Questions marked with a ⋆ are not a priority: search them only if you have time. For example, Section 3 helps you prepare for the next course.

1 Sequent calculus

We consider the sequent calculus as seen in the course slides, that is the cut-free sequent calculus. Its rules and the axiom rule and all right and left rules (including right and left contractions). We do not (yet) consider the cut rule presented in the course notes.

Exercise 1: Sequent calculus proofs
Give proofs in sequent calculus of the following formulas:

1. $A \lor (A \Rightarrow B)$
2. $A \lor (B \land C) \iff (A \lor B) \land (A \lor C)$
3. $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$
4. $\neg\neg A \Rightarrow A$
5. $(A \lor B) \iff \neg A \land \neg B$
6. $\forall x. R(x) \Rightarrow \exists x. \neg R(x)$
7. $\forall x. (Q \lor R(x)) \Rightarrow (Q \lor \forall x. R(x))$
8. $\exists x. [(R(a) \lor R(b)) \Rightarrow R(x)]$
9. Show that if $a \neq b$, there is no proof of 8. that does not use contractions.

Exercise 2: Interpolation theorem
If $\phi$ is a formula, we call $L(\phi)$ the set of free variables and function and predicate symbols appearing in $\phi$. By extension, if $\Gamma$ is a multiset of formulas, we write $L(\Gamma) = \bigcup_{\phi \in \Gamma} L(\phi)$. We want to show that if $L(\Gamma_1 \cup \Gamma_2 \cup \Delta_1 \cup \Delta_2)$ does not contain a function symbol and if $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$ is provable, then there exists a formula $\xi$ such that:

- $\Gamma_1 \vdash \xi, \Delta_1$ and $\Gamma_2, \xi \vdash \Delta_2$ are provable;
- $L(\xi) \subseteq L(\Gamma_1 \cup \Delta_1) \cap L(\Gamma_2 \cup \Delta_2)$
1. Prove this result. You will consider the following cases in detail: \( \Rightarrow \) right; \( \Rightarrow \) left; \( \forall \) left; \( \forall \) left.

2. Prove the interpolation theorem: if \( \phi \) and \( \psi \) are formulas that do not contain function symbols and if \( \vdash \phi \Rightarrow \psi \) is provable, then there exists a formula \( \xi \) such that:
   - \( \vdash \phi \Rightarrow \xi \) and \( \vdash \xi \Rightarrow \psi \) are provable;
   - \( L(\xi) \subseteq L(\phi) \cap L(\psi) \).

3. (⋆) We apply the interpolation theorem to prove the Beth theorem.
   Let \( P \) and \( P' \) be two unary predicates. Let \( \Gamma(\ P) \) be a set of closed formulas that do not contain the symbol \( P' \). We write \( \Gamma(\ P') \) the set of formulas generated by replacing the symbol \( P \) by the symbol \( P' \) in \( \Gamma(\ P) \).
   We say that \( \Gamma(\ P) \) implicitly defines \( P \) if \( \Gamma(\ P), \Gamma(\ P') \vdash \forall x. (P(x) \leftrightarrow P'(x)) \) is provable; we say that \( \Gamma(\ P) \) explicitly defines \( P \) if there exists a formula \( \phi(x) \) using neither \( P \) nor \( P' \) such that \( \Gamma(\ P) \vdash \forall x. (A(x) \leftrightarrow P'(x)) \).
   Prove that \( \Gamma(\ P) \) implicitly defines \( P \) iff \( \Gamma(\ P) \) explicitly defines \( P \).

4. (⋆⋆) In fact, the interpolation theorem holds even when \( \phi \) and \( \psi \) contain function symbols. Can you see how to use the previous result to treat this case?

## 2 Unification

A unification problem is a set \( E \) of equations of the form \( t \overset{?}{=} u \).
A unifier (i.e. solution to the unification problem) of a set \( E \) is a substitution \( \sigma \) such that for every equation \( t \overset{?}{=} u \) in \( E \), \( t\sigma = u\sigma \).

The unification procedure seen in class are reminded at the end of the exercise sheet (Algorithm 1).

**Exercise 3: Some examples**

Apply the procedure to the following unification problems (your answer should either be fail or the substitution returned by the procedure):

- \( E_1 = f(x,g(a,y)) \overset{?}{=} f(h(y),g(y,a)); g(x,h(y)) \overset{?}{=} g(z,z) \)
- \( E_2 = f(x,x) \overset{?}{=} f(g(y),z); h(z) \overset{?}{=} h(y) \)
- \( E_3 = f(x,a) \overset{?}{=} f(b,y); f(x) \overset{?}{=} f(y) \)

**Exercise 4: Studying the unification algorithm**

We study the properties of the unification algorithm \texttt{Unif}.

1. Is the procedure deterministic?
2. Show that it always terminates (in failure or by returning a substitution).
3. We now want to prove that it indeed calculates a unifier of the set of equations \( E \) given as entry. We will in fact prove a more general result.

(a) Show that if \( E \) contains \( f(u_1, \ldots, u_n) \overset{?}= g(t_1, \ldots, t_m) \), where \( f \neq g \), then it is not unifiable.

(b) Show that if \( E \) contains \( x \overset{?}= t \), where \( x \in \mathcal{X} \), \( v \notin \mathcal{X} \) and \( x \in \text{Var}(t) \), then it is not unifiable.

**A most general unifier (mgu) \( \sigma \) of a unification problem \( E \) is a unifier of \( E \) such that for every unifier \( \tau \) of \( E \), there exists \( \eta \) such that \( \tau = \eta \circ \sigma \).**

(c) Show that the unification problem \( x \overset{?}= f(y) \) has an infinity of unifiers (you can use a constant, i.e. 0-ary, function symbol \( a \)). Is there a most general one? When it exists, is there unicity of the mgu?

(d) Show that if \( E = E' \cup \{ x \overset{?}= x \} \), then \( \sigma \) unifies \( E \) iff it unifies \( E' \).

Observe that this implies that \( \sigma \) is a mgu of \( E \) iff it is a mgu of \( E' \).

(e) Show that if \( E = E' \cup \{ f(u_1, \ldots, u_n) \overset{?}= (t_1, \ldots, t_n) \} \), then \( \sigma \) unifies \( E \) iff it unifies \( E' \cup \{ u_1 \overset{?}= t_1, \ldots, u_n \overset{?}= t_n \} \).

Observe that this implies that \( \sigma \) is a mgu of \( E \) iff it is a mgu of \( E' \).

(f) Show that for every substitution \( \sigma \), variable \( x \) and term \( t \), if \( x\sigma = t\sigma \) then \( \sigma = \sigma \circ \{ x \mapsto t \} \).

(g) Show that if \( E = E' \cup \{ x \overset{?}= t \} \) where \( x \notin \text{var}(t) \), then \( \sigma \) unifies \( E' \{ x \mapsto t \} \) iff \( \sigma \circ \{ x \mapsto t \} \) unifies \( E \).

Prove using the previous question that: \( \sigma \) is a mgu of \( E' \{ x \mapsto t \} \) implies \( \sigma \circ \{ x \mapsto t \} \) is a mgu of \( E \).

(h) Show that if on input \( E \) the algorithm

- returns a substitution \( \sigma \), then \( \sigma \) is a mgu for \( E \);
- fails, then the unification problem \( E \) has no solution.
3 Additional exercises (⋆)

Exercise 5: A new rule
In this exercise we introduce the cut rule to the sequent calculus:

\[
\Gamma \vdash \psi \quad \Gamma, \psi \vdash \phi \\
\frac{}{\Gamma \vdash \phi} \text{ cut}
\]

Give two proofs in sequent of \(A \Rightarrow B, A \Rightarrow C, B \land C \Rightarrow D \vdash A \Rightarrow D\): a first with cuts, a second without cuts.

Exercise 6: Return to natural deduction
This exercise in an introduction to the proof a equivalence of natural deduction and sequent calculus.

1. Show that a proof in natural deduction of a sequent \(\Gamma, \phi \land \psi \vdash \xi\) can be transformed in a proof in natural deduction of the sequent \(\Gamma, \phi, \psi \vdash \xi\).
2. You can think of how to do the same transformation for other left rules.

The unification algorithm

The unification procedure is the following:

\begin{algorithm}
\textbf{Algorithm 1: Unif}\\
\textbf{Input} : a unification problem \(E\)\\
\textbf{if} \(E = E' \cup \{f(u_1, \ldots, u_n) = f(t_1, \ldots, t_n)\}\) then\\
\hspace{1cm} Unif\((E' \cup \{u_1 = t_1, \ldots, u_n = t_n\})\)\\
\textbf{else if} \(E = E' \cup \{f(u_1, \ldots, u_n) = g(t_1, \ldots, t_m)\}\) where \(f \neq g\) then\\
\hspace{1cm} \textbf{fail}\\
\textbf{else if} \(E = E' \cup \{x = x\}\) then\\
\hspace{1cm} Unif\((E')\)\\
\textbf{else if} \(E = E' \cup \{x = t\}\) where \(x \in \mathcal{X}\) and \(x \notin \text{Var}(t)\) then\\
\hspace{1cm} Unif\((E'\{x \mapsto t\}) \circ \{x \mapsto t\}\)\\
\textbf{else if} \(E = E' \cup \{x = t\}\) where \(x \in \mathcal{X}, t \notin \mathcal{X}\) and \(x \in \text{Var}(t)\) then\\
\hspace{1cm} \textbf{fail}\\
\textbf{else if} \(E = \emptyset\) then\\
\hspace{1cm} id
\end{algorithm}