1 Kripke structures and intuitionistic logic

Exercise 1: Kripke structures

In the following, \( X, Y, Z \) are constant predicates.

1. Is the structure:
   \[
   \begin{array}{c}
   P, R \\
   \bullet \Downarrow \bullet \\
   \bullet \Downarrow \bullet \\
   \bullet P \\
   \end{array}
   \]
   a model of \((\neg Q \land P) \Rightarrow R\)?

2. Let \( K = (W, \leq, \alpha) \) be a Kripke structure, and \( w \in W \) a world.
   (a) What does \( K, w \models \neg \neg X \) mean?
   (b) What does \( K, w \models \neg (\neg X \land \neg Y) \) mean?

3. Give Kripke counter-models or proofs for the following propositions:
   (a) \( X \Rightarrow \neg \neg X \)
   (b) \( \neg \neg X \Rightarrow X \)
   (c) \( \neg X \lor \neg \neg X \)
   (d) \( \neg (X \land Y) \Rightarrow (\neg X \lor \neg Y) \)
   (e) \( \neg X \lor \neg Y \Rightarrow \neg (Y \land X) \)
   (f) \( X \Rightarrow Y \Rightarrow (\neg X \lor Y) \)
   (g) \( \neg \neg (A \land B) \iff (\neg \neg A \land \neg \neg B) \)
   (h) \( \neg \neg (A \Rightarrow B) \iff (\neg \neg A \Rightarrow \neg \neg B) \)

Exercise 2: First-order intuitionnistic logic

We extend the Kripke semantics to first-order logic. Let \( \mathcal{K} = (W, \leq, \alpha) \) be a Kripke structure, \( w \in W \) a world. As in classical logic, we need to introduce valuations \( \sigma : X \rightarrow D_w \) to interpret free variables. It does not modify the rules of satisfiability for other connectives. Be careful, now we have \( n \)-ary predicates for \( n > 0 \) and terms again.

We remind/update some definitions: for every \( f \in F, w' \geq w \), we have \( D_w \subseteq D_{w'} \) and \( f_{w'}|_{D_w} = f_w \); for every \( P \in P, (t_1, \ldots, t_n) \in T(X, F)^n, w' \geq w \), we have \( \alpha(w, P) \subseteq \alpha(w', P) \) and \( (\mathcal{K}, w, \sigma \models P(t_1, \ldots, t_n) \iff ([t_1]_{w, \sigma}, \ldots, [t_n]_{w, \sigma}) \in \alpha(w, P)) \).

Try to think of possible definitions for \( \mathcal{K}, w, \sigma \models \exists x. \phi \) and \( \mathcal{K}, w, \sigma \models \forall x. \phi \) (answer on the next page).
— $\mathcal{K}, w, \sigma \models \exists x. \phi$ iff there exists $a \in D_w$ such that $\mathcal{K}, w, \sigma \{x \mapsto a\} \models \phi$.
— $\mathcal{K}, w, \sigma \models \forall x. \phi$ iff for every $w' \geq w$, and $a \in D_{w'}$, $\mathcal{K}, w', \sigma \{x \mapsto a\} \models \phi$.

1. Is the Kripke structure

\[
\begin{array}{c}
\bullet \mathbb{R} \\
\bullet \mathbb{N}
\end{array}
\]

a model of $\exists x. \exists y. x < y \land \neg(\exists z. x < z \land z < y)$?

2. State the new correction theorem.

3. Prove that this semantic is correct with respect to intuitionistic natural deduction (consider only the cases of quantifiers).

We go back to propositional intuitionistic logic.

Exercise 3: Independent connectives

We say that a binary connective $\otimes$ is independent from a set of connectives $C$ is there is no formula $A$ using $X$ and $Y$ and built only with connectives from $C$ such that $\vdash_i (X \otimes Y) \leftrightarrow A$.

1. Show that if $\lor$ is not independent from $\{\bot, \land, \neg, \Rightarrow\}$, then $\vdash_i \neg\neg(X \lor Y) \leftrightarrow (\neg\neg X \lor \neg\neg Y)$. Hint: use Exercise 1, questions (g) and (h). Conclude.

We consider the following Kripke structure $\mathcal{K} = (W, \alpha) : W = \{\omega_1, \omega_2, \omega_3\}$ with $\omega_1 \leq \omega_3$, $\omega_2 \leq \omega_3$ and $\alpha(\omega_1) = \{X\}$, $\alpha(\omega_2) = \{Y\}$, $\alpha(\omega_3) = \{X, Y\}$.

2. Show that for all proposition $A$ containing only $X, Y, \bot, \neg, \lor$ and $\Rightarrow$, if $\omega_3 \models A$ then $\omega_1 \models A$ or $\omega_2 \models A$. Conclude that $\land$ is independent from $\{\bot, \lor, \neg, \Rightarrow\}$.

Exercise 4: Excluded middle

Let $\mathcal{K}$ be the Kripke structure of worlds the partial interpretations, i.e. the ordered pairs $(I, f)$ where $I \subseteq P_0$, $f : I \rightarrow \{0, 1\}$, ordered by $(I, f) \sqsubseteq (J, g)$ iff $I \subseteq J$ and for every $X \in I$, $f(X) = g(X)$, and $\alpha(I, f) = \{X \in I \mid f(X) = 1\}$.

1. What does $\mathcal{K}, w \models \neg X$ mean, where $X \in I$?
2. Show that $\mathcal{K}$ is a counter-model to $X \lor \neg X$.
3. Give formula $A$ which is not provable intuitionistically, but satisfied by $\mathcal{K}$.

2 Additional exercises on intuitionistic logic

Exercise 5:

Let $A$ be a propositional formula, provable in classical logic (a tautology). We define $F_2$ the ordered set $\omega_1 \leq \omega_2$. We call structure of base $F_2$ every Kripke structure of underlying ordered set $F_2$. We call $LI+A$ the set of formulas provable when adding to natural deduction the following rule:

\[
\vdash A[X_1 \rightarrow B_1, \ldots, X_n \rightarrow B_n]
\]
For example, $\text{LI} + (X \lor \neg X)$ is LC, the set of formulas provable in classical logic. The goal of this exercise is to prove YANKOV’s theorem: for every tautology $A$, $\text{LI} + A = \text{LC}$ iff $A$ is not satisfied in a structure of base $F_2$.

1. Show that if $\text{LI} + A = \text{LC}$ then $A$ is not satisfied in a structure of base $F_2$. 
   Hint: you can use the counter-model to $\neg\neg X \Rightarrow X$ found in Exercise 1.
2. Let $K = (F_2, \alpha)$ be the Kripke structure such that $\alpha(\omega_1) = \emptyset$ and $\alpha(\omega_2) = \{X\}$. Assume that $A$ has a single propositional variable $X$. Show that if $A$ is not satisfied in $K$ then every structure $K' = (W', \alpha')$ satisfying $A$ is such that for every world $\omega \in W'$, $X \notin \alpha'(\omega)$ implies that for every world $\omega' \geq \omega$, $X \notin \alpha(\omega')$.
3. Conclude: if $A$ is a formula with a single propositional variable $X$ and $K$ does not satisfy $A$, then $A \vdash_i \neg\neg X \Rightarrow X$.
4. Let $A$ a proposition of propositional variables $X_1, \ldots, X_n$. Show that if $A$ is not satisfied by a structure of base $F_2$, then there are formulas $B_1, \ldots, B_n$ with one propositional variable $X$ and such that $K$ does not satisfy $A[X_1 \rightarrow B_1, \ldots, X_n \rightarrow B_n]$.
5. Conclude.
6. Show that if $A_1, \ldots, A_n$ are tautologies and $\text{LI} + A_1 \land \ldots \land A_n = \text{LC}$, then there is $i \in \{1, \ldots, n\}$ such that $\text{LI} + A_i = \text{LC}$.

**Exercise 6:**
We call Heyting arithmetic the intuitionistic theory of axioms those of Peano arithmetic, i.e. its theorems are all formulas provable in intuitionistic logic from the Peano axioms. We write $HA \vdash_i A$ when $A$ is a theorem of Heyting arithmetic.

1. Show that equality is decidable in Heyting arithmetic:
   $$HA \vdash_i \forall x.\forall y. (x = y \lor x \neq y)$$
   Hint: use induction and $\forall x.(x = 0 \lor \exists y. x = S(y))$
2. The goal is to show that Heyting arithmetic has the witness property, i.e. if $HA \vdash_i \exists x. A$, then there is $n \in \mathbb{N}$ such that $HA \vdash_i A\{x \rightarrow \overline{n}\}$, where $\overline{n}$ is the term $S^n(0) = S(\ldots(S(0)))$. By contradiction, let us assume that for every natural number $n$, $HA \not\vdash_i A\{x \rightarrow \overline{n}\}$. Then, for every $n$, there is a Kripke structure $K_n$ such that $K_n$ satisfies all Peano axioms but not $A\{x \rightarrow \overline{n}\}$. We built the structure $K = \{\omega\} \sqcup \bigcup_{n \in \mathbb{N}} K_n$ with smallest element $\omega$, in which $D_\omega = \mathbb{N}$, $S$ is interpreted as the successor, $0$ as $0$, $+$ as the addition, $\times$ as the multiplication and $=\land$ as the equality.
   (a) Show that $K$ is a Kripke structure.
   (b) Show that $K, \omega \not\models \exists x. A$.
   (c) Show that $K$ satisfies all Peano axioms (you can restrict to the induction scheme).
3. We want to show that if $HA \vdash_i A \lor B$ then $HA \vdash_i A$ or $HA \vdash_i B$.
   (a) Show that for all formulas $A$ and $B$ not containing the variable $x$,
   $$HA \vdash_i (A \lor B) \leftrightarrow \exists x. (x = 0 \Rightarrow A) \land (x \neq 0 \Rightarrow B)$$
   (b) Conclude.
3 Preparing for the exam

Exercise 7: Provability

1. \(A, B\) are constant predicates. Are the following sequents provable intuitio-
nistically and/or classically?

(a) \(B \vdash A \Rightarrow B\)
(b) \((A \lor B) \Rightarrow B \vdash B \Rightarrow A\)
(c) \(-A \lor B \vdash A \lor B\)
(d) \(A \Rightarrow B \vdash B \Rightarrow A\)

2. Are there two formulas \(\phi\) and \(\psi\) such that \(\vdash \phi\) provable iff \(\vdash \psi\) provable but \(\vdash \phi \leftrightarrow \psi\) not provable?

Exercise 8: Inductions

Let \(\mathcal{M}\) be a structure over a signature \((F, P)\).

1. Prove the substitution lemma: for all term \(t \in T(X, F)\), all variable \(x \in X\), valuation \(\sigma\), and formula \(\phi\):

\[\mathcal{M}, \sigma \models \phi\{x \rightarrow t\} \iff \mathcal{M}, \sigma\{x \mapsto [t]_{\mathcal{M}, \sigma}\} \models \phi\]

2. We fix \(F = \emptyset\) and \(P = \{\leq (2)\}\), and work with the theory of orders \(\text{TO}\); we assume that \(\mathcal{M}\) is a \(\text{TO-interpretation}\). Let \(\mathcal{M}'\) be a \(\text{TO-interpretation}\) such that there exists a function \(h : \mathcal{D}_\mathcal{M} \rightarrow \mathcal{D}_{\mathcal{M}'}\) preserving the order, i.e. for all \(a, b \in \mathcal{D}_\mathcal{M}\), \(a \leq_\mathcal{M} b\) iff \(h(a) \leq_{\mathcal{M}'} h(b)\). Show that for every quantifier-free formula \(\phi\) and valuation \(\sigma\):

\[\mathcal{M}, \sigma \models \phi\] iff \(\mathcal{M}', h \circ \sigma \models \phi\)

What about formulas with quantifiers?

Exercise 9: True or false

Justify if the following statements are true or false:

1. Every theory is incoherent or incomplete.
2. Every complete theory is decidable.
3. Every incoherent theory is decidable.
4. Peano arithmetic has no finite model.
5. There is a formula of Peano arithmetic that is neither satisfied nor not satisfied by the standard model \(N\).
6. Every subset of a complete theory is recursive.
7. The intersection of two decidable theories is decidable.