Exercise 1: Using completeness

1. Prove without induction that if $\Gamma \vdash \phi$ is provable in natural deduction, then $\Gamma, \psi \vdash \phi$ is also provable.
2. Prove without using natural deduction rules that there is a natural deduction proof of $\Gamma \vdash \phi \lor \neg \phi$ for every formula $\phi$.

Exercise 2: The compactness theorem

Prove the compactness theorem:

If all the finite subtheories $T'$ of a theory $T$ have a model, then $T$ has a model.

Exercise 3: Unary predicates

Let $\mathcal{F} = \emptyset$ and $\mathcal{P} = \{P_1(1), \ldots, P_n(1)\}$.

1. Prove that for all structure $\mathcal{M}$, there is a structure $\mathcal{N}$ of domain of size smaller than $2^n$ and a surjective function $f : D_M \to D_N$ such that for every formula $\phi$ of free variables $x_1, \ldots, x_m$ and parameters $a_1, \ldots, a_m \in D_M^m$:

   $\mathcal{M}, \{x_1 \to a_1, \ldots, x_m \to a_m\} \models \phi$ iff $\mathcal{N}, \{x_1 \to f(a_1), \ldots, x_m \to f(a_m)\} \models \phi$

2. Conclude that if $\phi$ is a closed formula, $\vdash \phi$ is provable iff $\phi$ is satisfied in every structure of domain smaller than $2^n$. What is the consequence of this result?
Applications of compactness

Exercise 4: A theorem of LOWENHEIM-SKOLEM
We consider a signature such that \( \mathcal{P} = (2) \in \mathcal{P} \). Let \( n \in \mathbb{N}^* \).

1. Give a formula \( \phi \) such that an egalitarian structure is a model of \( \phi \) iff it has a domain of size smaller than \( n \).

2. Give a formula \( \phi \) such that an egalitarian structure is a model of \( \phi \) iff it has a domain of size greater than \( n \).

3. Using compactness, prove that if a theory has finite models of arbitrarily large size, then it has an infinite model (weak version of upwards LOWENHEIM-SKOLEM).

   Observe that there is no formula \( \phi \) such an egalitarian structure is a model of \( \phi \) iff it has a finite domain.

4. Using compactness, prove that if a theory \( T \) has a model of infinite countable size, then it has an infinite uncountable model.

5. Conclude that there is an uncountable model of PEANO’s arithmetic.

Exercise 5: A sequence of theories
Let \( (T_n)_{n \in \mathbb{N}} \) a sequence of theories such that for every \( n \), \( T_n \subseteq T_{n+1} \) and there is a structure \( M_n \) such that \( M_n \models T_n \) but \( M_n \not\models T_{n+1} \). Let \( T = \bigcup_{n \in \mathbb{N}} T_n \).

1. Show that \( T \) is consistent.

2. Prove that there is no finite subtheory \( T' \) of \( T \) such that for all structure \( M, M \models T \) iff \( M \models T' \).

Exercise 6: Other applications

1. We call an axiomatization of a theory \( T \) a set of formulas \( \Psi \) such that for every formula \( A \), \( T \vdash A \) is provable iff \( \Psi \vdash A \) is provable. Show that a theory admits a finite axiomatization iff from any of its axiomatizations we can extract a finite axiomatization of \( T \).

2. A graph \( G \) is an ordered pair \( \langle V, E \rangle \), with \( V \) a set of vertices and \( E \subseteq V \times V \) a set of edges. A k-coloring is a function \( f : V \rightarrow \{1, \ldots, k\} \) such that \( (q_1, q_2) \in E \) implies \( f(q_1) \neq f(q_2) \). Show that an infinite graph \( G \) is \( k \)-colorable iff all its finite subgraphs are \( k \)-colorable.

3. We consider the theory of fields \( T \).
   (a) Show that there is no formula \( \phi \) such that all models of \( T, \phi \) are fields of characteristic 0.
   (b) Show that if a closed formula is true on all fields of characteristic 0, then there exists a natural number \( p \) such that for all fields of characteristic greater than \( p \) satisfy this formula.

4. Let \( \mathcal{F} = \emptyset \), \( \mathcal{P} = \{ = (2), \leq (2) \} \). Show that there is no formula \( F \) such that egalitarian models of \( F \) are well founded orders.
Additional exercises

Exercise 7: Isomorphism v.s. elementary equivalence (flashback)

Two structures are *elementarily equivalent* iff they satisfy the same closed formulas.

1. Give two elementarily equivalent structures that are not isomorphic.
2. Same question when \( P \) contains a binary predicate interpreted like the equality.
3. Show that if \( F = \emptyset \) and \( P = \{ = (2) \} \), two elementarily equivalent egalitarian structures with countable domains are isomorphic.
4. Let \( F = \{ 0(0), \ s(1) \} \) and \( P = \{ = (2) \} \). Give two elementarily equivalent structures with countable domains that are not isomorphic (without proof).

Exercise 8: Tarski-Vaught’s test (open exercise)

Let \( F, P \) be two sets of symbols. Let \( M \) be an \( F, P \)-structure and \( N \) be a substructure of \( M \). Show that if for every formula \( \phi \) with free variables \( x, x_1, \ldots, x_n \) and parameters \( a_1, \ldots, a_n \in D_N \) such that \( M, \{ x_1 \to a_1, \ldots, x_n \to a_n \} \models \exists x. \phi \), there exists \( a \in D_N \) such that \( M, \{ x_1 \to a_1, \ldots, x_n \to a_n, x \to a \} \models \phi \), then \( M \) and \( N \) are elementarily equivalent.

Exercise 9: Theorem of Lowenheim-Skolem (open exercise)

Let \( F \) and \( P \) countable sets of symbols. Prove that if a theory \( T \) written on \( F \) and \( P \) is consistent, then it has a countable model.

Exercise 10: A direct proof of compactness

The goal of this exercise is to give an alternate proof of the compactness theorem for the propositional calculus, i.e. quantifier-free formulas, built with symbols \( F = \emptyset \) and \( P \) a set of constant predicates, called *propositional variables*. This proof does not use completeness: it is a purely topological proof, based on Tychonoff’s theorem.

A product of compact spaces is compact with respect to the product topology.

1. Show that the space of all predicate interpretations is compact.
2. Show that the space the predicate interpretations that satisfy a formula is closed and open.
3. Prove the compactness theorem for the propositional calculus.