Homework 1 - Probabilistic aspects of computer science

The objective of this homework is to establish the existence and unicity of a steady-state distribution for an ergodic Markov chain without using the fundamental theorem of renewal process.

1 Renewal process

In order to handle the case of a defective distribution over $\mathbb{N}^*$, we consider that a distribution is defined by $\{f_i\}_{i \in \mathbb{N}^* \cup \infty}$. Its mean value $\mu$ is defined by:

$$\mu = \sum_{k \in \mathbb{N}} \rho_k \text{ where } \rho_k \overset{\text{def}}{=} \sum_{i > k} f_i$$

We recall an important renewal equation where $u_i$ is the probability that $i$ is a renewal instant:

$$\forall n \in \mathbb{N} \quad \rho_0 u_n + \rho_1 u_{n-1} + \ldots + \rho_n u_0 = 1$$

Let us define $u_n \overset{\text{def}}{=} \frac{1}{n+1} \sum_{i \leq n} u_i$.

**Question 1.** Show that $\lim_{n \to \infty} u_n = \frac{1}{\mu}$.

*Hint: sum equation (2) from 0 to $N$ and then consider the cases where $\mu$ is infinite and finite.*

Let $b$ be a non-defective distribution over $\mathbb{N}$, and define $v_i$ as the probability that $i$ is a renewal instant in the delayed renewal process where $b$ is the delay distribution and $f$ is the renewal distribution.

$$\forall n \in \mathbb{N} \quad v_n = b_0 u_n + b_1 u_{n-1} + \ldots + b_n u_0$$

Let us define $v_n \overset{\text{def}}{=} \frac{1}{n+1} \sum_{i \leq n} v_i$.

**Question 2.** Using equation (3), show that $\lim_{n \to \infty} v_n = \frac{1}{\mu}$.

2 Stationary distribution

Let $M$ be a DTMC with initial state $i$. Define:

- $p_{ij}^{(k)}$ as the probability to be in state $j$ at time $k$;
- $f_{ij}^{(k)}$ as the probability to be in state $j$ at time $k$ for the first time excluding time 0;
- $f_{ij}^{(\infty)}$ the probability to never reach $j$;
- $\mu_i \overset{\text{def}}{=} \sum_{k \in \mathbb{N} \cup \infty} k f_{ii}^{(k)}$ the mean return time to state $i$.

A state $i$ is positive recurrent if and only if $\mu_i$ is finite.

In all the homework, $M$ is irreducible: for all states $i, j$ there exists $k$ such that $p_{ij}^{(k)} > 0$.

**Question 3.** Show that in an irreducible DTMC, if $i$ is recurrent then for any state $j$ the probability to reach $j$ from $i$ is 1.

Observe that $p_{ij}^{(n)}$ corresponds to the probability that $n$ is a renewal instant where the renewal is defined by meeting state $i$. Using question 1, a state $i$ is positive recurrent if and only if $\lim_{n \to \infty} p_{ii}^{(n)} > 0$ with $p_{ii}^{(n)} \overset{\text{def}}{=} \frac{1}{n+1} \sum_{m \leq n} p_{ii}^{(m)}$.

**Question 4.** Using the previous characterization, prove that if some state $i$ is positive recurrent, any state $j$ is positive recurrent.
A distribution $\pi$ over the states of $\mathcal{M}$ is said to be stationary if:

$$\pi = \pi P$$  \hspace{1cm} (4)

**Question 5.** Using equation (4), show that if $\pi$ is a stationary distribution, for all state $i$ and all $n \in \mathbb{N}$, $\pi[i] = \sum_{j \in S} \pi[j] p_{ji}^{(n)}$.

Using questions 2 and 3, in a positive recurrent DTMC for all states $i,j$ and all $n \in \mathbb{N}$,

$$\lim_{n \to \infty} p_{ji}^{(n)} = \frac{1}{\mu_i}. \hspace{1cm} (5)$$

**Question 6.** Using the previous characterization and question 5, show that if $\mathcal{M}$ is positive recurrent then there is at most one stationary distribution $\pi$ which, when defined, is obtained by $\pi[i] = \frac{1}{\mu_i}$.

Given a positive recurrent DTMC $\mathcal{M}$, one defines another DTMC $\mathcal{M}'$ where $i$ is some state of $\mathcal{M}$. In words, $\mathcal{M}'$ behaves as $\mathcal{M}$ except that states of $\mathcal{M}'$ keep trace of the execution since the last visit of $i$ (here we assume that $\mathcal{M}$ starts in $i$). More formally:

- $S^i = \{ w \mid w \in i(S \setminus \{i\})^* \}$ i.e. words where $i$ only occurs at the first position;
- for $k \neq i$ $P^i[wj, wjk] = P[j, k]$ and $P^i[wj, i] = P[j, i]$;
- All other transition probabilities are null.

**Question 7.** Using equation (1), show that:

$$\mu_i = \sum_{k=0}^{\infty} \sum_{w_1 \ldots w_k \in (S \setminus \{i\})^*} p_{w_1} \prod_{m=1}^{k-1} p_{w_m w_{m+1}} \mu_i$$

**Question 8.** Show that the distribution $\pi^i$ over $S^i$ defined by:

$$\pi^i[i] \overset{\text{def}}{=} \frac{1}{\mu_i} \text{ and } \pi^i[iw_1 \ldots w_k] \overset{\text{def}}{=} \frac{p_{iw_1} \prod_{m=1}^{k-1} p_{w_m w_{m+1}}}{\mu_i}$$

is a stationary distribution of $\mathcal{M}'$.

**Question 9.** Deduce that the distribution $\pi$ over $S$ defined by $\pi[j] \overset{\text{def}}{=} \sum_{w \in S_j} \pi^i[j]$ is a stationary distribution of $\mathcal{M}$.

**Question 10.** Conclude that an irreducible DTMC has a (unique) stationary distribution if and only if it is positive recurrent.

### 3 Steady-state distribution

Given a DTMC $\mathcal{M}$, one defines the *square* DTMC $\mathcal{M}'$ with transition matrix $P'$ over state space $S^2$ by:

$$P'[(i, j), (i', j')] = P[i, j'] P[j', i']$$

**Question 11.** Exhibit a finite irreducible DTMC such that its square Markov chain is not irreducible.

By definition, when $\mathcal{M}$ is irreducible and aperiodic, $\gcd(n \mid f^n_{ii} > 0) = 1$. Thus using the arithmetic lemma of the lectures, there exists $n_i$ such that for all $n \geq n_i$, $p^n_{ii} > 0$.

**Question 12.** Using the previous observation, show that given an irreducible and aperiodic DTMC, its square Markov chain is irreducible and aperiodic.

**Question 13.** Using questions 10 and 12, show that if $\mathcal{M}$ is ergodic then $\mathcal{M}'$ is ergodic. \textbf{Hint: Find an invariant distribution for $\mathcal{M}'$ using the one of $\mathcal{M}$.}

Given a DTMC $\mathcal{M}$, one defines the *synchronized square* DTMC $\mathcal{M}''$ with transition matrix $P''$ over state space $S^2$ by:
• For all $i \neq j, i', j'$, $\mathbf{P}''[(i, j), (i', j')]' = \mathbf{P}[i, i'][j, j']$

• For all $i, i'$, $\mathbf{P}''[(i, i), (i', i')] = \mathbf{P}[i, i']$

• All other transition probabilities are null.

In words, as long as the states of the pair are different, $\mathcal{M}''$ behaves as $\mathcal{M}'$. When they are identical, $\mathcal{M}''$ behaves as $\mathcal{M}$ maintaining the identity. Define $(I_n, J_n)$ the random state of $\mathcal{M}''$ at time $n$.

**Question 14.** Let $i \in S$ and assume that $\mathcal{M}$ has an invariant distribution $\pi$. Define the initial distribution $\pi''$ of $\mathcal{M}''$ by:

$$\forall i \neq i', j \pi''[i, j] = \pi[j] \text{ and } \pi''[i', j] = 0$$

Prove that $|p_{ij}^{(n)} - \pi[j]| \leq \text{Pr}(I_n \neq J_n)$.

*Hint: consider $\text{Pr}(I_n) = j$ and $\text{Pr}(J_n) = j$.*

**Question 15.** Show that $\text{Pr}(I_n \neq J_n)$ is equal to the probability in $\mathcal{M}'$ with initial distribution $\pi''$ that the chain has not visited during time interval $[0, n]$ the subset of states $\{(s, s) \mid s \in S\}$.

Using questions 3, 13 and 14 deduce that when $\mathcal{M}$ is ergodic, $\lim_{n \to \infty} |p_{ij}^{(n)} - \pi[j]| = 0$ (hence establishing that $\pi$ is the steady-state distribution).