Logique
TD n°8

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In the following, we will use the result:

> Let ϕ be a quantifier-free formula. There is an equivalent formula of the form $\bigvee_{i \in I} \left(\bigwedge_{j \in J} L_{i,j} \right)$ where $L_{i,j}$ are literals. This form is called DNF (disjunctive normal forms).

Exercise 1 : Another theory of numbers

We work over signature $\mathcal{F} = \{ \ 0(0) \ , \ S(1) \ \}$, $\mathcal{P} = \{ \ = (2) \ \}$ and define the theory of axioms the axioms of equality and:

$$
\begin{array}{rll}
 & (F_1) & \forall x. \ S(x) \neq 0 \\
 & (F_2) & \forall x, y. \ S(x) = S(y) \Rightarrow x = y \\
 & (F_3) & \forall x. \ \exists y. \ x = 0 \vee x = S(y) \\
\text{For every } n > 0, & (C_n) & \forall x. \ S^n(x) \neq x
\end{array}
$$

Let $T' = \{ \ F_1 \ , \ F_2 \ , \ F_3 \ \}$ and $T = T' \cup \{ \ C_n \ \}_{n>0}$.

1. Show that any model of T is infinite.
2. Find a model of T of domain $\mathbb{Q}$.
3. Show that, for every n, $T' \nvdash C_n$. Conclude that T' and T are not equivalent.
4. Let A be the set of boolean combinations of atomic formulas.
   (a) Let F be a conjunction of literals containing x on only one side of equality. Give an algorithm transforming formula $\exists x. \ F$ into a formula G such that $T \vdash \exists x. \ F \Leftrightarrow G$.
   (b) Show that T admits quantifier elimination.
5. Show that T is complete.
6. Let $T" = \{ \ F_1 \ , \ F_2 \ \} \cup \{ \ Ind_{F,x} \mid F$ a formula with at least one free variable $\}$ where $Ind_{F,x}$ is the induction applied to formula F and variable x. Prove that T and $T"$ are equivalent.
7. Show that T has a model $\mathcal{M}$ of domain $\{0\} \times \mathbb{N} \cup \{1\} \times \mathbb{R}$. What can we conclude?

Exercise 2 : PRESBURGER arithmetic

We study the first order theory of natural numbers and addition called PRESBURGER arithmetic. More precisely, it is the first order theory over the language containing the binary predicate symbol $=$ and function symbols 0, S, and $+$ and of axioms every formula true over natural numbers, i.e. every formula Φ such that for every valuation $\sigma : \mathcal{X} \to \mathbb{N}$ we have $\mathbb{N}, \sigma \models \Phi$. In the following, two formulas ϕ_1, ϕ_2 are said to be equivalent if for any valuation σ, $\mathbb{N}, \sigma \models \phi_1$ iff $\mathbb{N}, \sigma \models \phi_2$.

1. Show that any formula can be transformed in polynomial time in an equivalent formula of atomic formulas of the form $x = 0$, $x = S(y)$ or $x + y = z$ (where x, y, z are variables) without any universal quantifiers. We say such a formula is *reduced*.

We encode natural numbers in base 2, little-endian convention (the heaviest byte is on the right). We define a decoding function $\nu : \{0,1\}^* \to \mathbb{N}$ by:

$$\nu(\epsilon) = 0 \qquad \nu(0w) = 2\nu(w) \qquad \nu(1w) = 1 + 2\nu(w)$$

This function is surjective but not injective. Let $\mathcal{V} \subseteq \mathcal{X}$ be a subset of variables. Valuations $\sigma : \mathcal{V} \to \mathbb{N}$ are coded by words on the alphabet $\Sigma_{\mathcal{V}} = \{0,1\}^{\mathcal{V}}$. If w is a word over $\Sigma_{\mathcal{V}}$, we define w_x the projection on its x^{th} component. The function ν can be extended to a function from $\Sigma_{\mathcal{V}}^*$ to valuations over $\mathcal{V}$ by:

$$\nu(w) = (x \mapsto \nu(w_x))_{x \in \mathcal{V}}$$

If Φ is a formula and $\mathcal{V}$ contains the free variables of Φ, we write $[\Phi]_{\mathcal{V}} = \{w \in \Sigma_{\mathcal{V}}^* \mid \mathbb{N}, \nu(w) \vDash \Phi\}$.

2. Show that a formula Φ is satisfied by $\mathbb{N}$ iff $[\Phi]_{fv(\Phi)} = \Sigma_{fv(\Phi)}^*$ where $fv(\Phi)$ is the set of the free variables of Φ.
3. Show that for any reduced formula Φ, there exists a finite automaton A_Φ over alphabet $\Sigma_{fv(\Phi)}$ of language $[\Phi]_{fv(\Phi)}$.
4. Show that PRESBURGER arithmetic is decidable. What is the complexity of this procedure?

Exercise 3 : Theory of total dense orders without borders

We work over the language containing the binary predicate symbols $<$ and $=$.

The theory $\mathcal{T}_O$ is defined with the axioms of equality and:

$$
\begin{array}{lrl}
(O_1) & \forall x \forall y. & \neg(x < y \wedge y < x) \\
(O_2) & \forall x \forall y \forall z. & x < y \wedge y < z \Rightarrow x < z \\
(O_3) & \forall x \forall y. & x < y \vee x = y \vee y < x \\
(O_4) & \forall x \forall y \exists z. & x < y \Rightarrow x < z \wedge z < y \\
(O_5) & \forall x \exists y. & x < y \\
(O_6) & \forall x \exists y. & y < x
\end{array}
$$

Models of $\mathcal{T}_O$ are sets with a total, dense order without borders.

1. Let us familiarize ourselves with this theory:
   (a) Show that models of $\mathcal{T}_O$ are infinite.
   (b) Give two models of $\mathcal{T}_O$ that are not isomorphic.
   (c) Show that $\mathcal{T}_O$ is consistent.

The goal of this exercise is to prove that this theory is decidable, by proving it satisfies the elimination of quantifiers. We want to show that for every formula ψ of the form $\exists x.\ \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} L_{i,j}$ of free variables $x_1, \ldots, x_n$ where L_i is a literal, there exists a quantifier-free formula ϕ of free variables in $x_1, \ldots, x_n$ such that $\mathcal{T}_O \vdash \forall x_1, \ldots, x_n.\ [\phi \Leftrightarrow \psi]$.

2. Show that we can consider that ψ contains only literals of the form $x = x_i$, $x_i = x_j$, $x_i < x_j$, $x_i < x$, $x < x_i$.
3. Show that proving the result on formulas of the form $\exists x.\ \bigwedge_{j=1}^{m} K_j$ where K_j is of the form $x = x_i$, $x_i = x_j$, $x_i < x_j$, $x_i < x$, or $x < x_i$ is enough to conclude.
   Hint: use the equivalence $\vdash \exists x.\ [\phi_1 \vee \phi_2] \Leftrightarrow (\exists x.\ \phi_1) \vee (\exists x.\ \phi_2)$.

We will consider that ψ is of the form described in Question 3. in the following.

4. Show that if ψ contains a literal of the form $x = x_i$, we can conclude.
5. Else, show that ψ is equivalent to a formula of the form $K_1 \wedge \exists x.\ K_2$ where:
   - $K_1 = \bigwedge_r K_r$ of free variables in $x_1, \ldots, x_n$,
   - K_2 is of the form
     $$\bigwedge_{i \in I} x_i < x \quad \wedge \quad \bigwedge_{j \in J} x < x_j$$
     where I and J are subsets of $\{ 1, \ldots, n \}$.
6. Show that if $I \cap J \neq \emptyset$ then ψ is equivalent to $\bot$.
7. Show that if $I \cap J = \emptyset$ then ψ is also equivalent to a quantifier-free formula.
8. Conclude that $\mathcal{T}_O$ is complete, and decidable.