k'=1}^{\infty} f_{k'} = w_0$$

implying $w_{k+k'} = w_0$

for all $k, k' \in A$.

Iterating the process, one gets $w_k = w_0$ for all $k \in B$.

$$w_{n_0-1} = \sum_{k=1}^{\infty} f_k w_{n_0-1+k} = w_0 \sum_{k=1}^{\infty} f_k = w_0$$

Iterating this process, one concludes.
The dominated convergence theorem

Let \((a_m)_{m \in \mathbb{N}}, (v_m)_{m \in \mathbb{N}}\) and \((u_{m,n})_{m,n \in \mathbb{N}}\) be sequences of non negative reals.

Assume that:

- \[\sum_{m \in \mathbb{N}} a_m v_m < \infty;\]
- \(\forall m, n \ u_{m,n} \leq v_m;\)
- \(\forall m \ \lim_{n \to \infty} u_{m,n} = \ell_m.\)

**Interpretation.**

\((a_m)_{m \in \mathbb{N}}\) is a measure over \(\mathbb{N}\), \((v_m)_{m \in \mathbb{N}}\) maps \(\mathbb{N}\) to \(\mathbb{R}_+\) integrable w.r.t. \((a_m)_{m \in \mathbb{N}}\).

For all \(n\), \((u_{m,n})_{m,n \in \mathbb{N}}\) maps \(\mathbb{N}\) to \(\mathbb{R}_+\) and is bounded by \((v_m)_{m \in \mathbb{N}}\).

These mappings converge to \(\ell_m\).

Then:

\[
\lim_{n \to \infty} \sum_{m \in \mathbb{N}} a_m u_{m,n} = \sum_{m \in \mathbb{N}} a_m \ell_m
\]
Proof of the renewal theorem ($\mu$ finite)

Let $\nu \overset{\text{def}}{=} \limsup_{n \to \infty} u_n$ and $(r_m)_{m \in \mathbb{N}}$ such that $\nu = \lim_{m \to \infty} u_{r_m}$.
Define $u_{n,m} = u_{r_m - n}$ if $n \leq r_m$ and $u_{n,m} = 0$ otherwise.

There exist $m_1, m_2, \ldots$ such that for all $n$, $(u_{n,m_k})_{k \in \mathbb{N}}$ converges to a limit $w_n \leq w_0 = \nu$.

Equation (1) can be rewritten as: $u_{n,m_k} = \sum_{i=1}^{\infty} f_i u_{n+i,m_k}$
Letting $k$ go $\infty$ yields: $w_n = \sum_{i=1}^{\infty} f_i w_{n+i}$ (by dominated convergence theorem)
Hence for all $n$, $w_n = \nu$.

Rewriting (2) for $n = m_k$ gives: $\rho_0 u_{0,m_k} + \rho_1 u_{1,m_k} + \cdots + \rho_{r_{m_k}} u_{r_{m_k},m_k} = 1$

For all fixed $r$, $\rho_0 u_{0,m_k} + \rho_1 u_{1,m_k} + \cdots + \rho_r u_{r,m_k} \leq 1$
Letting $m_k$ go to $\infty$, $\nu \sum^{r}_{r'=0} \rho_{r'} \leq 1$ and letting $r$ go to $\infty$, $\nu \mu \leq 1 \iff \nu \leq \eta$. 
Generalizations (1)

Periodicity
Nothing more than a change of scale

Let \( f \) be a distribution of period \( p \) with (non necessarily finite) mean \( \mu \). Then \( \lim_{n \to \infty} u_{np} = p\mu^{-1} \) and for all \( n \) such that \( n \mod p \neq 0 \), \( u_n = 0 \).

Defective renewal process: \( \sum_{n \in \mathbb{N}} f_n \leq 1 \)

\( 1 - \sum_{n \in \mathbb{N}} f_n \) is the probability that there will be no next renewal instant. (e.g. a perfect bulb)

The mean number of renewal instants, \( \sum_{n \in \mathbb{N}} u_n \), is finite iff \( \sum_{n \in \mathbb{N}} f_n < 1 \). In this case,

\[
\sum_{n \in \mathbb{N}} u_n = \frac{1}{1 - \sum_{n \in \mathbb{N}} f_n} \quad \text{(so } \lim_{n \to \infty} u_n = 0)\]
Generalizations (2)

**Delayed renewal process:** The first renewal instant is no more 0 but follows a possibly defective distribution \( \{b_n\} \) with \( B \overset{\text{def}}{=} \sum_{n \in \mathbb{N}} b_n \leq 1 \).

*(e.g. initially there is a possibly perfect bulb in the lamp)*

Let \( v_n \) be the probability of a renewal instant at time \( n \) in the delayed process.

- If \( \lim_{n \to \infty} u_n = \omega \) then \( \lim_{n \to \infty} v_n = B\omega \)
- If \( U \overset{\text{def}}{=} \sum_{n \to \infty} u_n \) is finite then \( \sum_{n \to \infty} v_n = BU \)

**Instantaneous renewal process:** \( 0 < f_0 < 1 \)

*(e.g. \( f_0 \) is the probability that a new bulb is initially faulty)*

\( u_n \) is now the mean number of renewals at time \( n \).

Let \( f'_n \) be a standard renewal process with \( f'_n = \frac{f_n}{1-f_0} \) for \( n > 0 \). Then:

\[
\frac{u_n}{1-f_0}
\]
Discrete Event Systems

Renewal Processes with Arithmetic Distribution

3 Discrete Time Markov Chains (DTMC)

Finite DTMC
Notations for a DTMC

\( p^{n}_{i,j} \), the probability to reach in \( n \) steps state \( j \) from state \( i \).

\( f^{n}_{i,j} \), the probability to reach in \( n \) steps state \( j \) from state \( i \) for the first time.

\[
    f_{i,j} \overset{\text{def}}{=} \sum_{n \in \mathbb{N}} f^{n}_{i,j}, \text{ the probability to reach } j \text{ from } i
\]

\[
    \mu_{i} \overset{\text{def}}{=} \sum_{n \in \mathbb{N}} n f^{n}_{i,i}, \text{ the mean return time in } i \text{ (only relevant if } f_{i,i} = 1). \]

A renewal-like equation

\[
    \forall n > 0 \quad p^{n}_{i,j} = \sum_{m=0}^{n} f^{m}_{i,j} p^{n-m}_{j,j}
\]
DTMC and renewal processes

Some renewal processes

- The visits of a state $i$ starting from $i$, is a renewal process with renewal distribution $\{f_{i,i}^n\}_n$. $p_{i,i}^n$ is the probability of a renewal instant at time $n$.

- The visits of a state $i$ starting from $j$, is a delayed renewal process with delay distribution $\{f_{j,i}^n\}_n$. $p_{j,i}^n$ is the probability of a renewal instant at time $n$.

Classification of states w.r.t. the associated renewal process

- A state $i$ is transient if $f_{i,i} < 1$, the probability of a return after a visit is strictly less than one.

- A state is null recurrent if $f_{i,i} = 1$ and $\mu_{i,i} = \infty$, the probability of a return after a visit is 1 and the mean return time is $\infty$.

- A state is positive recurrent if $f_{i,i} = 1$ and $\mu_{i,i} < \infty$, the probability of a return after a visit is 1 and the mean return time is finite.

- A state is aperiodic if its associated process is aperiodic.

- A state is ergodic if it is aperiodic and positive recurrent.
The value of $p$ is critical.

- All states have period 2.
- If $p > \frac{1}{2}$ then all states are transient.
- If $p = \frac{1}{2}$ then all states are null recurrent.
- If $p < \frac{1}{2}$ then all states are positive recurrent.
Structure of a DTMC

A closed subset

The closure of a state $i$

An irreducible DTMC: for all $i, j$
Irreducibility

All states of an irreducible DTMC are of the same kind.

Sketch of proof

Let $i, j$ be two states.
There exist $r$ and $s$ such that $p_{i,j}^r > 0$ and $p_{j,i}^s > 0$. Observe that:

$$p_{i,i}^{n+r+s} \geq p_{i,j}^r p_{j,j}^n p_{j,i}^s$$  \hspace{1cm} (4)

Transient vs recurrent. If $\sum_{n \in \mathbb{N}} p_{i,i}^n$ is finite then $\sum_{n \in \mathbb{N}} p_{j,j}^n$ is finite.

Null recurrence. If $\lim_{n \to \infty} p_{i,i}^n = 0$ then $\lim_{n \to \infty} p_{j,j}^n = 0$.

Periodicity. Assume $i$ has periodicity $t \geq 1$.
Using (4), $r + s$ is a multiple of $t$.
So if $n$ is not a multiple of $t$, then $p_{j,j}^n = 0$.
Thus the periodicity of $j$ is a multiple of $t$. 
Periodicity

Let $\mathcal{C}$ be irreducible with periodicity $p$. Then $S = S_0 \cup S_1 \cup \ldots \cup S_{p-1}$ with:

$$\forall k \forall i \in S_k \forall j \in S \ p_{i,j} > 0 \Rightarrow j \in S_{(k+1) \mod p}$$

Furthermore $p$ is the greatest integer fulfilling this property.

Sketch of proof

Let $i$ be a state. For all $0 \leq k < p$,

$$S_k \overset{\text{def}}{=} \{ j \mid \text{there is a path from } i \text{ to } j \text{ with length equal to } lp + k \text{ for some } l \}$$

Irreducibility implies $S = S_0 \cup S_1 \cup \ldots \cup S_{p-1}$

By definition, $\forall j \in S_k \forall j' \in S \ p_{j,j'} > 0 \Rightarrow j' \in S_{(k+1) \mod p}$

Assume there exists $j \in S_k \cap S_{k'}$ with $k \neq k'$.

Since there is a path from $j$ to $i$,

there exists at least a path from $i$ to $i$ whose length is not a multiple of $p$,

leading to a contradiction.

Assume there is some $p' > p$ fulfilling this property.

Then the periodicity of the renewal process is a multiple of $p'$,

leading to another contradiction.
Characterization of ergodicity

Definitions.

• A distribution $\pi$ is *invariant* if $\pi P = \pi$.
• There is a *steady-state* distribution $\pi$ for $\pi_0$ if $\pi = \lim_{n \to \infty} \pi_0 P^n$.

Let $C$ be an irreducible DTMC whose states are aperiodic.

Existence of an invariant steady-state distribution.

Assume that the states of $C$ are positive recurrent.

Then the limits $\lim_{n \to \infty} p_{j,i}^n$ exist and are equal to $\mu_i^{-1}$.

Furthermore:

$$\sum_{i \in S} \mu_i^{-1} = 1 \quad \text{and} \quad \forall i \, \mu_i^{-1} = \sum_{j \in S} \mu_j^{-1} p_{j,i}$$

Unicity of an invariant distribution.

Conversely, assume there exists $u$ such that:

For all $i$, $u_i \geq 0$, $\sum_{i \in S} u_i = 1$ and $u_i = \sum_{j \in S} u_j p_{j,i}$.

Then for all $i$, $u_i = \mu_i^{-1}$.
Proof of existence

Applying renewal theory on the delayed renewal process:
\[ \lim_{n \to \infty} p^n_{j,i} = f_{j,i} \mu_i^{-1} = \mu_i^{-1} \] (recurrence of states implies \( f_{j,i} = 1 \))

Since \( P^n \) is a transition matrix, \( \forall n \ \forall i \ \sum_{j \in S} p^n_{i,j} = 1 \)
Letting \( n \) go to infinity, yields: \( \sum_{j \in S} \mu_j^{-1} \leq 1 \)

One has: \( \sum_{j \in S} p^n_{k,j} p_{j,i} = p^{n+1}_{k,i} \)
Letting \( n \) go to infinity, yields: \( \sum_{j \in S} \mu_j^{-1} p_{j,i} \leq \mu_i^{-1} \)

Summing \( \sum_{j \in S} \mu_j^{-1} p_{j,i} \leq \mu_i^{-1} \) over \( i \), one obtains:
\[
\sum_{i \in S} \mu_i^{-1} \geq \sum_{i \in S} \sum_{j \in S} \mu_j^{-1} p_{j,i} = \sum_{j \in S} \mu_j^{-1} \sum_{i \in S} p_{j,i} = \sum_{j \in S} \mu_j^{-1}
\]

One has equality of sums, so also equality of terms: \( \mu_i^{-1} = \sum_{j \in S} \mu_j^{-1} p_{j,i} \)

By iteration: \( \mu_i^{-1} = \sum_{j \in S} \mu_j^{-1} p^n_{j,i} \)
Letting \( n \) go to infinity, yields: (by dominated convergence theorem)
\( \mu_i^{-1} = \sum_{j \in S} \mu_j^{-1} \mu_i^{-1} \) implying \( \sum_{j \in S} \mu_j^{-1} = 1 \)
Proof of unicity

Assume there exists \( u \) such that:

\[
\sum_{i \in S} u_i = 1 \quad \text{and for all } i, u_i \geq 0, \quad u_i = \sum_{j \in S} u_j p_{j,i}
\]

Let us pick some \( u_i > 0 \), by iteration of the last equation:

\[
u_i = \sum_{j \in S} u_j p_{j,i}^n
\]

Since the states of the chain are aperiodic, \( \lim_{n \to \infty} p_{j,i}^n \) exists for all \( j \).

Letting \( n \) go to \( \infty \),

\[
u_i = \sum_{j \in S} u_j \lim_{n \to \infty} p_{j,i}^n \quad \text{(by dominated convergence theorem)}
\]

There exists \( j \) such that \( \lim_{n \to \infty} p_{j,i}^n > 0 \).

So \( i \) is positive recurrent and then ergodic (since aperiodic).

So all states are ergodic implying that the limits of \( p_{j,i}^n \) are \( \mu_i^{-1} \).

The previous equation can be rewritten as:

\[
u_i = \sum_{j \in S} u_j \mu_j^{-1} = \mu_i^{-1}
\]
Characterization of positive recurrence

Let $C$ be irreducible. Then $C$ is positive recurrent iff there exists $u$ such that:

For all $i$, $u[i] > 0$, $u = u \cdot P$ and $\sum_{i \in S} u[i] < \infty$

Sketch of necessity proof

Let $p$ be the periodicity of $C$ and $S = S_0 \cup \ldots \cup S_{p-1}$.

$P^p$ defines an ergodic chain over $S_0$.

So there exists a positive vector $u_0$ indexed by $S_0$ with:

$$\sum_{s \in S_0} u_0[s] = 1 \text{ and } u_0 \cdot P^p = u_0$$

Define for $0 < k < p$ and $s \in S_k$:

$$u_k[s] = \sum_{s' \in S_{k-1}} u_{k-1}[s'] P[s', s]$$

Then $u = (u_0, \ldots, u_{p-1})$ is the required vector.
Sufficiency proof

Let $p$ be the periodicity of $C$ and $S = S_0 \cup \ldots \cup S_{p-1}$.

Assume there exists $u$ such that:

For all $i$, $u[i] > 0$, $u = u \cdot P$ and $\sum_{i \in S} u[i] < \infty$

Decompose $u$ as $(u_0, \ldots, u_{p-1})$.
Then $P^p$ defines an aperiodic irreducible chain $C'$ over $S_0$ and $u_0 \cdot P^p = u_0$.
So $C'$ is ergodic thus positive recurrent.

Let $s \in S_0$

- the probability of a return to $s$ is the same in $C$ and $C'$.
  So $s$ is recurrent in $C$.
- the mean return time $s$ in $C$ is $p$ times the mean return time to $s$ in $C'$.
  So $s$ is positive recurrent in $C$. 
Characterization of recurrence

Let $C$ be irreducible ($S=\mathbb{N}$). Then it is recurrent iff $0$ is the single solution of:

$$\forall i > 0 \ x[i] = \sum_{j>0} P[i, j] x[j] \land 0 \leq x[i] \leq 1$$  \hspace{1cm} (5)

Sketch of proof

Let $pin^n[i]$ be the probability to stay in $\mathbb{N}^*$ during $n$ steps starting from $i$.

$$pin[i] \overset{\text{def}}{=} \lim_{n\to\infty} pin^n[i]$$ is the probability to never meet $0$ starting from $i$.

By one-step reasoning: $pin^{n+1}[i] = \sum_{j>0} P[i, j] pin^n[j]$

By dct: $pin[i] = \sum_{j>0} P[i, j] pin[j]$

Let $x$ be a solution of (5)

One has: $\forall i \ x[i] \leq 1 = pin^0[i]$

By induction, the equality holds for all $n$:

$$x[i] = \sum_{j>0} P[i, j] x[j] \leq \sum_{j>0} P[i, j] pin^n[j] = pin^{n+1}[i]$$

Letting $n$ goes to infinity, $\forall i \ x[i] \leq pin[i]$
Solving $u = u \cdot P$ in a recurrent chain

Let $rP^{(n)}_{ij}$ be the probability that starting from $i$ one reaches $j$ after $n$ transitions without ever visiting $r$ except initially (so $rP^{(0)}_{ij} = 1_{i=j}$).

Let $r\pi_{ij} \overset{\text{def}}{=} \sum_{n \in \mathbb{N}} rP^{(n)}_{ij}$ be the mean number of visits of $j$ without visiting $r$. (finite in an irreducible chain)

Let $\mathcal{C}$ be irreducible and recurrent.
Then, up to a scalar factor, $(r\pi_{ri})_{i \in \mathcal{S}}$ is the single solution of:

$$u = u \cdot P$$  \hspace{1cm} (6)
Proof

Existence

Due to the initial visit, $r \pi_{rr} = 1$

For $i \neq r$, $r P^{(n+1)}_{ri} = \sum_{j \in S} r P^{(n)}_{rj} p_{ji}$

Summing over $n$: $\sum_{n \geq 1} r P^{(n)}_{ri} = \sum_{n \geq 0} \sum_{j \in S} r P^{(n)}_{rj} p_{ji}$

Since $r P^{(0)}_{ri} = 0$: $r \pi_{ri} = \sum_{j \in S} r \pi_{rj} p_{ji}$

$\sum_{j \in S} r P^{(n)}_{rj} p_{jr}$ is the probability of a first return to $r$ at the $n + 1^{th}$ transition.

So $\sum_{j \in S} r \pi_{rj} p_{jr}$ is the probability of a return to $r$.

Since $r$ is recurrent: $\sum_{j \in S} r \pi_{rj} p_{jr} = 1 = r \pi_{rr}$

Unicity

Let $u$ fulfill $u = u \cdot P$ and for all $i$, $u_i \geq 0$.

So $u_i = 0$ implies $u_j = 0$ for all $j$ such that $p_{ji} > 0$.

Using irreducibility, either $u$ is null or all its components are strictly positive.

Let $u$ be strictly positive and apply a scalar factor so that $u_r = 1$. For $i \neq r$:

$u_i = p_{ri} + \sum_{j \neq r} u_j p_{ji} = p_{ri} + \sum_{j \neq r} \left( p_{rj} + \sum_{k \neq r} u_k p_{kj} \right) p_{ji} = p_{ri} + r P^{(2)}_{ri} + \sum_{k \neq r} u_k \cdot r P^{(2)}_{ki}$

By induction: $u_i = p_{ri} + r P^{(2)}_{ri} + \cdots + r P^{(n)}_{ri} + \sum_{j \neq r} u_j \cdot r P^{(n)}_{ji}$

So $u_i \geq r \pi_{ri}$ and $(u_i - r \pi_{ri})_{i \in S}$ is another solution.

Since $u_r - r \pi_{rr} = 0$, for all $i$, $u_i = r \pi_{ri}$.
### Summary of characterizations

<table>
<thead>
<tr>
<th>Status</th>
<th>Characterization</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Recurrent</strong></td>
<td>0 is the single solution of: ( \forall i \in S \setminus s_0 \ u_i = \sum_{j \in S \setminus {s_0}} p_{i,j} u_j ) and ( 0 \leq u_i \leq 1 )</td>
</tr>
<tr>
<td></td>
<td>In this case given any ( r \in S ), ( \forall u \geq 0 \ u \cdot P = u \iff \exists \alpha \geq 0 \ \forall s \ u[s] = \alpha \cdot r \pi_r s )</td>
</tr>
<tr>
<td><strong>Positive recurrent</strong></td>
<td>( \exists! u &gt; 0 \ u \cdot P = u \land \sum_{i \in S} u_i = 1 ) ( (u \text{ is the steady-state distribution when the DTMC is aperiodic}) )</td>
</tr>
<tr>
<td>**Period is } p )</td>
<td>( S = S_0 \uplus S_1 \uplus \ldots \uplus S_{p-1} ) with ( \forall r &lt; p \ \forall i \in S_r \ \forall j \in S \ p_{i,j} &gt; 0 \Rightarrow j \in S_{r+1 \mod p} ) and } p \text{ is the greatest integer fulfilling this property.}</td>
</tr>
</tbody>
</table>
The chain has period 2.

**Recurrence versus transience**

\[ x_1 = px_2 \text{ and } \forall i \geq 2 \quad x_i = px_{i+1} + (1-p)x_{i-1} \]

It can be rewritten as:

\[ x_1 = px_2 \text{ and } \forall i \geq 2 \quad x_{i+1} - x_i = \frac{1-p}{p}(x_i - x_{i-1}) \]

By induction:

\[ x_i = x_1 + (x_2 - x_1) \sum_{j=0}^{i-2} \left(\frac{1-p}{p}\right)^j = x_1 \left(1 + \frac{1-p}{p} \sum_{j=0}^{i-2} \left(\frac{1-p}{p}\right)^j\right) = x_1 \left(\sum_{j=0}^{i-1} \left(\frac{1-p}{p}\right)^j\right) \]

Thus if \( p \leq \frac{1}{2} \) then the \( x_i \)'s are unbounded.

Otherwise the \( x_i \)'s are bounded by \( x_1 \frac{p}{2p-1} \).
Analysis of a random walk (2)

Positive versus null recurrence

\[ x_0 = (1 - p)x_1 \text{ and } x_1 = (1 - p)x_2 + x_0 \text{ and } \forall i \geq 2 \ x_i = (1 - p)x_{i+1} + px_{i-1} \]

So \( x_2 = \frac{px_1}{1-p} \) and by induction:

\[ x_{i+1} = \frac{x_i - px_{i-1}}{1-p} = \frac{x_i - (1 - p)x_i}{1-p} = \frac{px_i}{1-p} \]

**Assume** \( x_0 > 0 \). (otherwise \( x = 0 \) is not a distribution)

- When \( p = \frac{1}{2} \), \( x_2 = x_1 \) and by induction, for \( i \geq 1 \), \( x_i = x_1 \). So \( \sum_{i \in \mathbb{N}} x_i = \infty \).

- When \( p < \frac{1}{2} \) for \( i \geq 1 \), \( x_i = x_1 \left( \frac{p}{1-p} \right)^{i-1} \)

Thus \( \sum_{i \in \mathbb{N}} x_i = x_1 (1 - p + \frac{1-p}{1-2p}) \) is finite.
Plan

Discrete Event Systems

Renewal Processes with Arithmetic Distribution

Discrete Time Markov Chains (DTMC)

Finite DTMC
A scc $S'$ is a maximal subset of vertices such that:
for all $i, j \in S'$ there is a path from $i$ to $j$.

A scc $S'$ is terminal if there is no path from $S'$ to $S \setminus S'$.

Computation of scc's in linear time by the algorithm of Tarjan.
### Classification of states

Every terminal scc is an irreducible DTMC.
The transient states are the states of the non terminal scc's.
The states of terminal scc’s are positive recurrent.

**Sketch of proof**

Let $i$ belonging to a non terminal scc.
There is a path from $i$ to $j$ outside the scc.
There is no path from $j$ to $i$.
So $i$ is transient.

Let $\mathbf{P}$ be the transition matrix of a terminal scc $S'$.
Pick some state $i \in S'$.
For all $n$, $\sum_{j \in S'} p_{i,j}^n = 1$.
Since this is a finite sum, there exists $j$ such that:
$p_{i,j}^n$ does not converge to 0 when $n$ goes to infinity.
So $j$ is positive recurrent.
Computing the periodicity: an example

Adaptation of a tree covering construction

periodicity = \text{gcd}(0, 2, 4) = 2
Computing the periodicity: the algorithm

Input $G$, a strongly connected graph whose set of vertices is $\{1, \ldots, n\}$
Output $p$, the periodicity of $G$
Data $i, j$ integers, $Height$ an array of size $n$, $Q$ a queue

For $i$ from $1$ to $n$ do $Height[i] \leftarrow \infty$

$p \leftarrow 0$
$Height[0] \leftarrow 0$

InsertQueue($Q, 0$)

While not EmptyQueue($Q$) do
    $i \leftarrow$ ExtractQueue($Q$)
    For $(i, j) \in G$ do
        If $Height[j] = \infty$ then
            $Height[j] \leftarrow Height[i] + 1$
            InsertQueue($Q, j$)
        Else
            $p \leftarrow \gcd(p, Height[i] - Height[j] + 1)$
    
Return($p$)
Proof of the algorithm

Let $p$ be the periodicity, $p'$ the gcd of the edge labels and $r$ the root.

Given two paths with same source and destination, the difference between the lengths of these paths must be a multiple of $p$.

Let $(u, v)$ be an edge with non null label. Let $\sigma_u$ (resp. $\sigma_v$), the path from $r$ to $u$ (resp. $v$) along the tree. $\sigma_u(u, v)$ is a path from $r$ to $v$. The difference between the lengths of the two paths is: $\text{Height}[u] - \text{Height}[v] + 1$

Thus $p | \text{Height}[u] - \text{Height}[v] + 1$ and so $p | p'$.

Let $S'_{i}$ for $0 \leq i < p'$ be defined by:

$$s \in S'_{i} \text{ iff } \text{Height}[s] \mod p' = i$$

An edge of the tree joins a vertex of $S'_{i}$ to a vertex of $S'_{i+1 \mod p'}$. An edge $(u, v)$ out of the tree joins $u \in S'_{\text{Height}[u] \mod p'}$ to $v \in S'_{\text{Height}[v] \mod p'}$. $\text{Height}[u] - \text{Height}[v] + 1 \mod p' = 0 \Rightarrow \text{Height}[v] \mod p' = \text{Height}[u] + 1 \mod p'$

Using the characterization of periodicity, $p' \leq p$. 
Matrices and vectors of a DTMC

Let \( \{C_1, \ldots, C_k\} \) be the subchains associated with terminal scc. Let \( \pi_i \) be the steady-state distribution of \( C_i \) supposed to be aperiodic. Let \( T \) be the set of transient states.
Let \( P_{T,T} \) (resp. \( P_{T,i} \)) be the transition matrix from \( T \) to \( T \) (resp. \( S_i \)).

\[
P_{T,T} = \begin{pmatrix}
0.0 & 0.7 & 0.0 \\
0.1 & 0.0 & 0.8 \\
0.0 & 0.2 & 0.0
\end{pmatrix}
\]

\[
P_{T,1} \cdot \mathbf{1}^T = \begin{pmatrix}
0.0 & 0.3 & 1.0 \\
0.0 & 0.0 & 1.0 \\
0.0 & 0.4 & 1.0
\end{pmatrix} \begin{pmatrix}
0.3 \\
0.0 \\
0.4
\end{pmatrix} = \begin{pmatrix}
0.0 \\
0.0 \\
0.0
\end{pmatrix}
\]

\[
P_{T,2} \cdot \mathbf{1}^T = \begin{pmatrix}
0.0 & 0.0 & 0.0 & 1.0 \\
0.0 & 0.1 & 0.0 & 1.0 \\
0.4 & 0.0 & 0.0 & 1.0
\end{pmatrix} \begin{pmatrix}
0.0 \\
0.0 \\
0.0
\end{pmatrix} = \begin{pmatrix}
0.0 \\
0.1 \\
0.4
\end{pmatrix}
\]

\( T=\{1, 2, 3\}, C_1=\{4, 5\}, C_2=\{6, 7, 8\} \)

\( \pi_1=(2/3, 1/3) \)

\( \pi_2=(1/8, 7/16, 7/16) \)
Steady-state distribution

Let $C$ be a finite DTMC with initial distribution $\pi_0$ whose terminal scc’s are aperiodic. Then there exists a steady-state distribution:

$$\pi_\infty \overset{\text{def}}{=} \sum_{i=1}^{k} \left( \left( \pi_{0,i} + \pi_{0,T} (\text{Id} - P_{T,T})^{-1} \cdot P_{T,i} \right) \cdot 1^T \right) \pi_i$$

where $\pi_{0,i}$ (resp. $\pi_{0,T}$) is $\pi_0$ restricted to states of $C_i$ (resp. $T$) and $\pi_i$ is the steady-state distribution of $C_i$.

Sketch of proof

$$\pi_\infty \overset{\text{def}}{=} \sum_{i=1}^{k} \Pr(\text{to reach } C_i) \cdot \pi_i$$

$$\Pr(\text{to reach } C_i) = \sum_{s \in S} \pi_0(s) \cdot \pi'_{C_i}(s) \text{ where } \pi'_{C_i}(s) = \Pr(\text{to reach } C_i \mid S_0 = s)$$

• When state $s \in C_i$, then $\pi'_{C_i}(s) = 1$ and $\pi'_{C_j}(s) = 0$ for $j \neq i$

• The probability of paths from a transient state $s$ along $T$ to $C_i$ of length $n + 1$ is:

$$\left( (P_{T,T})^n \cdot P_{T,i} \cdot 1^T \right)[s]$$

$$(\sum_{n \geq 0} (P_{T,T})^n)[i,j]$$ is the (finite) mean number of visits of $j$ starting from $i$.

For every $n_0$: $\left( \sum_{n \leq n_0} (P_{T,T})^n \right)(\text{Id} - P_{T,T}) = \text{Id} - (P_{T,T})^{n_0+1}$

Since $\lim_{n \to \infty} (P_{T,T})^n = 0$, letting $n_0$ go to infinity establishes the result.
Regular matrices

A matrix $M$ is *positive* if for all $i, j$, $M[i, j] > 0$.

A matrix $M$ is *non negative* if for all $i, j$, $M[i, j] \geq 0$.

A non negative square matrix $M$ is *regular* if for some $k$, $M^k$ is positive.

The transition matrix of an ergodic DTMC is regular.

**Sketch of proof**

Let $s \overset{\text{def}}{=} |S|$ and $i \in S$.

There is a $n_0$ such that for all $n \geq n_0$, $p_{ii}^n > 0$.

Furthermore for all $j, j'$, there are $m, m' \leq s - 1$ such that $p_{ji}^m > 0$ and $p_{ij'}^{m'} > 0$.

So for all $j, j'$ and $n \geq n_0 + 2(s - 1)$, one has $p_{jj'}^n > 0$. 

Let $\mathcal{C}$ be a finite ergodic DTMC, $\pi_n$ its distribution at time $n$ and $\pi_\infty$ its steady-state distribution. Then there exists some $0 < \lambda < 1$ such that:

$$\|\pi_\infty - \pi_n\| = O(\lambda^n)$$

Sketch of proof

Let $\Pi_\infty$ be the square matrix where every row is a copy of $\pi_\infty$.

$\Pi_\infty P = \Pi_\infty$ since $\pi_\infty P = \pi_\infty$ and for every transition matrix $P'$, $P'\Pi_\infty = \Pi_\infty$

Since $P^k$ is positive, there is some $0 < \delta < 1$ such that $\forall i, j \ P^k[i, j] \geq \delta \Pi_\infty[i, j]$

Let $\theta \overset{\text{def}}{=} 1 - \delta$ and $Q \overset{\text{def}}{=} \frac{1}{\theta} P^k - \frac{1 - \theta}{\theta} \Pi_\infty$

$Q$ is a transition matrix and fulfills: $P^k = \theta Q + (1 - \theta) \Pi_\infty$

Let us prove that: $\forall n \ P^{kn} = \theta^n Q^n + (1 - \theta^n) \Pi_\infty \ (\Leftrightarrow P^{kn} - \Pi_\infty = \theta^n (Q^n - \Pi_\infty))$

$$\Pi_\infty Q = \frac{1}{\theta} \Pi_\infty P^k - \frac{1 - \theta}{\theta} \Pi_\infty \Pi_\infty = \frac{1}{\theta} \Pi_\infty - \frac{1 - \theta}{\theta} \Pi_\infty = \Pi_\infty$$

$$P^{kn+k} = (\theta^n Q^n + (1 - \theta^n) \Pi_\infty)(\theta Q + (1 - \theta) \Pi_\infty)$$

$$= \theta^{n+1} Q^{n+1} + ((1 - \theta^n)\theta + (1 - \theta)\theta^n + (1 - \theta^n)(1 - \theta)) \Pi_\infty$$

Multiplying by $P^j$ with $0 \leq j < k$: $\forall n \ \forall j < k \ P^{kn+j} - \Pi_\infty = \theta^n (Q^n P^j - \Pi_\infty)$

Multiplying by $\pi_0$: $\forall n \ \forall j < k \ \pi_{kn+j} - \pi_\infty = \theta^n (\pi_0 Q^n P^j - \pi_\infty)$