Logique
TD n°7

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In the following, we will write $\Gamma \vdash_{\mathsf{LK}} \Delta$, resp $\Gamma \vdash_{\mathsf{NK}} \phi$ when $\Gamma \vdash \Delta$ is provable in sequent calculus, resp when $\Gamma \vdash \phi$ is provable in natural deduction.

Exercise 1 : LK with cuts

We study the sequent calculus with cuts. In the following, P, Q are unary predicate symbols.

1. Prove formula $\phi_a = \exists x.\, (P(x) \Rightarrow \forall y.\, P(y))$ in sequent calculus.
   Hint: use a cut on $\forall x.\, P(x) \vee \exists x.\, \neg P(x)$.

2. Eliminate cuts in the following proof. Detail the procedure.

   First, let's do a little recap on the cut-elimination procedure, from a practical point of view.
   The goal is to make cuts go up in the proof, or reduce them to cuts over smaller formulas, until the cuts can be replaced directly with structural rules (contraction, axiom). There are three main cases, depending on the first rules of the two proofs above the cut rule we want to eliminate:

   (a) The two rules are applied to the cut formula: we divide the cut in cuts over the subformula(s) of the cut formula. See the course notes to know what to do depending on the form of the cut formula.

   Observation: if both are axiom rules, replace the cut directly with an axiom rule.

   (b) One of the rules is not applied to the cut formula: we move the cut above this rule.

   Observation: if the rule is an axiom, this eliminates the cut and ends the proof with an axiom.

   (c) One is an axiom over the cut formula: we replace the cut with a structural rule (axiom if the second is also an axiom over the cut formula, else a contraction).

$$\dfrac{\dfrac{}{\phi \vdash \phi, \psi}\,\mathrm{ax}}{\phi \vdash \phi \vee \psi}\,\vee_{right} \qquad \dfrac{\dfrac{\dfrac{}{\phi \vdash \psi, \phi}\,\mathrm{ax} \qquad \dfrac{}{\psi \vdash \psi, \phi}\,\mathrm{ax}}{\phi \vee \psi \vdash \psi, \phi}\,\vee_{left}}{\phi \vee \psi \vdash \psi \vee \phi}\,\vee_{right}$$
$$\overline{\qquad\qquad \phi \vdash \psi \vee \phi \qquad\qquad}\,\mathrm{cut}$$

3. Is there a term t such that $\vdash_{\mathsf{LK}} P(t) \Rightarrow \forall y.\, P(y)$?

Exercise 2 : Subformula property

In the previous exercise sheet, we have seen the interpolation theorem. We will now study the *subformula property.*

1. Prove that if $\Gamma \vdash_{\mathsf{LK}} \Delta$, then there is a proof of $\Gamma \vdash \Delta$ containing only formulas of the form $\phi\{x_1 \to t_1, \ldots, x_n \to t_n\}$, where ϕ is a subformula of a formula in Γ, Δ, and $t_1, \ldots, t_n$ are terms (not necessarily appearing in $\Gamma \cup \Delta$). Treat at least structural rules, right and left rules for disjunction and all cases for quantifiers.
2. We call LKP the propositionnal fragment of sequent calculus. It contains all LK rules except for the left and right rules of quantifiers.
   (a) Prove that if $\Gamma \vdash_{\mathsf{LK}} \Delta$ and $\Gamma \cup \Delta$ contains only propositionnal formulas, then there is a proof of $\Gamma \vdash \Delta$ in LKP, using the subformula property.
   (b) Show syntactically that provability in LKP is decidable.

Exercise 3 : LK is equivalent to NK

1. In class, you have proven that if $\Gamma \vdash_{\mathsf{LK}} \Delta$ then $\Gamma, \neg\Delta \vdash_{\mathsf{NK}} \bot$ by induction. Write the cases of $\exists_{left}$, $\exists_{right}$.
2. You have also proven that if $\Gamma \vdash_{\mathsf{NK}} \phi$, then $\Gamma \vdash_{\mathsf{LK}} \phi$. Write the cases of RAA, $\Rightarrow_{elim}$.
3. Here are two natural deduction proofs.

$$\dfrac{\dfrac{}{\neg\neg\neg\phi, \phi \vdash \neg\neg\neg\phi}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\neg\neg\neg\phi, \neg\phi, \phi \vdash \neg\phi}\,\text{ax} \qquad \dfrac{}{\neg\neg\neg\phi, \neg\phi, \phi \vdash \phi}\,\text{ax}}{\neg\neg\neg\phi, \neg\phi, \phi \vdash \bot}\,\neg_{elim}}{\neg\neg\neg\phi, \phi \vdash \neg\neg\phi}\,\neg_{intro}}{\dfrac{\neg\neg\neg\phi, \phi \vdash \bot}{\neg\neg\neg\phi \vdash \neg\phi}\,\neg_{intro}}\,\neg_{elim}$$

(a) First proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \phi}\,\text{ax} \qquad \dfrac{\dfrac{}{\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \neg\psi}\,\text{ax} \qquad \dfrac{}{\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \neg\psi \Rightarrow \neg\phi}\,\text{ax}}{\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \neg\phi}\,\Rightarrow_{elim}}{\neg\psi \Rightarrow \neg\phi, \phi, \neg\psi \vdash \bot}\,\neg_{elim}}{\neg\psi \Rightarrow \neg\phi, \phi \vdash \neg\neg\psi}\,\neg_{intro}}{\neg\psi \Rightarrow \neg\phi, \phi \vdash \psi}\,\text{RAA}}{\neg\psi \Rightarrow \neg\phi \vdash \phi \Rightarrow \psi}\,\Rightarrow_{intro}$$

(b) Second proof

Transform them into sequent calculus proofs (you can just use left rules instead of elimination rules when it works, else use what seen in class).

4. Transform this proof into a natural deduction proof:

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{ax}}{\vdash A, \neg A}\,\neg_{right}}{\vdash A \vee \neg A}\,\vee_{right}$$

5. Prove that if $\Gamma \vdash_{LK} \phi_1, \dots, \phi_n$ then $\Gamma \vdash_{NK} \phi_1 \vee \dots \vee \phi_n$.

Exercise 4 : Exercise 1, ctd

You can use the same cut as in Exercise 1. question 1) to do the two following proofs.

1. Prove $(\forall x.\, P(x) \Rightarrow \exists x.\, Q(x)) \Rightarrow \exists x.\, (P(x) \Rightarrow Q(x))$ in sequent calculus.
2. Prove $\exists x. \forall y. [((P(y) \Rightarrow P(x)) \Rightarrow P(x)) \Rightarrow P(y)]$ in sequent calculus.
3. Eliminate cuts in the proof of Exercise 1. question 1.a). (no need to detail the procedure).