Logique
TD n°7

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In the following, we will write $\Gamma \vdash_{\mathsf{LK}} \Delta$, resp $\Gamma \vdash_{\mathsf{NK}} \phi$ when $\Gamma \vdash \Delta$ is provable in sequent calculus, resp when $\Gamma \vdash \phi$ is provable in natural deduction.

Exercise 1 : LK with cuts

We study the sequent calculus with cuts. In the following, P, Q are unary predicate symbols.

1. Prove formula $\phi_a = \exists x. (P(x) \Rightarrow \forall y. P(y))$ in sequent calculus.
   Hint: use a cut on $\forall x. P(x) \vee \exists x. \neg P(x)$.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\pi}{P(x) \vdash P(x), \phi_a}\,\text{ax}}{\vdash P(x), \neg P(x), \phi_a}\,\neg_{right}}{\vdash P(x), \exists x. \neg P(x), \phi_a}\,\exists_{right}}{\vdash \forall x. P(x), \exists x. \neg P(x), \phi_a}\,\forall_{right}}{\vdash \forall x. P(x) \vee \exists x. \neg P(x), \phi_a}\,\vee_{right}
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{}{\forall x. P(x), P(x) \vdash \forall y. P(y)}\,\text{ax}}{\forall x. P(x) \vdash P(x) \Rightarrow \forall y. P(y)}\,\Rightarrow_{right}}{\forall x. P(x) \vdash \phi_a}\,\exists_{right}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash \forall y. P(y), P(x)}\,\text{ax}}{\vdash P(x) \Rightarrow \forall y. P(y), P(x)}\,\neg_{left}}{\neg P(x) \vdash P(x) \Rightarrow \forall y. P(y)}\,\neg_{left}}{\neg P(x) \vdash \phi_a}\,\exists_{right}}{\exists x. \neg P(x) \vdash \phi_a}\,\exists_{left}
}{\forall x. P(x) \vee \exists x. \neg P(x) \vdash \phi_a}\,\vee_{left}
}{\vdash \phi_a}\,\text{cut}
$$

2. Eliminate cuts in the following proof. Detail the procedure.

First, let's do a little recap on the cut-elimination procedure, from a practical point of view.
The goal is to make cuts go up in the proof, or reduce them to cuts over smaller formulas, until the cuts can be replaced directly with structural rules (contraction, axiom). There are three main cases, depending on the first rules of the two proofs above the cut rule we want to eliminate:

(a) The two rules are applied to the cut formula: we divide the cut in cuts over the subformula(s) of the cut formula. See the course notes to know what to do depending on the form of the cut formula.

Observation: if both are axiom rules, replace the cut directly with an axiom rule.

(b) One of the rules is not applied to the cut formula: we move the cut above this rule.

Observation: if the rule is an axiom, this eliminates the cut and ends the proof with an axiom.

(c) One is an axiom over the cut formula: we replace the cut with a structural rule (axiom if the second is also an axiom over the cut formula, else a contraction).

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi, \psi}\,\mathrm{ax}}{\phi \vdash \phi \lor \psi}\,\lor_{right} \qquad \dfrac{\dfrac{\dfrac{}{\phi \vdash \psi, \phi}\,\mathrm{ax} \quad \dfrac{}{\psi \vdash \psi, \phi}\,\mathrm{ax}}{\phi \lor \psi \vdash \psi, \phi}\,\lor_{left}}{\phi \lor \psi \vdash \psi \lor \phi}\,\lor_{right}}{\phi \vdash \psi \lor \phi}\,\mathrm{cut}$$

We want to eliminate the cut over fomula $\phi \lor \psi$. We are in situation (b): the $\lor_{right}$ rule on the right hand proof is not applied to the cut formula $\phi \lor \psi$. The cut is moved above this rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi, \psi}\,\mathrm{ax}}{\phi \vdash \phi \lor \psi}\,\lor_{right} \qquad \dfrac{\dfrac{}{\phi \vdash \psi, \phi}\,\mathrm{ax} \quad \dfrac{}{\psi \vdash \psi, \phi}\,\mathrm{ax}}{\phi \lor \psi \vdash \psi, \phi}\,\lor_{left}}{\phi \vdash \psi, \phi}\,\mathrm{cut}}{\phi \vdash \psi \lor \phi}\,\lor_{right}$$

Now we are in the situation (a), as on both sides the first rule is applied to the cut formula $\phi \lor \psi$. We divide the cut in two smaller cuts over formulas ϕ and ψ:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \phi \vdash \psi, \psi, \phi}\,\mathrm{ax} \quad \dfrac{}{\phi, \phi, \psi \vdash \psi, \phi}\,\mathrm{ax}}{\phi, \phi \vdash \psi, \phi}\,\mathrm{cut} \qquad \dfrac{}{\phi \vdash \phi, \psi, \phi}\,\mathrm{ax}}{\phi \vdash \psi, \phi}\,\mathrm{cut}}{\phi \vdash \psi \lor \phi}\,\lor_{right}$$

We are now in the case of (b), more specifically the observation, as the axiom on the top left hand side of the proof is applied to formula ϕ, which is not the cut formula. We replace the cut with an axiom rule:

$$\dfrac{\dfrac{\dfrac{}{\phi, \phi \vdash \psi, \phi}\,\text{ax} \qquad \dfrac{}{\phi \vdash \phi, \psi, \phi}\,\text{ax}}{\phi \vdash \psi, \phi}\,\text{cut}}{\phi \vdash \psi \lor \phi}\,\lor_{right}$$

We are now in the case of (a), more specifically the observation. We replace the cut with an axiom rule:

$$\dfrac{\dfrac{}{\phi \vdash \psi, \phi}\,\text{ax}}{\phi \vdash \psi \lor \phi}\,\lor_{right}$$

As some of you noticed, I cheated a bit on this proof and procedure, more specifically on the cut rule. The cut rule in the original proof of 2. is the following:

$$\dfrac{\Gamma' \vdash A, \Delta' \qquad \Gamma, A \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{cut'}$$

As the most attentive have probably already observed, it is the rule you have proven to be admissible in the proof of the admissibility of the cut rule. With weakening and contraction, the two rules are equivalent. We can simulate cut' with cut:

$$\dfrac{\dfrac{\Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash A, \Delta, \Delta'}\,\text{weakenings} \qquad \dfrac{\Gamma, A \vdash \Delta}{\Gamma, \Gamma', A \vdash \Delta, \Delta'}\,\text{weakenings}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{cut'}$$

We can simulate cut with cut':

$$\dfrac{\dfrac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma, \Gamma \vdash \Delta, \Delta}\,\text{cut'}}{\Gamma \vdash \Delta}\,\text{contractions}$$

And that is exactly how, in practice, we will use the proof to eliminate cuts: adding contractions. Let us see on the proof of question 2, using this time the real LK cut rule:

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi, \psi, \psi \lor \phi}\,\text{ax}}{\phi \vdash \phi \lor \psi, \psi \lor \phi}\,\lor_{right} \qquad \dfrac{\dfrac{\dfrac{}{\phi, \phi \vdash \psi, \phi}\,\text{ax} \qquad \dfrac{}{\phi, \psi \vdash \psi, \phi}\,\text{ax}}{\phi, \phi \lor \psi \vdash \psi, \phi}\,\lor_{left}}{\phi, \phi \lor \psi \vdash \psi \lor \phi}\,\lor_{right}}{\phi \vdash \psi \lor \phi}\,\text{cut}$$

Here, we are in situation (b). However, we cannot make the cut rule "go above" the $\lor_{right}$ rule, as in the left side of the proof, the rule is not applied to this formula. Formulas are copied instead of distributed in the LK cut rule: they can be treated differently in the two proofs. The solution is to add a contraction. The two copies of the formula represent the two copies in each proof above the cut, and we can continue with the procedure accordingly:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\phi \vdash \phi, \psi, \psi \lor \phi, \psi, \phi}\,\text{ax}}{\phi \vdash \phi \lor \psi, \psi \lor \phi, \psi, \phi}\,\lor_{right} \qquad \dfrac{\dfrac{}{\phi, \phi \vdash \psi \lor \phi, \psi, \phi}\,\text{ax} \qquad \dfrac{}{\phi, \psi \vdash \psi \lor \phi, \psi, \phi}\,\text{ax}}{\phi, \phi \lor \psi \vdash \psi \lor \phi, \psi, \phi}\,\lor_{left}}{\phi \vdash \psi \lor \phi, \psi, \phi}\,\text{cut}}{\phi \vdash \psi \lor \phi, \psi \lor \phi}\,\lor_{right}}{\phi \vdash \psi \lor \phi}\,\text{contraction}$$

Now, we reduce the cut into two smaller cuts (a).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \phi \vdash \psi \lor \phi, \psi, \psi, \phi}\text{ax} \quad \dfrac{}{\phi, \phi, \psi \vdash \psi \lor \phi, \psi, \phi}\text{ax}}{\phi, \phi \vdash \psi \lor \phi, \psi, \phi}\text{cut} \quad \dfrac{}{\phi \vdash \phi, \psi \lor \phi, \psi, \phi}\text{ax}}{\phi \vdash \psi \lor \phi, \psi, \phi}\text{cut}}{\dfrac{\phi \vdash \psi \lor \phi, \psi \lor \phi}{\phi \vdash \psi \lor \phi}\text{contraction}}\lor_{right}$$

We replace the cut by an axiom, as there are two axioms above:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\phi, \phi \vdash \psi \lor \phi, \psi, \phi}\text{ax} \quad \dfrac{}{\phi \vdash \phi, \psi \lor \phi, \psi, \phi}\text{ax}}{\phi \vdash \psi \lor \phi, \psi, \phi}\text{cut}}{\phi \vdash \psi \lor \phi, \psi \lor \phi}\lor_{right}}{\phi \vdash \psi \lor \phi}\text{contraction}$$

And again:

$$\dfrac{\dfrac{\dfrac{}{\phi \vdash \psi \lor \phi, \psi, \phi}\text{ax}}{\phi \vdash \psi \lor \phi, \psi \lor \phi}\lor_{right}}{\phi \vdash \psi \lor \phi}\text{contraction}$$

Of course, the contraction is not necessary here.

3. Is there a term t such that $\vdash_{\mathsf{LK}} P(t) \Rightarrow \forall y.\, P(y)$?

$P(t) \Rightarrow \forall y.\, P(y)$ is not satisfied in the model $\mathcal{M}$ defined by: $\mathcal{D}_{\mathcal{M}} = \{ m_1, m_2 \}$, $P_{\mathcal{M}} = \{ m_2 \}$ and for all term u and valuation σ, $[\![u]\!]_{\mathcal{M},\sigma} = m_1$.

Exercise 2 : Subformula property

In the previous exercise sheet, we have seen the interpolation theorem. We will now study the *subformula property.*

1. Prove that if $\Gamma \vdash_{\mathsf{LK}} \Delta$, then there is a proof of $\Gamma \vdash \Delta$ containing only formulas of the form $\phi\{x_1 \to t_1, \ldots, x_n \to t_n\}$, where ϕ is a subformula of a formula in Γ, Δ, and $t_1, \ldots, t_n$ are terms (not necessarily appearing in $\Gamma \cup \Delta$). Treat at least structural rules, right and left rules for disjunction and all cases for quantifiers.

By a straightforward induction on the cut-free proof of $\Gamma \vdash_{\mathsf{LK}} \Delta$. The substitutions come from the rules for quantifiers.

2. We call LKP the propositionnal fragment of sequent calculus. It contains all LK rules except for the left and right rules of quantifiers.

   (a) Prove that if $\Gamma \vdash_{\mathsf{LK}} \Delta$ and $\Gamma \cup \Delta$ contains only propositionnal formulas, then there is a proof of $\Gamma \vdash \Delta$ in LKP, using the subformula property.

   If there is a proof of $\Gamma \vdash_{\mathsf{LK}} \Delta$, then there is a cut-free proof of $\Gamma \vdash_{\mathsf{LK}} \Delta$. By the subformula property, there is a proof containing only propositional formulas, and so using only rules of LKP.
   We observe that as there are no quantifier rules applied, all formulas in the proof are subformulas of Γ, Δ.

(b) Show syntactically that provability in LKP is decidable.

Let $\Gamma \vdash \Delta$ be a sequent containing only propositional formulas. If it is provable, let π be the smallest proof in LKP of this sequent.
By the previous question and observation, there is only a limited number of formulas appearing in the proofs, and thus a limited number of sequents. The same sequent cannot appear twice in the same branch of π, because π is the smallest proof. Every rule has at most two premisses. This gives us a maximum size of the smallest proof of the sequent: we only have to test every proof smaller than this proof to conclude on the provability of $\Gamma \vdash \Delta$.

Exercise 3 : LK is equivalent to NK

1. In class, you have proven that if $\Gamma \vdash_{\mathsf{LK}} \Delta$ then $\Gamma, \neg\Delta \vdash_{\mathsf{NK}} \bot$ by induction. Write the cases of $\exists_{left}$, $\exists_{right}$.

- We write $\Gamma = \exists x.\, \phi, \Gamma'$, and $x \notin fv(\Gamma, \Delta)$. We have a proof of $\Gamma', \phi \vdash_{\mathsf{LK}} \Delta$. By induction hypothesis, let π be a proof of $\Gamma', \phi, \neg\Delta \vdash_{\mathsf{NK}} \bot$. We build the proof:

$$\dfrac{\dfrac{}{\Gamma, \neg\Delta \vdash \exists x.\, \phi}\text{ax} \qquad \dfrac{\dfrac{\pi}{\Gamma', \neg\Delta, \phi \vdash \bot}}{\Gamma, \neg\Delta, \phi \vdash \bot}\text{weakening}}{\Gamma, \neg\Delta \vdash \bot}\exists_{elim}$$

- We write $\Delta = \exists x.\, \phi, \Delta'$. We have a proof of $\Gamma \vdash_{\mathsf{LK}} \Delta', \phi\{x \to t\}$. By induction hypothesis, let π be a proof of $\Gamma, \neg\Delta', \neg\phi\{x \to t\} \vdash_{\mathsf{NK}} \bot$. We build the proof:

$$\dfrac{\dfrac{}{\Gamma, \neg\Delta \vdash \neg\exists x.\, \phi}\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\pi}{\Gamma', \neg\Delta', \neg\phi\{x \to t\} \vdash \bot}}{\Gamma, \neg\Delta' \vdash \neg\neg\phi\{x \to t\}}\mathsf{RAA}}{\Gamma, \neg\Delta' \vdash \exists x.\, \phi}\exists_{intro}}{\Gamma, \neg\Delta \vdash \exists x.\, \phi}\text{weakening}}{\Gamma, \neg\Delta \vdash \bot}\neg_{elim}$$

2. You have also proven that if $\Gamma \vdash_{\mathsf{NK}} \phi$, then $\Gamma \vdash_{\mathsf{LK}} \phi$. Write the cases of RAA, $\Rightarrow_{elim}$.

- We have a proof of $\Gamma \vdash_{\mathsf{NK}} \neg\neg\phi$. By induction hypothesis, let π be a proof of $\Gamma \vdash_{\mathsf{LK}} \neg\neg\phi$. We build the proof:

$$\dfrac{\dfrac{\pi}{\Gamma \vdash \neg\neg\phi} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \phi \vdash \phi}\text{ax}}{\Gamma \vdash \phi, \neg\phi}\neg_{right}}{\Gamma, \neg\neg\phi \vdash \phi}\neg_{left}}{\Gamma \vdash \phi}\text{cut}$$

- We have proofs of $\Gamma \vdash_{\mathsf{NK}} \phi \Rightarrow \psi$ and $\Gamma \vdash_{\mathsf{NK}} \phi$. By induction hypothesis, let π_1, π_2 be proofs of $\Gamma \vdash_{\mathsf{LK}} \phi \Rightarrow \psi$ and $\Gamma \vdash_{\mathsf{LK}} \phi$. We build the proof:

$$\dfrac{\dfrac{\pi_1}{\Gamma \vdash \phi \Rightarrow \psi} \qquad \dfrac{\dfrac{\pi_2}{\Gamma \vdash \phi} \qquad \dfrac{}{\Gamma, \psi \vdash \psi}\text{ax}}{\Gamma, \phi \Rightarrow \psi \vdash \psi}\Rightarrow_{left}}{\Gamma \vdash \psi}\text{cut}$$

3. Here are two natural deduction proofs.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\neg\neg\phi, \phi \vdash \neg\neg\neg\phi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\neg\neg\phi, \neg\phi, \phi \vdash \neg\phi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\neg\neg\phi, \neg\phi, \phi \vdash \phi$}
\RightLabel{$\neg_{elim}$}
\BinaryInfC{$\neg\neg\neg\phi, \neg\phi, \phi \vdash \bot$}
\RightLabel{$\neg_{intro}$}
\UnaryInfC{$\neg\neg\neg\phi, \phi \vdash \neg\neg\phi$}
\RightLabel{$\neg_{elim}$}
\BinaryInfC{$\neg\neg\neg\phi, \phi \vdash \bot$}
\RightLabel{$\neg_{intro}$}
\UnaryInfC{$\neg\neg\neg\phi \vdash \neg\phi$}
\end{prooftree}
$$

(a) First proof

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \phi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \neg\psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \neg\psi \Rightarrow \neg\phi$}
\RightLabel{$\Rightarrow_{elim}$}
\BinaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \neg\phi$}
\RightLabel{$\neg_{elim}$}
\BinaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \bot$}
\RightLabel{$\neg_{intro}$}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi \vdash \neg\neg\psi$}
\RightLabel{RAA}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi \vdash \psi$}
\RightLabel{$\Rightarrow_{intro}$}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi \vdash \phi \Rightarrow \psi$}
\end{prooftree}
$$

(b) Second proof

Transform them into sequent calculus proofs (you can just use left rules instead of elimination rules when it works, else use what seen in class).

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\phi \vdash \phi$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\neg\phi, \phi \vdash \bot$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\phi \vdash \neg\neg\phi$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\neg\neg\neg\phi, \phi \vdash \bot$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\neg\neg\neg\phi \vdash \neg\phi$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{2.}
\UnaryInfC{$\neg\neg\psi \vdash \psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\phi \vdash \phi$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\phi, \neg\phi \vdash$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\psi \vdash \neg\psi$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg\phi, \neg\psi \vdash \neg\phi$}
\RightLabel{$\Rightarrow_{left}$}
\BinaryInfC{$\neg\psi \Rightarrow \neg\phi, \neg\psi \vdash \neg\phi$}
\RightLabel{cut}
\BinaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \bot$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi \vdash \neg\neg\psi$}
\RightLabel{cut}
\BinaryInfC{$\neg\psi \Rightarrow \neg\phi, \phi \vdash \psi$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\neg\psi \Rightarrow \neg\phi \vdash \phi \Rightarrow \psi$}
\end{prooftree}
$$

4. Transform this proof into a natural deduction proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$A \vdash A$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\vdash A, \neg A$}
\RightLabel{$\vee_{right}$}
\UnaryInfC{$\vdash A \vee \neg A$}
\end{prooftree}
$$

$$\dfrac{\dfrac{}{\neg(A \vee \neg A) \vdash \neg(A \vee \neg A)}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A \vee \neg A) \vdash \neg(A \vee \neg A)}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A, \neg\neg A \vdash \neg\neg A}\,\text{ax} \qquad \dfrac{}{\neg A, \neg\neg A \vdash \neg A}\,\text{ax}}{\neg A, \neg\neg A \vdash \bot}\,\neg_{elim}}{\neg A \vdash \neg\neg\neg A}\,\neg_{intro}}{\neg A \vdash \neg A}\,\mathsf{RAA}}{\neg A \vdash A \vee \neg A}\,\vee_{intro}}{\neg(A \vee \neg A), \neg A \vdash \bot}\,\neg_{elim}}{\neg(A \vee \neg A) \vdash \neg\neg A}\,\neg_{intro}}{\neg(A \vee \neg A) \vdash A}\,\mathsf{RAA}}{\neg(A \vee \neg A) \vdash A \vee \neg A}\,\vee_{intro}}{\neg(A \vee \neg A) \vdash \bot}\,\neg_{elim}$$

5. Prove that if $\Gamma \vdash_{LK} \phi_1, \ldots, \phi_n$ then $\Gamma \vdash_{NK} \phi_1 \vee \cdots \vee \phi_n$.

If $\Gamma \vdash_{LK} \phi_1, \ldots, \phi_n$, then $\Gamma, \neg\phi_1, \ldots, \neg\phi_n \vdash_{NK} \bot$.

By applying RAA and $\neg_{intro}$ to $\Gamma \vdash \phi_1 \vee \cdots \vee \phi_n$, we show that a proof of $\Gamma, \neg(\phi_1 \vee \cdots \vee \phi_n) \vdash \bot$ would be enough.

We know that we can simulate $\vee_{intro}$ "on the left side", as we did in the equivalence proof of sequent calculus and natural deduction (and in the previous question), so we have a proof of $\Gamma, \neg\phi_1, \ldots, \neg\phi_n \vdash \bot$

Exercise 4 : Exercise 1, ctd

You can use the same cut as in Exercise 1. question 1) to do the two following proofs.

1. Prove $(\forall x. P(x) \Rightarrow \exists x. Q(x)) \Rightarrow \exists x. (P(x) \Rightarrow Q(x))$ in sequent calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x. P(x) \vdash \forall x. P(x)}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall x. P(x), P(x), Q(x) \vdash Q(x)}\,\text{ax}}{\forall x. P(x), Q(x) \vdash P(x) \Rightarrow Q(x)}\,\Rightarrow_{right}}{\forall x. P(x), Q(x) \vdash \exists x. (P(x) \Rightarrow Q(x))}\,\exists_{right}}{\forall x. P(x), \exists x. Q(x) \vdash \exists x. (P(x) \Rightarrow Q(x))}\,\exists_{left}}{\forall x. P(x), \forall x. P(x) \Rightarrow \exists x. Q(x) \vdash \exists x. (P(x) \Rightarrow Q(x))}\,\Rightarrow_{left} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall x. P(x) \Rightarrow \exists x. Q(x), P(x) \vdash Q(x), P(x)}\,\text{ax}}{\neg P(x), \forall x. P(x) \Rightarrow \exists x. Q(x), P(x) \vdash Q(x)}\,\neg_{left}}{\neg P(x), \forall x. P(x) \Rightarrow \exists x. Q(x) \vdash P(x) \Rightarrow Q(x)}\,\Rightarrow_{right}}{\neg P(x), \forall x. P(x) \Rightarrow \exists x. Q(x) \vdash \exists x. (P(x) \Rightarrow Q(x))}\,\exists_{right}}{\exists x. \neg P(x), \forall x. P(x) \Rightarrow \exists x. Q(x) \vdash \exists x. (P(x) \Rightarrow Q(x))}\,\exists_{left}}{\forall x. P(x) \vee \exists x. \neg P(x), \forall x. P(x) \Rightarrow \exists x. Q(x) \vdash \exists x. (P(x) \Rightarrow Q(x))}\,\vee_{left}}{\forall x. P(x) \vee \exists x. \neg P(x) \vdash (\forall x. P(x) \Rightarrow \exists x. Q(x)) \Rightarrow \exists x. (P(x) \Rightarrow Q(x))}\,\Rightarrow_{right}}{\vdash (\forall x. P(x) \Rightarrow \exists x. Q(x)) \Rightarrow \exists x. (P(x) \Rightarrow Q(x))}\,\text{cut}$$

2. Prove $\exists x. \forall y. [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]$ in sequent calculus.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y), (P(y) \Rightarrow P(x)) \Rightarrow P(x) \vdash P(y)}\,\text{ax}}{P(y) \vdash ((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)}\,\Rightarrow_{right}}{\forall x.\, P(x) \vdash ((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)}\,\forall_{left}}{\forall x.\, P(x) \vdash \forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}\,\forall_{right}}{\forall x.\, P(x) \vdash \exists x.\forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}\,\exists_{right} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(y) \vdash P(y), P(x)}\,\text{ax}}{\vdash P(y), P(y) \Rightarrow P(x)}\,\Rightarrow_{right} \quad \dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{(P(y) \Rightarrow P(x)) \Rightarrow P(x) \vdash P(y), P(x)}\,\Rightarrow_{left}}{\neg P(x), (P(y) \Rightarrow P(x)) \Rightarrow P(x) \vdash P(y)}\,\neg_{left}}{\neg P(x) \vdash ((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)}\,\Rightarrow_{right}}{\neg P(x) \vdash \forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}\,\forall_{right}}{\neg P(x) \vdash \exists x.\forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}\,\exists_{right}}{\exists x.\, \neg P(x) \vdash \exists x.\forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}\,\exists_{left}}{\dfrac{\forall x.\, P(x) \vee \exists x.\, \neg P(x) \vdash \exists x.\forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}{\vdash \exists x.\forall y.\, [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]}\,\text{cut}}\,\vee_{left}
$$

3. Eliminate cuts in the proof of Exercise 1. question 1.a). (no need to detail the procedure).

But I will detail the procedure here. The original proof was:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \phi_a}\,\text{ax}}{\vdash P(x), \neg P(x), \phi_a}\,\neg_{right}}{\vdash P(x), \exists x.\, \neg P(x), \phi_a}\,\exists_{right}}{\vdash \forall x.\, P(x), \exists x.\, \neg P(x), \phi_a}\,\forall_{right}}{\vdash \forall x.\, P(x) \vee \exists x.\, \neg P(x), \phi_a}\,\vee_{right} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)}\,\text{ax}}{\forall x.\, P(x) \vdash P(x) \Rightarrow \forall y.\, P(y)}\,\Rightarrow_{right}}{\forall x.\, P(x) \vdash \phi_a}\,\exists_{right} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash \forall y.\, P(y), P(x)}\,\text{ax}}{\vdash P(x) \Rightarrow \forall y.\, P(y), P(x)}\,\neg_{left}}{\neg P(x) \vdash P(x) \Rightarrow \forall y.\, P(y)}\,\neg_{left}}{\neg P(x) \vdash \phi_a}\,\exists_{right}}{\exists x.\, \neg P(x) \vdash \phi_a}\,\exists_{left}}{\forall x.\, P(x) \vee \exists x.\, \neg P(x) \vdash \phi_a}\,\vee_{left}}{\vdash \phi_a}\,\text{cut}
$$

Here, we are directly in situation (a). We reduce the cut into two smaller cuts:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \phi_a}\,\text{ax}}{\vdash P(x), \neg P(x), \phi_a}\,\neg_{right}}{\vdash P(x), \exists x.\, \neg P(x), \phi_a}\,\exists_{right}}{\vdash \forall x.\, P(x), \exists x.\, \neg P(x), \phi_a}\,\forall_{right} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash \forall x.\, P(x), \forall y.\, P(y), P(x)}\,\text{ax}}{\vdash \forall x.\, P(x), P(x) \Rightarrow \forall y.\, P(y), P(x)}\,\Rightarrow_{right}}{\neg P(x) \vdash \forall x.\, P(x), P(x) \Rightarrow \forall y.\, P(y)}\,\neg_{left}}{\neg P(x) \vdash \forall x.\, P(x), \phi_a}\,\exists_{right}}{\exists x.\, \neg P(x) \vdash \forall x.\, P(x), \phi_a}\,\exists_{left}}{\vdash \forall x.\, P(x), \phi_a}\,\text{cut} \quad \dfrac{\dfrac{\dfrac{}{\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)}\,\text{ax}}{\forall x.\, P(x) \vdash P(x) \Rightarrow \forall y.\, P(y)}\,\Rightarrow_{right}}{\forall x.\, P(x) \vdash \phi_a}\,\exists_{right}}{\vdash \phi_a}\,\text{cut}
$$

We start by eliminating the highest cut. We are in situation (b), we make the cut go above the $\forall_{right}$, using a contraction:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \phi_a}\,\text{ax}}{\vdash P(x), \neg P(x), \phi_a}\,\neg_{right}}{\vdash P(x), \exists x.\, \neg P(x), \phi_a}\,\exists_{right} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash \forall x.\, P(x), \forall y.\, P(y), P(x)}\,\text{ax}}{\vdash \forall x.\, P(x), P(x) \Rightarrow \forall y.\, P(y), P(x)}\,\Rightarrow_{right}}{\vdash \forall x.\, P(x), P(x) \Rightarrow \forall y.\, P(y)}\,\neg_{left}}{\neg P(x) \vdash \forall x.\, P(x), \phi_a}\,\exists_{right}}{\exists x.\, \neg P(x) \vdash \forall x.\, P(x), \phi_a}\,\exists_{left}}{\vdash \forall x.\, P(x), P(x), \phi_a}\,\text{cut}}{\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a}\,\forall_{right}}{\vdash \forall x.\, P(x), \phi_a}\,\text{contraction} \quad \dfrac{\dfrac{\dfrac{}{\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)}\,\text{ax}}{\forall x.\, P(x) \vdash P(x) \Rightarrow \forall y.\, P(y)}\,\Rightarrow_{right}}{\forall x.\, P(x) \vdash \phi_a}\,\exists_{right}}{\vdash \phi_a}\,\text{cut}
$$

We are in situation (a), we reduce the cut in a smaller cut:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), P(x), \phi_a$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \neg P(x), \phi_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), \forall y.\, P(y), P(x)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x) \Rightarrow \forall y.\, P(y), P(x)$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\neg P(x) \vdash \forall x.\, P(x), P(x) \Rightarrow \forall y.\, P(y)$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\neg P(x) \vdash \forall x.\, P(x), \phi_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash P(x) \Rightarrow \forall y.\, P(y)$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

We make the cut go above the $\exists_{right}$ rule (b). We write ϕ'_a for $P(x) \Rightarrow \forall y.\, P(y)$:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), P(x), \phi_a, \phi'_a$}
\RightLabel{$\neg_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \neg P(x), \phi_a, \phi'_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), \forall y.\, P(y), \phi_a, P(x), P(x)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi'_a, \phi_a, P(x), P(x)$}
\RightLabel{$\neg_{left}$}
\UnaryInfC{$\neg P(x) \vdash \forall x.\, P(x), \phi'_a, \phi_a, P(x)$}
\RightLabel{cut}
\BinaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a, \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

We reduce the cut in a smaller cut (a):

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), P(x), \phi_a, \phi'_a, P(x)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), \forall y.\, P(y), \phi_a, P(x), P(x)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi'_a, \phi_a, P(x), P(x)$}
\RightLabel{cut}
\BinaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a, \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

We make the cut rule go above the $\Rightarrow_{right}$ rule (b):

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), P(x), \phi_a, \phi'_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), \forall y.\, P(y), \phi_a, P(x)$}
\RightLabel{cut}
\BinaryInfC{$P(x) \vdash \forall x.\, P(x), P(x), \phi_a, \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi'_a, \phi_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

Finally we replace the cut by an axiom:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash \forall x.\, P(x), P(x), \phi_a, \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi'_a, \phi_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi_a$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

We are in situation (b), we need a contraction and we make the cut go above the $\exists_{right}$ rule.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(z) \vdash \forall x.\, P(x), P(z), \phi'_a, \phi_a, \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(x), \phi'_a\{x \to z\}, \phi_a, \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi_a, \phi_a, \phi'_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi_a, \phi'_a$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a, \phi'_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a, \phi'_a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \phi_a, \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\forall x.\, P(x) \vdash \phi_a, \phi'_a$}
\RightLabel{cut}
\BinaryInfC{$\vdash \phi_a, \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

Again, we are (b) with the $\Rightarrow_{right}$ rule:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(z) \vdash \forall x.\, P(x), P(z), \phi'_a, \phi_a, \forall y.\, P(y), , \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi'_a\{x \to z\}, \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi_a, \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a, \phi'_a, \forall y.\, P(y)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\forall x.\, P(x), P(x) \vdash \phi_a, \forall y.\, P(y)$}
\RightLabel{cut}
\BinaryInfC{$P(x) \vdash \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \phi_a, \phi'_a, \phi'_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \phi_a, \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

We are yet again in situation (b), but this time the last rule being an axiom over the cut formula, we finish the transformation with a contraction:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(z) \vdash \forall x.\, P(x), P(z), \phi'_a, \phi_a, \forall y.\, P(y), , \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi'_a\{x \to z\}, \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi_a, \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), P(z), \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$\vdash \forall x.\, P(x), \forall x.\, P(x), \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \forall x.\, P(x), \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{contraction}
\UnaryInfC{$P(x) \vdash \phi_a, \phi'_a, \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash \phi_a, \phi'_a, \phi'_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \phi_a, \phi'_a$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \phi_a, \phi_a$}
\RightLabel{contraction}
\UnaryInfC{$\vdash \phi_a$}
\end{prooftree}
$$

Some of the introduced contractions are not useful, some are. Taking them out gives the following, simpler, proof – this step is not mandatory, but it gives a better view of the proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x), P(y) \vdash P(y), \forall y.\, P(y), \forall y.\, P(y)$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$P(x) \vdash P(y), \forall y.\, P(y), P(y) \Rightarrow (\forall y.\, P(y))$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$P(x) \vdash P(y), \forall y.\, P(y), \exists y.\, (P(y) \Rightarrow (\forall y.\, P(y)))$}
\RightLabel{$\forall_{right}$}
\UnaryInfC{$P(x) \vdash \forall y.\, P(y), \forall y.\, P(y), \exists x.\, (P(x) \Rightarrow (\forall y.\, P(y)))$}
\RightLabel{right contraction}
\UnaryInfC{$P(x) \vdash \forall y.\, P(y), \exists x.\, (P(x) \Rightarrow (\forall y.\, P(y)))$}
\RightLabel{$\Rightarrow_{right}$}
\UnaryInfC{$\vdash P(x) \Rightarrow (\forall y.\, P(y)), \exists x.\, (P(x) \Rightarrow (\forall y.\, P(y)))$}
\RightLabel{$\exists_{right}$}
\UnaryInfC{$\vdash \exists x.\, (P(x) \Rightarrow (\forall y.\, P(y))), \exists x.\, (P(x) \Rightarrow (\forall y.\, P(y)))$}
\RightLabel{right contraction}
\UnaryInfC{$\vdash \exists x.\, (P(x) \Rightarrow (\forall y.\, P(y)))$}
\end{prooftree}
$$