Logique

TD n°3

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February 4, 2021

1 Warm-up

Exercise 1 : Definitions of inconsistency
Prove that the following definitions of the *inconsistency* of a theory $\mathcal{T}$ are equivalent:

(a) There exists A such that A and $\neg A$ are provable in $\mathcal{T}$.
(b) $\bot$ is provable in $\mathcal{T}$.
(c) Every formula is provable in $\mathcal{T}$.

1. Show that if $\mathcal{T}$ is inconsistent, then there exists an inconsistent theory $\mathcal{T}' \subseteq \mathcal{T}$ such that $\mathcal{T}'$ is finite.
2. Is an inconsistent theory complete? decidable?

2 Equality

Exercise 2 : Theory of equality
Let $\mathcal{L}$ be the language with a sort of terms s with a set $\mathcal{F}$ of function symbols and a set $\mathcal{P}$ of predicate symbols containing a binary predicate $=$. The theory of equality $\mathcal{E}$ can be expressed as the following set of formulas:

$$\begin{cases} \forall x \ (x = x) & \\ \forall x_1 \dots \forall x_i \forall x'_i \dots \forall x_n & (x_i = x'_i \Rightarrow \\ & f(x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, x'_i, \dots, x_n)) \\ \forall x_1 \dots \forall x_j \forall x'_j \dots \forall x_m & (x_j = x'_j \Rightarrow \\ & P(x_1, \dots, x_j, \dots, x_m) \Rightarrow P(x_1, \dots, x'_j, \dots, x_m)) \end{cases}$$

for all n-ary function $f \in \mathcal{F}$ and m-ary predicate $P \in \mathcal{P}$.

Write the formulas expressing the symmetry and transitivity of the equality and show that they are provable in this theory.

In all the following exercises, the equality symbols are given the reflexivity axiom and the substitivity scheme $\overline{\forall}\forall x, y \ (x = y \Rightarrow (x/z)A \Rightarrow (y/z)A)$

where z is a free variable of A.

3 Arithmetic

Exercise 3 : Peano arithmetic

We work on the language $\mathcal{L}_{PA}$ with a sort *nat* and function symbols 0 of arity $\langle nat \rangle$ s of arity $\langle nat, nat \rangle$, $+$ and $\times$ of arity $\langle nat, nat, nat \rangle$ (and equality).

The axioms of PA are the axioms of equality and:

$$\forall x \quad 0 + x = x \tag{1}$$
$$\forall x \forall y \quad s(x) + y = s(x + y) \tag{2}$$
$$\forall x \quad 0 \times x = 0 \tag{3}$$
$$\forall x \forall y \quad s(x) \times y = (x \times y) + y \tag{4}$$
$$\forall x \exists y \quad x = 0 \vee x = s(y) \tag{5}$$
$$\forall x \quad \neg s(x) = 0 \tag{6}$$
$$\forall x \forall y \quad s(x) = s(y) \Rightarrow x = y \tag{7}$$

Induction scheme (for every A with free variables included in $x, x_1, \dots, x_n$):

$$\forall x_1 \dots \forall x_n (0/x)A \Rightarrow \forall y.\ ((y/x)A \Rightarrow (s(y)/x)A) \Rightarrow \forall y.\ (y/x)A$$

1. Prove that axiom (5) is superfluous.
2. Prove in PA:
   (a) $\forall x . x + 0 = x$
   (b) $\forall x, y\ (s(x) + y) = s(x + y)$
   (c) $\forall x, y\ (x + y) = (y + x)$
3. Define formulas representing $x \leq y$ and $x < y$.
4. Express in PA that $\leq$ is an order relation. Prove it, or use it freely in the following.
5. Prove in PA, or use freely in the following:
   (a) $\forall x . (x \leq 0 \Rightarrow x = 0)$
   (b) $\forall x, y.\ (y \leq s(x) \Rightarrow y \leq x \vee y = s(x))$
   (c) $\forall x, y.\ (x \leq y \vee y \leq x)$
6. Prove the strong induction scheme, that is for any formula A of free variables $x, x_1, \dots, x_n$:

   $$\forall x_1 \dots \forall x_n (0/x)A \Rightarrow \forall x.\ (\forall y.\ (y \leq x \Rightarrow (y/x)A) \Rightarrow (s(x)/x)A) \Rightarrow \forall y.\ (y/x)A$$

7. ($\star$) We prove that any non empty set of natural numbers definable by a first-order formula has a minimal element, that is for any formula A with a free variable x:

   $$\mathsf{PA} \vdash \exists x.\ A \Rightarrow \exists x.\ (A \wedge \forall y.\ (y < x \Rightarrow \neg (y/x)A))$$

   You can freely use that

   $$\neg \exists x.\ (A \wedge \forall y.\ (y < x \Rightarrow \neg (y/x)A)) \vdash \forall x.\ (A \Rightarrow \exists y.\ (y < x \wedge (y/x)A))$$

4 Set theory

In the following, we define the language $\mathcal{L}_{ZF}$ containing the sort ι and the symbols $\in$ of arity $\langle \iota, \iota, Prop \rangle$. You can use the notations $\cup$ and $\mathcal{P}$ for the union and power set.

Exercise 4 : Russell's paradox

We define naive set theory over $\mathcal{L}_{ZF}$. The theory contains, for each formula A whose free variables are among $x_1, \ldots, x_n, y$, an axiom of the form:

$$\forall x_1 \ldots \forall x_n \exists a \forall y \ (y \in a \Leftrightarrow A)$$

Prove $\forall y (y \in a \Leftrightarrow \neg y \in y) \vdash \bot$.

Exercise 5 : Z

The Z set theory is made of the axioms of equality, and of the following axioms:

Extensionality: $\forall x \forall y \ ((\forall z \ (z \in x \Leftrightarrow z \in y)) \Rightarrow x = y)$

Union: $\forall x \exists z \forall w \ (w \in z \Leftrightarrow (\exists v \ (w \in v \wedge v \in x)))$

Pair: $\forall x \forall y \exists z \forall w \ (w \in z \Leftrightarrow (w = x \vee w = y))$

Power set: $\forall x \exists z \forall w \ (w \in z \Leftrightarrow (\forall v \ (v \in w \Rightarrow v \in x)))$

Restricted comprehension scheme: $\forall x_1 \ldots \forall x_n \forall v \exists w \forall x \ (x \in w \Leftrightarrow x \in v \wedge A)$

for every proposition A of free variables $x, x_1, \ldots, x_n$. The set w will be denoted by $\{ \ x \in v \ : \ A \ \}$.

Infinity axiom: We will define $\emptyset$ in a later question. We define the formula $Succ[x, y] = \forall z. \ (z \in y \Leftrightarrow (z \in x \vee z = y))$. The axiom of infinity is

$$\exists I. \ (\emptyset \in I) \wedge \forall x \forall y. \ ((x \in I \wedge Succ[x, y]) \Rightarrow y \in I)$$

1. Define a proposition $P[x, y]$ representing the inclusion, i.e. $x \subseteq y$ iff $P[x, y]$ provable. Rewrite the **Power set** axiom using this notation. You can use this notation freely.
2. Show that there exists a unique set $\emptyset$ which does not contain any element.
3. State the theorem "For any element x, there is a set containing only x". Which axiom(s) of Z would you use to prove it?
4. State the theorem "The union of any two sets is also a set". Which axiom(s) of Z would you use to prove it?
5. State the theorem "The intersection of any two sets is also a set". Which axiom(s) of Z would you use to prove it?
6. Show that the set of all sets does not exists, that is show that $\neg \exists x \forall y \ (y \in x)$ is provable in Z.

Exercise 6 : ZF

We define theory ZF by replacing the restricted comprehension scheme in theory Z by the **Replacement scheme**:

$$\forall x_1 \dots x_n x, y, y' \ (F[x,y] \wedge F[x,y'] \Rightarrow y = y') \Rightarrow \exists b \forall y \ (y \in b \Leftrightarrow \exists a \ (x \in a \wedge F[x,y]))$$

where F has free variables $x, y, x_1, \dots, x_n$.

1. Show that for all F, $\mathsf{Z} \vdash F$ implies $\mathsf{ZF} \vdash F$, i.e. that we can prove any instance of the **Restricted comprehension scheme** using the **Replacement scheme**.
2. ($\star$) We denote by (a, b) the set $\{ \{ a \} , \{ a , b \} \}$. Prove that this construction satisfies the properties of the couples (or give the necessary axioms):

$$\mathsf{ZF} \vdash \forall a, a', b, b' \ \{ \ (a, b) = (a', b') \Leftrightarrow a = a' \wedge b = b' \ \}$$

3. ($\star$) Prove that for every sets a and b, their cartesian product $a \times b$ exists, that is (or give the necessary axioms):

$$\mathsf{ZF} \vdash \forall a, b \ \exists c \ \forall z \ \{ \ z \in c \Leftrightarrow \exists x, y. \ x \in a \wedge y \in b \wedge z = (x, y) \ \}$$

4. *(open question)* How could we define functions?

Exercise 7 : Von Neumann numbers (open exercise)

Von Neumann numbers are defined by $0 = \emptyset$, $1 = \{ \emptyset \}$, $2 = \{ 0, 1 \}$, $3 = \{ 0, 1, 2 \}$, etc. As a consequence, a set is a natural number if it belongs to every set containing 0 and closed by successor.

1. Write a proposition N with a free variable x and using only symbols $=$ and $\in$ expressing that x is a natural number.

We write $N[t]$ the proposition $(t/x)N$. We observe that all natural numbers belong to the set I defined by the axiom of infinity.

1. Prove the proposition $\exists \mathbb{N} \ (x \in \mathbb{N} \Leftrightarrow N[x])$ in ZF.
2. Write a proposition which expresses the induction principle: if a set contains 0 and is closed by successor, then it contains every natural number. Show that this proposition is provable in the theory ZF.
3. Define the addition and multiplication by their graph. Show that the axioms of Peano's arithmetic is provable in ZF.