# Logique
## Exo n°5

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*Hand over before the end of 1st April*

**Exercise 1 : Total dense orders without borders**

We work over the language $\mathcal{L}$ containing the binary predicate symbols $<$ and $=$.

The theory $\mathcal{T}_O$ is defined with the axioms of equality and:

$$\begin{array}{lrl} (O_1) & \forall x \forall y. & \neg(x < y \wedge y < x) \\ (O_2) & \forall x \forall y \forall z. & x < y \wedge y < z \Rightarrow x < z \\ (O_3) & \forall x \forall y. & x < y \vee x = y \vee y < x \\ (O_4) & \forall x \forall y \exists z. & x < y \Rightarrow x < z \wedge z < y \\ (O_5) & \forall x \exists y. & x < y \\ (O_6) & \forall x \exists y. & y < x \end{array}$$

Models of $\mathcal{T}_O$ are sets with a total, dense order without borders. As a reminder, $\mathcal{T}_0$ is complete and decidable.

1. Show that all models of $\mathcal{T}_0$ satisfy the same closed formulas.
2. Using the fact that $\mathbb{R}$, and $\mathbb{R}^*$ are models of $\mathcal{T}_0$, show that connectedness (connexité) cannot be expressed by a (first-order) formula over $\mathcal{L}$.
3. Let $\mathcal{T}$ be the theory containing the axioms of equality, axioms $O_1, O_2, O_3, O_4, O_6$ and $\neg O_5$. Show that $\mathcal{T}$ is complete and decidable.

   *This proof is very close to the proof of completeness of $\mathcal{T}_0$: identify the step that needs to be changed and give the necessary modifications.*