Stochastic Petri Net

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1. Stochastic Petri Net
2. Markov Chain
3. Markovian Stochastic Petri Net
4. Generalized Markovian Stochastic Petri Net (GSPN)
5. Product-form Petri Nets
Outline

1. Stochastic Petri Net

Markov Chain

Markovian Stochastic Petri Net

Generalized Markovian Stochastic Petri Net (GSPN)

Product-form Petri Nets
Stochastic Petri Net versus Time Petri Net

- In TPN, the delays are *non deterministically* chosen.
- In Stochastic Petri Net (SPN), the delays are *randomly* chosen by sampling distributions associated with transitions.

... but these distributions are not sufficient to eliminate non determinism.

Policies for a net

One needs to define:

- The *choice* policy.
  What is the next transition to fire?

- The *service* policy.
  What is the influence of the enabling degree of a transition on the process?

- The *memory* policy.
  What become the samplings of distributions that have not be used?
Choice Policy

In the net, associate a distribution $D_i$ and a weight $w_i$ with every transition $t_i$.

**Preselection w.r.t. a marking $m$ and enabled transitions $T_m$**

- Normalize weights $w_i$ of the enabled transitions: $w'_i \equiv \frac{w_i}{\sum_{t_j \in T_m} w_j}$
- Sample the distribution defined by the $w'_i$'s.
- Let $t_i$ be the selected transition, sample $D_i$ giving the value $d_i$.

**versus**

**Race policy with postselection w.r.t. a marking $m$**

- For every $t_i \in T_m$, sample $D_i$ giving the value $d_i$.
- Let $T'$ be the subset of $T_m$ with the smallest delays.
  Normalize weights $w_i$ of transitions of $T'$: $w'_i \equiv \frac{w_i}{\sum_{t_j \in T'} w_j}$
- Sample the distribution defined by the $w'_i$'s yielding some $t_i$.

Priorities between transitions could added to refine the selection.
Choice Policy: Illustration

\[ t_1(D_1, w_1) \quad t_2(D_2, w_2) \quad t_3(D_3, w_3) \]
\[ w_1 = 1 \quad w_2 = 2 \quad w_3 = 2 \]

Preselection
Sample \( (1/5, 2/5, 2/5) \)
Outcome \( t_1 \)
Sample \( D_1 \)
Outcome \( 4.2 \)

Race Policy
Sample \( (D_1, D_2, D_3) \)
Outcome \( (3.2, 6.5, 3.2) \)
Sample \( (1/3, -, 2/3) \)
Outcome \( t_3 \)
Server Policy

A transition $t$ can be viewed as server for firings:

- A **single server** $t$ allows a single instance of firings in $m$ if $m[t]$.
- An **infinite server** $t$ allows $d$ instances of firings in $m$ where $d = \min\left(\left\lfloor \frac{m(p)}{Pre(p,t)} \right\rfloor \mid p \in \bullet t \right)$ is the enabling degree.
- A **multiple server** $t$ with bound $b$ allows $\min(b, d)$ instances of firings in $m$.

This can be generalised by marking-dependent services.
Resampling Memory

Every sampling not used is forgotten.

This could correspond to a “crash” transition.
Memory Policy (2)

What happens to $d_2$ and $d_3$?

Enabling Memory

- The samplings associated with still enabled transitions are kept and decremented ($d'_3 = d_3 - d_1$).
- The samplings associated with disabled transitions are forgotten (like $d_2$).

Disabling a transition could correspond to abort a service.
What happens to $d_2$ and $d_3$?

Age Memory

- All the samplings are kept and decremented ($d'_3 = d_3 - d_1$ and $d'_2 = d_2 - d_1$).
- The sampling associated with a disabled transition is frozen until the transition become again enabled (like $d'_2$).

Disabling a transition could correspond to suspend a service.
Memory Policy (4)

Specification of memory policy
To be fully expressive, it should be defined w.r.t. any pair of transitions.

What happens to $d_1, d_1'$ and $d_1''$?

Interaction between memory policy and service policy
Assume enabling memory for $t_1$ when firing $t_2$ and infinite server policy for $t_1$.
Which sample should be forgotten?

- The last sample performed,
- The first sample performed,
- The greatest sample, etc.

Warning: This choice may have a critical impact on the complexity of analysis.
Outline

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Product-form Petri Nets
A DTMC is a stochastic process which fulfills:

- For all $n$, $T_n$ is the constant 1
- The process is *memoryless*

\[
Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, ..., S_{n-1} = s_{i_{n-1}}, S_n = s_i) = Pr(S_{n+1} = s_j \mid S_n = s_i) = P[i,j]
\]

A DTMC is defined by $S_0$ and $P$
Analysis: the State Status

The transient analysis is easy and effective in the finite case:
\[ \pi_n = \pi_0 \cdot P^n \] with \( \pi_n \) the distribution of \( S_n \)

The steady-state analysis (\( \exists \lim_{n \to \infty} \pi_n \)) requires theoretical developments.

Classification of states w.r.t. the asymptotic behaviour of the DTMC

- A state is transient if the probability of a return after a visit is less than one. Hence the probability of its occurrence will go to zero. \((p < 1/2)\)

- A state is recurrent null if the probability of a return after a visit is one but the mean time of this return is infinite. Hence the probability of its occurrence will go to zero. \((p = 1/2)\)

- A state is recurrent non null if the probability of a return after a visit is one and the mean time of this return is finite. \((p > 1/2)\)
State Status in Finite DTMC

In a finite DTMC

- The status of a state only depends on the graph associated with the chain.
- A state is transient iff it belongs to a non terminal *strongly connected component* (scc) of the graph.
- A state is recurrent non null iff it belongs to a terminal scc.

![Transition Diagram]

T={1,2,3}
C_1={4,5}
C_2={6,7,8}
Irreducibility and Periodicity

- A chain is *irreducible* if its graph is strongly connected.
- The *periodicity* of an irreducible chain is the greatest integer $p$ such that:
  the set of states can be partitioned in $p$ subsets $S_0, \ldots, S_{p-1}$
  where every transition goes from $S_i$ to $S_{i+1 \% p}$ for some $i$.

**Computation of the periodicity**

```plaintext
4 7
2 3 6
1 5 8

periodicity = gcd(0, 2, 4) = 2
```

```
1
2
3 4
5 6 7 8
```

- height 0
- height 1
- height 2
- height 3
- height 4
A particular case

The chain is irreducible and *aperiodic* (i.e. its periodicity is 1)

- $\pi_\infty \equiv \lim_{n \to \infty} \pi_n$ exists and its value is independent from $\pi_0$.
- $\pi_\infty$ is the unique solution of $X = X \cdot P \land X \cdot 1 = 1$
  where one can omit an arbitrary equation of the first system.

\[
\begin{pmatrix}
0.3 & 0.7 & 0.2 & 0.8
\end{pmatrix}
\]

\[
\pi_\infty = \begin{pmatrix}
1/8 & 7/16 & 7/16
\end{pmatrix}
\]

\[
\pi_1 = 0.3\pi_1 + 0.2\pi_2 \quad \pi_2 = 0.7\pi_1 + 0.8\pi_3 \quad \pi_3 = \pi_2
\]
Analysis of a DTMC: the “General” Case

Almost general case: every terminal scc is aperiodic

- $\pi_\infty$ exists.
- $\pi_\infty = \sum_{s \in S} \pi_0(s) \sum_{i \in I} \text{preach}_i[s] \cdot \pi_i^\infty$ where:
  1. $S$ is the set of states,
  2. $\{C_i\}_{i \in I}$ is the set of terminal scc,
  3. $\pi_i^\infty$ is the steady-state distribution of $C_i$,
  4. and $\text{preach}_i[s]$ is the probability to reach $C_i$ starting from $s$.

Computation of the reachability probability for transient states

- Let $T$ be the set of transient states
  (i.e. not belonging to a terminal scc)
- Let $P_{T,T}$ be the submatrix of $P$ restricted to transient states
- Let $P_{T,i}$ be the submatrix of $P$ transitions from $T$ to $C_i$
- Then $\text{preach}_i = (\sum_{n \in \mathbb{N}} (P_{T,T})^n) \cdot P_{T,i} \cdot 1 = (\text{Id} - P_{T,T})^{-1} \cdot P_{T,i} \cdot 1$
Illustration: SCC and Matrices

\[ P_{T,T} = \begin{pmatrix} 0.0 & 0.7 & 0.0 \\ 0.1 & 0.0 & 0.8 \\ 0.0 & 0.2 & 0.0 \end{pmatrix} \]

\[ P_{T,1} \cdot 1 = \begin{pmatrix} 0.0 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.4 & 0.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.0 \\ 0.4 \end{pmatrix} \]

\[ P_{T,2} \cdot 1 = \begin{pmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.1 & 0.0 \\ 0.3 & 0.1 & 0.0 \end{pmatrix} \begin{pmatrix} 1.0 \\ 1.0 \\ 1.0 \end{pmatrix} = \begin{pmatrix} 0.0 \\ 0.1 \\ 0.4 \end{pmatrix} \]

\[ T=\{1, 2, 3\}, C_1=\{4, 5\}, C_2=\{6, 7, 8\} \]
A CTMC is a stochastic process which fulfills:

- **Memoryless state change**

\[
Pr(S_{n+1} = s_j \mid S_0 = s_{i_0}, ..., S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, ..., T_n < \tau_n, S_n = s_i) = Pr(S_{n+1} = s_j \mid S_n = s_i) \equiv P[i, j]
\]

- **Memoryless transition delay**

\[
Pr(T_n < \tau \mid S_0 = s_{i_0}, ..., S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, ..., T_{n-1} < \tau_{n-1}, S_n = s_i) = Pr(T_n < \tau \mid S_n = s_i) = 1 - e^{-\lambda_i \tau}
\]

### Notations and properties

- \(P\) defines an *embedded* DTMC (the chain of state changes)

- Let \(\pi(\tau)\) the distribution de \(X(\tau)\), for \(\delta\) going to 0 it holds that:
  \[
  \pi(\tau + \delta)(s_i) \approx \pi(\tau)(s_i)(1 - \lambda_i \delta) + \sum_j \pi(\tau)(s_j) \lambda_j \delta P[j, i]
  \]

- Hence, let \(Q\) the *infinitesimal generator* defined by:
  \[
  Q[i, j] \equiv \lambda_i P[i, j] \quad \text{for} \quad j \neq i \quad \text{and} \quad Q[i, i] \equiv -\sum_{j \neq i} Q[i, j]
  \]

Then:

\[
\frac{d\pi}{d\tau} = \pi \cdot Q
\]
The exponential distribution

Let \( F \) be defined by: \( F(\tau) = 1 - e^{-\lambda \tau} \)

Then \( F \) is the exponential distribution with rate \( \lambda > 0 \).

The exponential distribution is memoryless.

Let \( X \) be a random variable with a \( \lambda \)-exponential distribution.

\[
\Pr(X > \tau' \mid X > \tau) = \frac{\Pr(X > \tau')}{\Pr(X > \tau)} = \frac{e^{-\lambda \tau'}}{e^{-\lambda \tau}} = e^{-\lambda (\tau' - \tau)} = \Pr(X > \tau' - \tau)
\]

The minimum of exponential distributions is an exponential distribution.

Let \( Y \) be independent from \( X \) with \( \mu \)-exponential distribution.

\[
\Pr(\min(X, Y) > \tau) = e^{-\lambda \tau} e^{-\mu \tau} = e^{-(\lambda + \mu) \tau}
\]

The minimal variable is selected proportionally to its rate.

\[
\Pr(X < Y) = \int_0^{\infty} \Pr(Y > \tau) F_X \{d\tau\} = \int_0^{\infty} e^{-\mu \tau} \lambda e^{-\lambda \tau} d\tau = \frac{\lambda}{\lambda + \mu}
\]
Convoluting the exponential distribution

The $n^{th}$ convolution of a distribution $F$ is defined by:

$$F^{n*} \overset{\text{def}}{=} F \ast \cdots \ast F \quad (n \text{ times})$$

Let $f_n$ (resp. $F_n$) be the density (resp. distribution) of the $n^{th}$ convolution of the $\lambda$-exponential distribution. Then:

$$f_n(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} \quad \text{and} \quad F_n(x) = 1 - e^{-\lambda x} \sum_{0 \leq m < n} \frac{(\lambda x)^m}{m!}$$

Sketch of proof

Recall that: $f_1(x) = \lambda e^{-\lambda x}$.

$$f_{n+1}(x) = \int_0^x f_n(x - u) f_1(u) du = \int_0^x \lambda e^{-\lambda (x-u)} \frac{(\lambda (x-u))^{n-1}}{(n-1)!} \lambda e^{-\lambda u} du$$

$$= \lambda e^{-\lambda x} \int_0^x \lambda \frac{(\lambda (x-u))^{n-1}}{(n-1)!} du = \lambda e^{-\lambda x} \left( \frac{\lambda x)^n}{n!} \right)$$

Deduce $F_{n+1}$ by:

$$\frac{d}{dx} \left( 1 - e^{-\lambda x} \sum_{0 \leq m \leq n} \frac{(\lambda x)^m}{m!} \right) =$$

$$e^{-\lambda x} \left( \lambda \sum_{0 \leq m \leq n} \frac{(\lambda x)^m}{m!} - \sum_{0 \leq m \leq n-1} \lambda \frac{(\lambda x)^m}{m!} \right) = f_{n+1}(x)$$
A CTMC

λ

P

λ′

P′

A uniform version of the CTMC (equivalent w.r.t. the states)
Analysis of a CTMC

Transient Analysis

- Construction of a uniform version of the CTMC \((\lambda, P)\) such that \(P[i, i] > 0\) for all \(i\).
- Computation by case decomposition w.r.t. the number of transitions:

\[
\pi(\tau) = \pi(0) \sum_{n \in \mathbb{N}} \left( e^{-\lambda \tau} \frac{\tau^n}{n!} \right) P^n
\]

Steady-state analysis

- The steady-state distribution of visits is given by the steady-state distribution of \((\lambda, P)\) (by construction, the terminal scc are aperiodic) ...
- Equal to the steady-state distribution since the sojourn times follow the same distribution.
- A particular case: \(P\) irreducible

the steady-state distribution \(\pi\) is the unique solution of \(X \cdot Q = 0 \land X \cdot 1 = 1\) where one can omit an arbitrary equation of the first system.
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Product-form Petri Nets
Hypotheses

- The distribution of every transition $t_i$ has a density function $e^{-\lambda_i \tau}$ where the parameter $\lambda_i$ is called the rate of the transition.
- For simplicity reasons, the server policy is single server.

First observations

- The weights for choice policy are no more required since equality of two samples has a null probability. (due to continuity of distributions)
- The residual delay $d_j - d_i$ of transition $t_j$ knowing that $t_i$ has fired (i.e. $d_i$ is the shortest delay) has the same distribution as the initial delay. Thus the memory policy is irrelevant.
Markovian Net and Markov Chain

Key observation: given a marking $m$ with $T_m = t_1, \ldots, t_k$

- The sojourn time in $m$ is an exponential distribution with rate $\sum_i \lambda_i$.
- The probability that $t_i$ is the next transition to fire is $\frac{\lambda_i}{(\sum_j \lambda_j)}$.
- Thus the stochastic process is a CTMC whose states are markings and whose transitions are the transitions of the reachability graph.
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Product-form Petri Nets
Modelling delays with exponential distributions is **reasonable** when:

- Only mean value information is known about distributions.
- Exponential distributions (or combination of them) are enough to approximate the “real” distributions.

Modelling delays with exponential distributions is **not reasonable** when:

- The distribution of an event is known and is poorly approximable with exponential distributions:
  
  *a time-out of 10 time units*

- The delays of the events have different magnitude orders:
  
  *executing an instruction versus performing a database request*

In the last case, the 0-Dirac distribution is required.
Generalized Markovian Stochastic Petri Net (GSPN)

Generalized Markovian Stochastic Petri Nets (GSPN) are nets whose:

- *timed transitions* have exponential distributions,
- and *immediate transitions* have 0-Dirac distributions.

Their analysis is based on Markovian Renewal Process, a generalization of Markov chains.
A Markovian Renewal Process (MRP) fulfills:

- a *relative* memoryless property

\[
Pr(S_{n+1} = s_j, T_n < \tau \mid S_0 = s_{i_0}, ..., S_{n-1} = s_{i_{n-1}}, T_0 < \tau_0, ..., S_n = s_i) = Pr(S_{n+1} = s_j, T_n < \tau \mid S_n = s_i) \equiv Q[i, j, \tau]
\]

- The embedded chain is defined by: \( P[i, j] = \lim_{\tau \to \infty} Q[i, j, \tau] \)

- The sojourn time Soj has a distribution defined by:

\[
Pr(\text{Soj}[i] < \tau) = \sum_j Q[i, j, \tau]
\]

### Analysis of a MRP

- The steady-state distribution (if there exists) \( \pi \) is deduced from the steady-state distribution of the embedded chain \( \pi' \) by:

\[
\pi(s_i) = \frac{\pi'(s_i) E(\text{Soj}[i])}{\sum_j \pi'(s_j) E(\text{Soj}[j])}
\]

- Transient analysis is much harder ... but the reachability probabilities only depend on the embedded chain.
A GSPN is a Markovian Renewal Process

Observations

- Weights are required for immediate transitions.
- The *restricted* reachability graph corresponds to the embedded DTMC.

![Diagram of a GSPN with tangible and vanishing markings]
Steady-State Analysis of a GSPN (1)

Standard method for MRP

- Build the restricted reachability graph equivalent to the embedded DTMC
- Deduce the probability matrix $P$
- Compute $\pi^*$ the steady-state distribution of the visits of markings: $\pi^* = \pi^* P$
- Compute $\pi$ the steady-state distribution of the sojourn in tangible markings:

$$
\pi(m) = \frac{\pi^*(m) \text{Soj}(m)}{\sum_{m' \text{ tangible}} \pi^*(m') \text{Soj}(m')}
$$

How to eliminate the vanishing markings sooner in the computation?
Steady-State Analysis of a GSPN (2)

An alternative method

- As before, compute the transition probability matrix $P$.
- Compute the transition probability matrix $P'$ between tangible markings.
- Compute $\pi'^*$ the (relative) steady-state distribution of the visits of tangible markings: $\pi'^* = \pi'^* P'$.
- Compute $\pi$ the steady-state distribution of the sojourn in tangible markings:

\[
\pi(m) = \frac{\pi'^*(m)\text{Soj}(m)}{\sum_{m' \text{ tangible}} \pi'^*(m')\text{Soj}(m')}
\]

Computation of $P'$

- Let $P_{X,Y}$ the probability transition matrix from subset $X$ to subset $Y$.
- Let $V$ (resp. $T$) be the set of vanishing (resp. tangible) markings.
- $P' = P_{T,T} + P_{T,V} (\sum_{n \in \mathbb{N}} P^n_{V,V}) P_{V,T} = P_{T,T} + P_{T,V} (\text{Id} - P_{V,V})^{-1} P_{V,T}$
- Iterative (resp. direct) computations uses the first (resp. second) expression.
Steady-State Analysis: Illustration

\[ p_2 = \frac{w_2}{(w_2 + w_3)} \quad p_3 = \frac{w_3}{(w_2 + w_3)} \]
\[ p_4 = \frac{\lambda_4}{(\lambda_4 + \lambda_5)} \quad p_5 = \frac{\lambda_5}{(\lambda_4 + \lambda_5)} \]

“c” and “d” are normalizing constants
Outline

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Product-form Petri Nets
A (Markovian) queue is a CTMC

- Interarrival time: exponential distribution with parameter $\lambda$
- Service time: exponential distribution with parameter $\mu$

Let $\rho = \frac{\lambda}{\mu}$ be the utilization

- The steady-state distribution $\pi_\infty$ exists iff $\rho < 1$
- The probability of $n$ clients in the queue is $\pi_\infty(n) = \rho^n(1 - \rho)$
Observation. The associated Markov chain is more complex than the one corresponding to two isolated queues. However ...

Assume $\rho_1 = \frac{\lambda}{\mu} < 1$ and $\rho_2 = \frac{\lambda}{\delta} < 1$

- The steady-state distribution $\pi_\infty$ exists.
- The probability of $n_1$ clients in queue 1 and $n_2$ clients in queue 2 is $\pi_\infty(n_1, n_2) = \rho_1^{n_1}(1 - \rho_1)\rho_2^{n_2}(1 - \rho_2)$
- It is the product of the steady-state distributions corresponding to two isolated queues.
In a steady-state

- Define the (input and output) flow through queue 1 (resp. 2) as $\gamma_1$ (resp. $\gamma_2$).
- Then $\gamma_1 = \lambda + q\gamma_2$ and $\gamma_2 = p\gamma_1$. Thus $\gamma_1 = \frac{\lambda}{1-pq}$ and $\gamma_2 = \frac{p\lambda}{1-pq}$

Assume $\rho_1 = \frac{\gamma_1}{\mu} < 1$ and $\rho_2 = \frac{\gamma_2}{\delta} < 1$

- The steady-state distribution $\pi_\infty$ exists.
- The probability of $n_1$ clients in queue 1 and $n_2$ clients in queue 2 is $\pi_\infty(n_1, n_2) = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)$
- It is the product of the steady-state distributions corresponding to two isolated queues.
Analysis of a Closed Queuing Network

Visit ratios (up to a constant)

- Define the visit ratio flow of queue $i$ as $v_i$.
- Then $v_1 = v_3 + qv_2$, $v_2 = pv_1$ and $v_3 = (1 - p)v_1 + (1 - q)v_2$.
  Thus $v_1 = 1$, $v_2 = p$ and $v_3 = 1 - pq$.

Define $\rho_1 = \frac{v_1}{\mu}$, $\rho_2 = \frac{v_2}{\delta}$ and $\rho_3 = \frac{v_3}{\lambda}$

- The steady-state probability of $n_i$ clients in queue $i$ is $\pi_\infty(n_1, n_2, n_3) = \frac{1}{G} \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3}$ (with $n_1 + n_2 + n_3 = n$)
- where $G$ the normalizing constant can be efficiently computed by dynamic programming.
Observations

- A (single client class) queuing network can easily be represented by a Petri net.
- Such a Petri net is a *state machine*: every transition has at most a single input and a single output place.

Can we define a more general subclass of Petri nets with a product form for the steady-state distribution?
Product Form Stochastic Petri Nets (PFSPN)

Principles

- Transitions can be partitioned into subsets corresponding to several classes of clients with their specific activities.
- Places model resources shared between the clients.
- Client states are implicitly represented.
The resource graph

- The vertices are the input and the output bags of the transitions.
- Every transition of the net \( t \) yields a graph transition \( \bullet t \xrightarrow{t} t^\bullet \)
- Client classes correspond to the connected components of the graph.

First requirement: The connected components of the graph must be strongly connected.
Witnesses in PFSPN

Vector \(-p_2-p_3\) is a witness for bag \(p_1+p_4\):

\[ (-p_2-p_3) \cdot W(t_3) = 1 \]
\[ (-p_2-p_3) \cdot W(t_1) = -1 \]
\[ (-p_2-p_3) \cdot W(t) = 0 \text{ for every other } t \]

where \(W\) is the incidence matrix.

**Witness for a bag \(b\)**

- Let \(In(b)\) (resp. \(Out(b)\)) the transitions with input (resp. output) \(b\).
- Let \(v\) be a place vector, \(v\) is a **witness** for \(b\) if:
  - \(\forall t \in In(b) \ v \cdot W(t) = -1\) (where \(W(t)\) is the incidence of \(t\))
  - \(\forall t \in Out(b) \ v \cdot W(t) = 1\)
  - \(\forall t \not\in In(b) \cup Out(b) \ v \cdot W(t) = 0\)

**Second requirement:** Every bag must have a witness.
Steady-State Distributions of PFSPN

The reachability space:
\[ m(p_1) + m(p_2) + m(p_3) = 2 \]
\[ m(p_4) + m(p_5) + m(p_6) = m(p_1) + 1 \]

Steady-state distribution
- Assume the requirements are fulfilled, with \( w(b) \) the witness for bag \( b \).
- Compute the ratio visit of bags \( v(b) \) on the resource graph.
- The output rate of a bag \( b \) is \( \mu(b) = \sum_{t|\cdot t=b} \mu(t) \) with \( \mu(t) \) the rate of \( t \).
- Then: \[ \pi_\infty(m) = \frac{1}{G} \prod_b \left( \frac{v(b)}{\mu(b)} \right)^{w(b) \cdot m} \]

Observation. The normalizing constant can be efficiently computed if the reachability space is characterized by linear place invariants.
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