Exercise 1: Minimization by Moore’s algorithm

1. Minimize the two following automata, using Moore’s algorithm:

(a) Automaton $\mathcal{A}_1$

(b) Automaton $\mathcal{A}_2$

2. Give a minimal automaton for $((a(a + b)^2 + b)^* a(a + b))^*$. 
Exercise 2: Minimization by Brzozowski inversion
Show that the determinized of a co-deterministic co-accessible automaton which recognizes a language $L$ is (isomorphic to) the minimal automaton of $L$. Using this result, devise a procedure to minimize an automaton. What is the complexity of this method?

Exercise 3: State complexity of a language
Given a recognizable language $L$, we define its state complexity $\text{Sc}(L)$ by the number of states of its minimal automaton. Show that the following inequalities hold ($L^t$ is the transposed of $L$, the language of the mirror images of words of $L$):

1. $\text{Sc}(L \cap K) \leq \text{Sc}(L) \text{Sc}(K)$;
2. $\text{Sc}(L \cup K) \leq \text{Sc}(L) \text{Sc}(K)$;
3. $\text{Sc}(L^t) \leq 2^{\text{Sc}(L)}$;
4. $\text{Sc}(LK) \leq (2^{\text{Sc}(L)} - 1)2^{\text{Sc}(K)-1}$.

We will now show that some of these bounds have the right order of magnitude. Let $\Sigma = \{a, b\}$.

5. Consider $L_n = \{ |w|_a + |w|_b = 2n \}$ and $L'_n = \{ |w|_a + 2|w|_b = 3n \}$ for the bound for intersection.
6. Consider $L_n = \Sigma^n - 1 a \Sigma^* b$ for the bound for transposition.

Exercise 4: Brzozowski-McCluskey algorithm
The goal of this exercise is to translate a finite automaton into a rational expression, giving an alternate proof of the associated implication of Kleene’s theorem. We will proceed by successive transformations of the automaton.

1. We call strongly normalized every automaton which has a unique initial state to which no transition leads and a unique final state with no exiting transition, i.e. an automaton $A = \langle Q, \Sigma, \{i\}, \{f\}, \delta \rangle$ such that for every state $q$ and letter $a$, $(q, a, i) \notin \delta$ and $(f, a, q) \notin \delta$. Show that for all finite automaton, there is a strongly normalized automaton which recognizes the same language.

We will use a generalization of the definition of finite automata: the transition function will be a subset of $Q \times 2^\Sigma \times Q$. An execution of such an automaton recognizes the concatenation of languages of the transitions’ labels. The automaton recognizes the union of the languages of all its accepting executions.

2. Show that every generalized automaton is equivalent to a generalized automaton in which there exists exactly one transition between each pair of states: $q' \in \delta(q, L)$ et $q' \in \delta(q, L')$ implies $L = L'$.

3. Let $A$ be a strongly normalized generalized automaton with initial state $i$ and final state $f$. Let $q \in Q_A$, $q \notin \{i, f\}$. Show that there exists an automaton equivalent to $A$ with set of states $Q_A \setminus \{q\}$. 


4. Conclude that if $L$ is recognized by a strongly normalized generalized automaton $\mathcal{A}$, then $L$ belongs to the rational closure of the labels of the transitions of $\mathcal{A}$.

5. Show that every finite automaton has an equivalent generalized automaton.

6. Give a procedure which, given a finite automaton, outputs a rational expression of same language.

7. Apply the construction to compute a rational expression corresponding to the following automaton:

![Automaton Diagram]

8. We consider the alphabet $\Sigma_n = [1; n] \times [1; n]$ and define:

$$L = \{ (a_1, a_2)(a_2, a_3) \ldots (a_m, a_{m+1}) : m \geq 1, a_1 = 1, a_{m+1} = n \}$$

(a) Give an automaton of quadratic size recognizing $L$.

(b) What is the size of the expression obtained by this construction on this automaton?

Exercise 5: Congruences and monoids

$R$ is a congruence if $uRv$ implies $xuyRxvy$ for all $x, y$.

1. Show that a congruence is an equivalence relation. We will call congruence classes the equivalence classes of a congruence.

2. Prove that a language is regular iff it is the union of some of the congruence classes of a congruence relation of finite index, i.e. with a finite number of congruence classes.

A congruence $c_1$ is coarser (i.e. "grossière") than another congruence $c_2$ if every congruence class of $c_2$ is included in a congruence class of $c_1$.

3. Let $L$ be a language. Find a characterization of the coarsest congruence $\equiv_L$ such that $L$ is the union of some of its congruence classes.

This congruence is called the syntactic congruence of $L$.

4. Give a more precise criterion for the recognizability of a language. Apply it to prove that $\{ a^n b^n : n \in \mathbb{N} \}$ is not recognizable.
5. Show that a language of $\Sigma^*$ is regular iff there exists a finite monoid $(M, \times)$, a morphism $\mu : (\Sigma^*, \cdot) \to (M, \times)$, and a set $P \subseteq M$ such that $L = \mu^{-1}(P)$. Find a characterization of the smallest such monoid for a regular language $L$.

6. Prove that if $L$ is regular, $\sqrt{L} = \{ u : u^2 \in L \}$ is regular.

7. What is the link between the syntactic congruence, this smallest monoid, and the minimal automaton?