A few things on Noetherian spaces

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Noetherian spaces

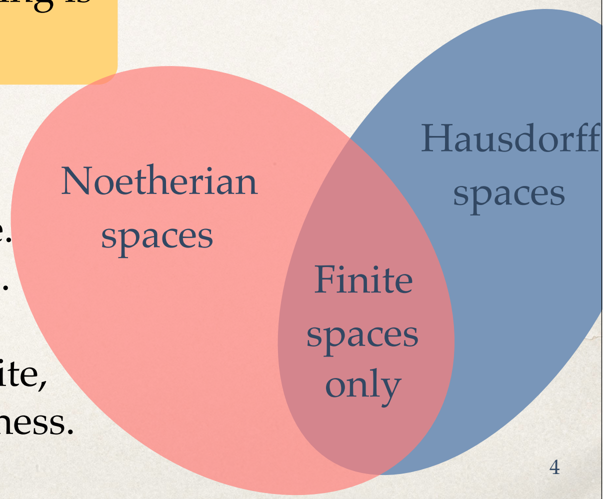
- **Defn.** A space is **Noetherian** iff every open is compact.
- Here compact does not entail any kind of separation.
- **Fact.** The following are equivalent:
  (1) X is Noetherian
  (2) Every subspace of X is compact
  (3) Ascending sequences $U_1 \subseteq U_2 \subseteq ... \subseteq U_n \subseteq ...$ of opens stabilize
  (4) Descending sequences $C_1 \supseteq C_2 \supseteq ... \supseteq C_n \supseteq ...$ of closed sets stabilize
- We shall see other characterizations later.

Outline

- Characterizations of Noetherian spaces (half of them well-known)
- Transfering results from wqo theory to topology
- Applications in software verification
- Representations
- Conclusion

Noetherian spaces, classically

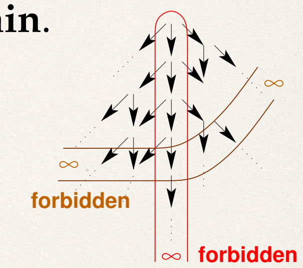
- **Defn.** A space is **Noetherian** iff every open is compact.
- **Prop.** The spectrum of a Noetherian ring is a Noetherian space.
- E.g., the spectrum of a polynomial ring over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.
  Not my first source of inspiration here.
  We shall see many (simpler) examples.
- **Note.** Noetherian + Hausdorff ⇔ finite, so we shall definitely **drop** Hausdorffness.



Well-quasi-orders

- **Fact.** The following are equivalent for a quasi-ordering ≤:
  (1) Every sequence $(x_n)_{n\in\mathbb{N}}$ is **good**: $x_m \le x_n$ for some $m<n$
  (2) Every sequence $(x_n)_{n\in\mathbb{N}}$ is **perfect**: has a monotone subsequence
  (3) ≤ is **well-founded** and has **no infinite antichain**.

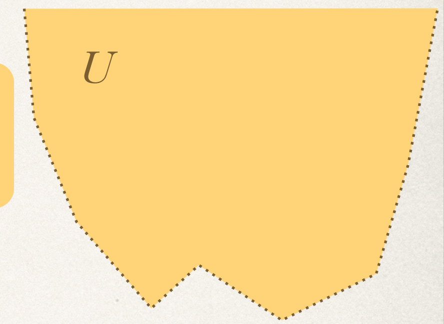
- **Defn.** Such a quasi-ordering ≤ is called a **well-quasi-order (wqo).**



- Applications:
  classification of graphs (Kuratowski, Robertson-Seymour)
  verification (computer science)
  model theory (logic: Fraïssé, Jullien, Pouzet)

The starting observation

- Given a qo $(X, \leq)$, its **Alexandroff topology** has as opens U all upwards-closed subsets of X.



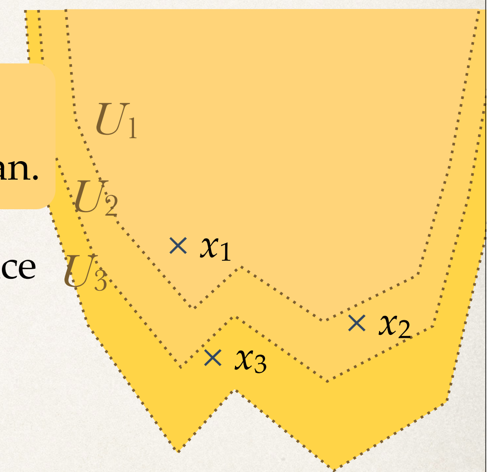
- **Prop.** Let $(X, \leq)$ be wqo.
  With its Alexandroff topology, X is Noetherian.

The starting observation

- Given a qo $(X, \leq)$, its **Alexandroff topology** has as opens U all upwards-closed subsets of X.

- **Prop.** Let $(X, \leq)$ be wqo.
  With its Alexandroff topology, X is Noetherian.

- *Proof*. Consider an infinite ascending sequence
  $U_1 \subsetneq U_2 \subsetneq \ldots \subsetneq U_n \subsetneq \ldots$ of opens.
  Pick x_n in U_n, not in any previous U_m.
  By wqo, $x_m \leq x_n$ for some $m<n$.
  Since $x_m \in U_m$ upwards-closed, $x_n \in U_m$:
  contradiction. □



- Plenty of wqos ⇨ plenty of Noetherian spaces.

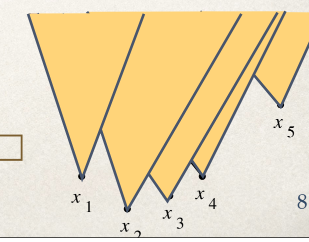
Noetherian + Alexandroff

- **Prop.** Let $(X, \leq)$ be wqo.
  With its Alexandroff topology, X is Noetherian.

- There are also Noetherian spaces that are not Alexandroff:
  — spectra of rings, with the Zariski topology
  — powersets (see later)

- **Conversely**, the qo sets $(X, \leq)$ that are Noetherian
  in their Alexandroff topology
  are exactly the **wqo** sets.

- *Proof*. From $(x_n)_{n\in\mathbb{N}}$ define $U_n = \uparrow\{x_1, \ldots, x_n\}$.
  This stabilizes at n: $U_{n-1} = U_n$, so $x_n \in \uparrow\{x_1, \ldots, x_{n-1}\}$. □



Basic constructions

- **Prop.** (1) Every **wqo** is Noetherian in its Alexandroff topology
  (2) The **spectrum** of a Noetherian ring is Noetherian
  (3) Finite **products** of Noetherian spaces are Noetherian
  (4) Finite **coproducts** of Noetherian spaces are Noetherian
  (5) **Subspaces** of Noetherian spaces are Noetherian
  (6) Topologies **coarser** than a Noetherian topology are Noetherian
  (7) Continuous **images** of Noetherian spaces are Noetherian
  (in particular, quotients)

- We shall see other constructions that preserve Noetherianness.

- We need additional characterizations of Noetherianness.

Cluster points

- **Prop.** X is Noetherian iff every net $(x_i)_{i \in I}$ *contains* a cluster point x_i.
  (The important point is: the cluster point x_i *belongs* to the net.)

- *Proof.*
  (⇒) If X Noetherian, then subspace $K=\{x_i \mid i \in I\}$ is compact,
  hence $(x_i)_{i \in I}$ has a cluster point **in K.**
  (⇐) Let U be open in X.
  Every net $(x_i)_{i \in I}$ inside U has a cluster point in U, *viz.* some x_i.
  So U is compact. □

- **Note:** in Alexandroff spaces, x_i cluster point means that for some i, cofinally [infinitely] many entries x_j are above x_i. (Take the open $\uparrow x_i$.)
  ... hence all sequences are **good**.

Self-convergent nets

- A net $(x_i)_{i \in I}$ is **self-convergent** iff it converges to **every** x_i.
  (A very much non-Hausdorff notion!)

- **Thm.** X is Noetherian iff every net $(x_i)_{i \in I}$ has a self-convergent subnet.

- *Proof.* (⇒) Let J be $\{i \in I \mid x_i$ is a cluster point of the net$\}$.
  By previous Prop., J is non-empty.
  Check: J is cofinal and directed in I; so $(x_j)_{j \in J}$ is a subnet.
  By Kelley's Theorem, $(x_j)_{j \in J}$ has a further subnet that is an ultranet.
  Check that this ultranet is self-convergent.
  (⇐) Obvious, using previous Prop. □

- In Alexandroff spaces, $(x_i)_{i \in I}$ self-convergent iff eventually monotone
  .... hence all sequences are **perfect**.

Ultrafilters

- A similar characterization (lim $\mathcal{U}$ =set of limits of $\mathcal{U}$):
- **Thm.** X is Noetherian iff every ultrafilter $\mathcal{U}$ is compact: lim $\mathcal{U} \in \mathcal{U}$.
- *Proof*. (⇒) Let U be (open) complement of lim $\mathcal{U}$.
  If lim $\mathcal{U}$ not in $\mathcal{U}$, then U is in $\mathcal{U}$ (ultrafilter).
  Since U is compact, $\mathcal{U}$ has a limit in U.
  So lim $\mathcal{U}$ intersects U: contradiction.
  (⇐) Fix an open U. Let $\mathcal{U}$ be an arbitrary ultrafilter containing U.
  Since lim $\mathcal{U} \in \mathcal{U}$, lim $\mathcal{U} \cap U \in \mathcal{U}$, so lim $\mathcal{U} \cap U \neq \emptyset$.
  Hence $\mathcal{U}$ has a limit in U: U is compact. □

Application: finite products

- Well-known: finite products of Noetherian spaces are Noetherian. Here is a simple proof.
  (Warning: I'm lying a bit about what a subnet is.)

- Let X, Y be Noetherian.
  Let $(x_i, y_i)_{i \in I}$ be a net in $X \times Y$.
  Extract a self-convergent subnet $(x_j)_{j \in J}$.
  From $(y_j)_{j \in J}$ extract a further self-convergent subnet $(y_k)_{k \in K}$.
  Then $(x_k, y_k)_{k \in K}$ is a self-convergent subnet of the original net.

- This is a topological version of the Ramsey argument behind the classical wqo proofs.

Stone duality in a nutshell

- There is a functor **O** : **Top** → **Frame**^{op} that:
  — maps each space X to its frame **O**X of opens
  — maps f:X → Y to **O**f : **O**Y → **O**X : $V \mapsto f^{-1}(V)$.

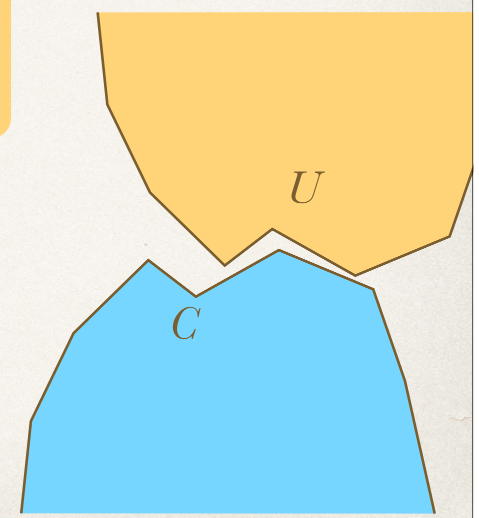
- **O** is left-adjoint to a functor **pt** : **Frame**^{op} → **Top**.

- **S**=**pt O** is the **sobrification** functor.

- **Defn.** A space X is **sober** iff it is of the form **pt** L for some frame L
  iff it is of the form **S**Y for some space Y
  iff X=**S**X (all that, up to iso.)

The specialization quasi-ordering

- **Defn** (specialization, $\leq$). In a space X, $x \leq y$
  iff every open containing x also contains y
  iff $x \in \mathrm{cl}(\{y\})$.
- X is T_0 iff $\leq$ is antisymmetric (an ordering).
- Every open U is upwards-closed.
  Every closed set C is downwards-closed.
- The closure $\mathrm{cl}(\{x\})$ is $\downarrow x = \{z \mid z \leq x\}$.



Sober spaces

- Call C **irreducible closed** iff closed and:
  if $C \subseteq \cup_{i=1}^{n} C_i$ then $C \subseteq C_i$ for some i.
  E.g., $\downarrow x = \mathrm{cl}(\{x\})$ is irreducible closed, for every point x.
- **Fact.** X is sober iff T_0 and all irreducible closed sets are of this form.
- All Hausdorff spaces are sober, but there are more (e.g., continuous and quasi-continuous dcpos in domain theory).

Sobrification

- The sobrification functor can be described more concisely as:
  — $\mathbf{S}X$ = {irreducible closed subsets of X}
  Opens $\Diamond U = \{C \mid C \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$
  — For $f : X \to Y$ to, $\mathbf{S}f : \mathbf{S}X \to \mathbf{S}Y : C \mapsto \mathrm{cl}(f(C))$.
  — X **embeds** into $\mathbf{S}X$ through $\eta : X \to \mathbf{S}X : x \mapsto \downarrow x$.

- **Fact.** X is Noetherian iff $\mathbf{S}X$ is Noetherian.

- *Proof*. $\Diamond : \mathbf{O}X \to \mathbf{OS}X$ iso, and Noetherianness is a property of opens (ascending sequences of opens stabilize). □

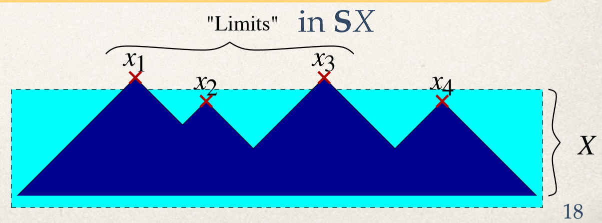
- **Fact.** The Noetherian sober spaces X are the Stone duals $\mathbf{pt}\, L$ of distributive lattices L with the ascending chain condition.

Sober Noetherian spaces

- An **order-theoretic** characterization.
  Call sets of the form $\downarrow\{x_1, \ldots, x_n\}$ **finitary**.

- **Thm.** A sober space X is Noetherian iff:
  (1) $\leq$ is **well-founded**, and
  (2) the set of lower bounds of any finite set is **finitary**.
  Then:
  (3) The topology of X is the upper topology of $\leq$
  (4) **The closed sets are the finitary sets**.

- *Proof*. Folklore,
  or see (JGL 2013)
  or (Dickmann,Schwartz&Tressl). □



Outline

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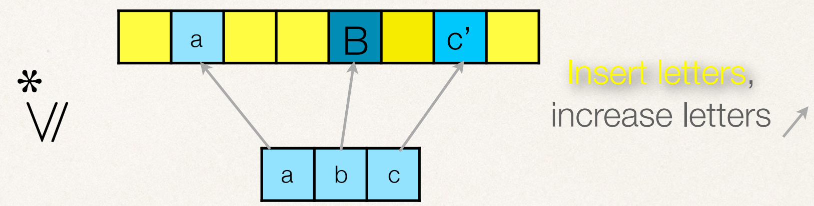
Higman’s Lemma



- **Lemma** (Higman 1954). If $X, \leq$ is wqo, then so is the qoset X^* of finite words under **subword** relation $\leq^*$.
- **Thm** (Topological Higman Lemma, JGL 2013). If X is Noetherian, then so is X^* with the **subword** topology.
- My aim here is to give you a proof of that, imitating Nash-Williams’ classic proof (1963) of Higman’s Lemma.

The subword topology

- **Defn.** For opens $U_1, U_2, \ldots, U_n$ in X, let $[U_1 U_2 \ldots U_n]$ be set of words
  $\ldots a_1 \ldots a_2 \ldots$ (etc.) $\ldots a_n \ldots$
  with $a_1 \in U_1, a_2 \in U_2, \ldots, a_n \in U_n$.
  The **subword topology** on X^* is generated by those sets $[U_1 U_2 \ldots U_n]$.
- Specialization quasi-ordering is $\leq^*$
- If X is Alexandroff, then subword topology=Alexandroff on X^*,
  so Higman's Lemma is a special case
  of the topological Higman Lemma.

Bad sequences

- Let X be a topological space, with subbase $\mathcal{B}$.
  A **bad sequence** is a sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that no U_n is included in a union $\bigcup_{m<n} U_n$ of previous elements.

- **Lemma.** If X is not Noetherian, then (whatever the subbase) it has a bad sequence.

- *Proof*. Let U be non-compact open.
  By Alexander's Subbase Lemma, U has a cover $(U_i)_{i\in I}$ by elements of $\mathcal{B}$ that has no finite subcover.
  Pick some U_{i1}. Some point x_1 in U is not in U_{i1}.
  Pick U_{i2} containing x_1. Some point x_2 in U is not in $U_{i1} \cup U_{i2}$.
  Pick U_{i3} containing x_2. Some point ... etc. □

Minimal bad sequences

- Assume additionally a well-founded $\sqsubseteq$ ordering on $\mathcal{B}$.
  A **bad sequence** $(U_n)_{n\in\mathbb{N}}$ is **minimal** iff every $\sqsubseteq$-lexicographically smaller sequence $(V_n)_{n\in\mathbb{N}}$ is **good** (i.e., not bad).
  [i.e., $V_0=U_0$, $V_1=U_1$, ..., $V_{n-1}=U_{n-1}$, and $V_n \sqsubset U_n$ (strictly) for some n]

- **Lemma.** If X is not Noetherian, then (whatever $\mathcal{B}$ and $\sqsubseteq$) it has a minimal bad sequence.

- *Proof.* Find U_0 $\sqsubseteq$-minimal so that it starts a bad sequence.
  Given U_0, find U_1 $\sqsubseteq$-minimal so that U_0, U_1 start a bad sequence.
  Given U_0, U_1, find U_2 $\sqsubseteq$-minimal so that ... etc. □

- **Note:** Similar to wqos, where bad sequences are sequences of **points**.

Proof plan

- On X^*, let $\mathcal{B}$ consist of the subbasic opens $[U_1\ U_2 \ldots\ U_m]$.
  Let $[U_1\ U_2 \ldots\ U_m] \sqsubseteq [V_1\ V_2 \ldots\ V_n]$ iff
  there is a (strictly) increasing map $f:\{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$
  such that $U_k = V_{f(k)}$.
  (I.e., the V_ps are obtained by inserting new opens in the list of U_ks.)

- If X^* is not Noetherian, then extract some minimal bad sequence.

- Using the zoom-in Lemma (next slide), find a smaller sequence: that one must be good, leading to a contradiction.

The zoom-in Lemma

- **Lemma.** Let X be Noetherian, and $a_n \in U_n$ open for each $n \in \mathbb{N}$. There is a subsequence $(a_{n(k)})_k$ s.t. $a_{n(k)} \in U_{n(0)} \cap \ldots \cap U_{n(k)}$ for every k.

- *Proof*. Pick a cluster point $a_{n(0)}$ (inside the sequence itself).

  Infinitely many a_ns with $n > n(0)$ are in $U_{n(0)}$, forming a subsequence. Pick a cluster point $a_{n(1)}$ from that subsequence.

  Infinitely many a_ns with $n > n(1)$ from that subsequence are in $U_{n(0)} \cap U_{n(1)}$, forming a sub-subsequence. Pick a cluster point $a_{n(2)}$ from that sub-subsequence... etc. □

Proving the topological Higman Lemma

- **Thm** (Topological Higman Lemma, JGL 2013). If X is Noetherian, then so is X^* with the **subword** topology.

- *Proof* (1/3).
  Imagine X^* is not Noetherian, and let
  $\mathcal{U}_n = [U_{n1}\ U_{n2} \ldots U_{nm} \ldots]$ form a minimal bad sequence.
  Pick a word w_n in $\mathcal{U}_n$ that is in no previous $\mathcal{U}_m$.
  Let $\mathcal{R}_n = [U_{n2} \ldots U_{nm} \ldots]$ be « $\mathcal{U}_n$ without its first open U_{n1}».

  By definition, $w_n = l_n\ a_n\ r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  By zoom-in, extract $(a_{n(k)})_k$ s.t. $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$ for every k.
  By minimality, $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_{n(0)-1}, \mathcal{R}_{n(0)}, \mathcal{R}_{n(1)}, \ldots$ is **good**. So, for some k:

$$\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)-1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k-1)}.$$

Proving the topological Higman Lemma

- **Thm** (Topological Higman Lemma, JGL 2013). If X is Noetherian, then so is X^* with the **subword** topology.

- *Proof* (2/3). Recall: $\mathcal{U}_n = [U_{n1}\ U_{n2} \ldots U_{nm} \ldots]$, w_n in $\mathcal{U}_n$, in no previous $\mathcal{U}_m$.
  $\mathcal{R}_n = [U_{n2} \ldots U_{nm} \ldots]$
  $w_n = l_n\ a_n\ r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$
  $\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)-1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k-1)}$.

- Note $r_{n(k)} \in \mathcal{R}_{n(k)}$.

Proving the topological Higman Lemma

- **Thm** (Topological Higman Lemma, JGL 2013). If X is Noetherian, then so is X^* with the **subword** topology.

- *Proof* (2/3). Recall: $\mathcal{U}_n = [U_{n1}\ U_{n2} \ldots\ U_{nm} \ldots]$, w_n in $\mathcal{U}_n$, in no previous $\mathcal{U}_m$.
  $\mathcal{R}_n = [U_{n2} \ldots\ U_{nm} \ldots]$
  $w_n = l_n\ a_n\ r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$
  $\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)\text{-}1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k\text{-}1)}$.

- Note $r_{n(k)} \in \mathcal{R}_{n(k)}$. **Case 1:** $r_{n(k)} \in \mathcal{U}_m$ for some m in 0, ..., $n(0)$-1.
  Then the larger $w_{n(k)}$ is in $\mathcal{U}_m$, too. (Opens are upwards-closed.)
  Impossible since $\mathcal{U}_m$ is previous ($m < n(k)$).

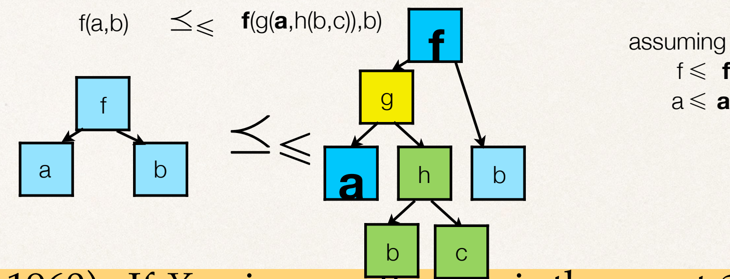
Proving the topological Higman Lemma

- **Thm** (Topological Higman Lemma, JGL 2013). If X is Noetherian, then so is X^* with the **subword** topology.

- *Proof* (3/3). Recall: $\mathcal{U}_n = [U_{n1}\, U_{n2} \ldots U_{nm} \ldots]$, w_n in $\mathcal{U}_n$, in no previous $\mathcal{U}_m$.
  $\mathcal{R}_n = [U_{n2} \ldots U_{nm} \ldots]$
  $w_n = l_n\, a_n\, r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$
  $\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)\text{-}1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k\text{-}1)}$.

- Note $r_{n(k)} \in \mathcal{R}_{n(k)}$. **Case 2:** $r_{n(k)} \in \mathcal{R}_{n(j)}$ for some j in 0, ..., k-1.
  Note that $a_{n(k)} \in U_{n(j)1}$.
  Hence $w_{n(k)} = l_{n(k)}\, a_{n(k)}\, r_{n(k)}$ is in $\mathcal{U}_{n(j)}$, too.
  Impossible since $\mathcal{U}_{n(j)}$ is previous ($j<k$). □

Kruskal's Theorem



- **Thm** (Kruskal 1960). If $X, \leq$ is wqo, then so is the qoset $\mathcal{T}(X)$ of finite trees labeled by X under **homeomorphic embedding** relation $\preceq_{\leq}$.
- **Thm** (Topological Kruskal Theorem, JGL 2013). If X is Noetherian, then so is $\mathcal{T}(X)$ with the **tree** topology.
- Admitted. Slightly more complex.

Powersets

- Let $\mathbb{P}(X)$ come with the lower Vietoris topology, with subbase $\Diamond U = \{A \mid A \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.
- **Thm** (JGL, 2007). If X is Noetherian, then so is $\mathbb{P}(X)$.
- *Proof*. If $\mathbb{P}(X)$ not Noetherian, let $(\Diamond U_n)_{n \in \mathbb{N}}$ be a bad sequence:

  no $\Diamond U_n$ is included in $\bigcup_{m<n} \Diamond U_m = \Diamond \bigcup_{m<n} U_m$.

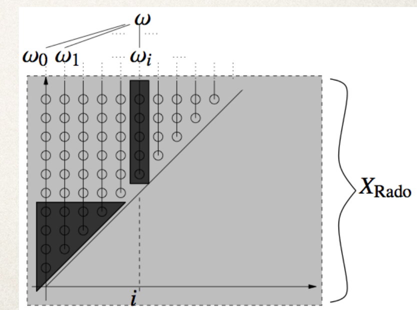
  Since $\Diamond$ is monotonic, no U_n is included in $\bigcup_{m<n} U_m$.
  Therefore $(U_n)_{n \in \mathbb{N}}$ is bad: contradiction. □
- Specialization qo: $A \leq^\flat B$ iff every $a \in A$ is below some $b \in B$.

Powersets, or: beyond wqos

- Let $\mathbb{P}(X)$ come with the lower Vietoris topology, with subbase $\Diamond U = \{A \mid A \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.
- **Thm** (JGL, 2007). If X is Noetherian, then so is $\mathbb{P}(X)$.
- Specialization qo: $A \leq^\flat B$ iff every $a \in A$ is below some $b \in B$.

  Pretty remarkable, since:
- **Prop** (Rado, 1957). There are wqos $X, \leq$ such that $\mathbb{P}(X), \leq^\flat$ is **not** wqo.

  $X_{\text{Rado}} = \{(i,j) \mid i<j\}$, $(i,j) \sqsubseteq (k,l)$ iff $i=k$ and $j \leq l$ or $j<k$.



A catalogue of Noetherian spaces

D	$::=$	A	finite poset
	$\mid$	$\mathbb{N}$	natural numbers
	$\mid$	$D_1 \times D_2 \times \ldots \times D_n$	products
	$\mid$	$D_1 + D_2 + \ldots + D_n$	sums
	$\mid$	$\mathcal{S}(D)$	sobrification
	$\mid$	$\mathbb{P}(D)$	powerset
	$\mid$	$\mathbb{P}^*(D)$	non-empty powerset
	$\mid$	$\mathcal{H}(D)$	extended Hoare powerspace
	$\mid$	$\mathcal{H}_\emptyset(D)$	Hoare powerspace
	$\mid$	$\mathrm{Spec}(R)$	spectrum of a ring
	$\mid$	$\mathbb{C}^k$	complex vector space (Zariski)
	$\mid$	D^*	words / embedding
	$\mid$	$D^{\circledast}$	multisets
	$\mid$	$\rhd_{n=1}^{+\infty} D_n$	words / prefix
	$\mid$	$\mathcal{T}(D)$	trees / embedding

Noetherian, not wqo

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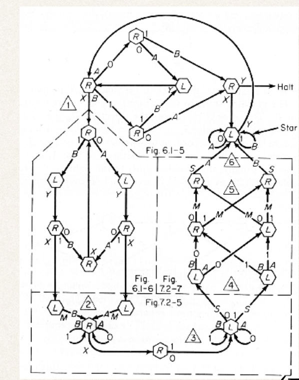
Verification

- How do you ensure a software/hardware system **Sys** is **correct**?
- **Testing**: fine and useful, but not exhaustive
- **Verification**: given a desirable property **P**, check that
  $\mathbf{Sys} \models \mathbf{P}$
- That check should be done by an **algorithm**.

Sys



$\models \Box\,(E \Leftrightarrow F)$

P

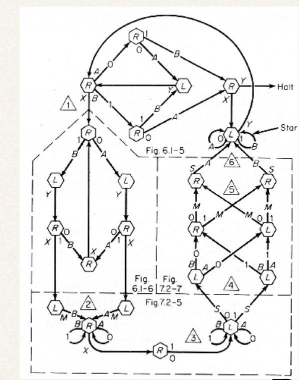
Verification

- A paradigmatic case is given by:
  — **Sys** is a transition system
  (a directed graph, vertices=states)
  — **P** is a (non-)reachability property
  «can **Sys** evolve from an initial state *s*
  to a state in the set **Bad**?»

- Verification is **undecidable** in general.
  Decidable for **Sys** finite.
  But most systems are infinite.

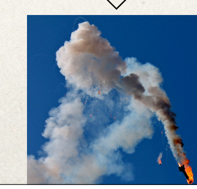
- Classes of infinite-state systems/properties
  for which verification would be decidable?

Sys



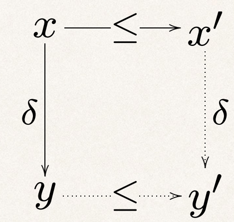
$\models \Diamond$**Bad**

Bad=



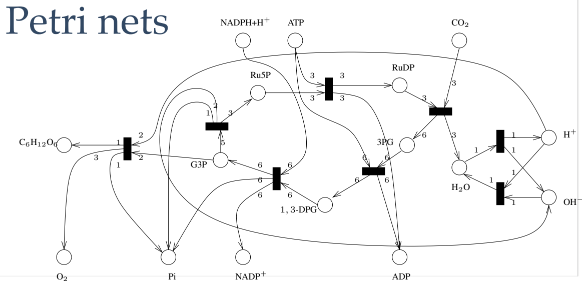
Well-Structured Transition Systems (WSTS)

- A transition system: state space *X*, transition relation $\xrightarrow{\delta}$
- with a **wqo** ≤
- satisfying **monotonicity**

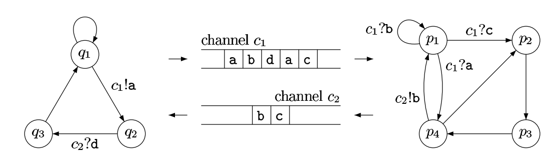


(Finkel 1990,
Abdulla,Čerāns,
Jonsson&Tsay 2000,
Finkel&Schnoebelen 2001)

Petri nets



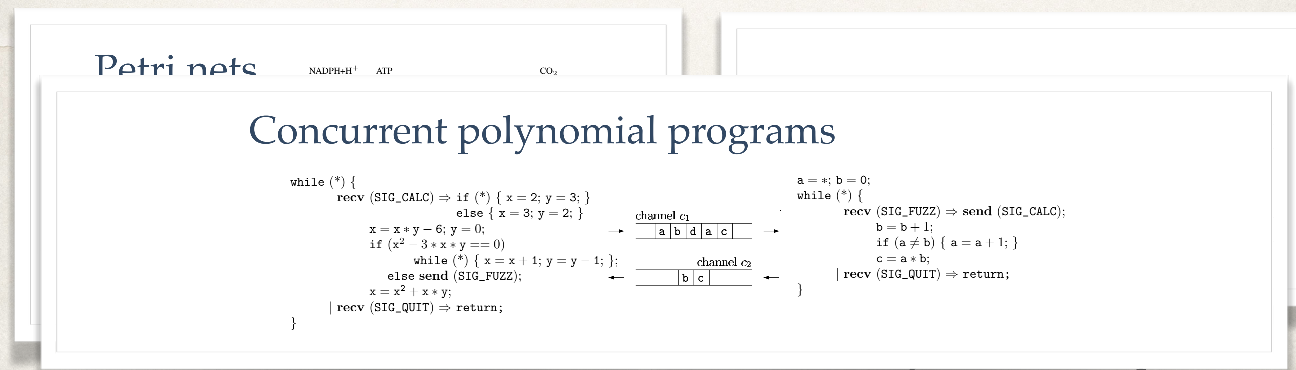
Lossy channel systems



... and many other examples

Topological WSTS

- A transition system: state space X, transition relation $\xrightarrow{\delta}$
- with a **Noetherian** topology
- satisfying **lower semi-continuity**: for every open U, $\delta^{-1}(U)$ open.



The standard backward algorithm

- **Defn** (coverability).
  INPUT: state s, and open state set **Bad**
  QUESTION: $s \rightarrow_\delta^* \textbf{Bad}$?

- **Prop.** Given an effective topological WSTS, coverability is **decidable**.

- *Proof*. The function `pre*` computes
  $U_0 = \textbf{Bad}$, $U_{n+1} = U_n \cup \delta^{-1}(U_n)$.
  This **terminates** because $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n \subseteq ..$ stabilizes (Noetherianness).
  At the end, $U_n = \{s \mid s \rightarrow_\delta^* \textbf{Bad}\}$. □

```
fun pre* U =
    let V = pre U
    in
       if V⊆U
          then U
       else pre* (U ∪ V)
    end;

fun coverability (s, bad) =
    s in pre* (bad);
```

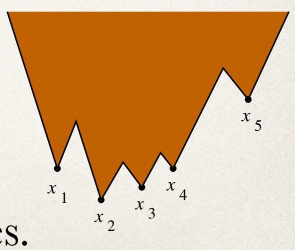
(Don't be fooled by the simplicity of the algorithm: complexity is not even primitive recursive in general
—and I mean the complexity of the problem, independently of the algorithm.)

Effective?

- By an **effective** topological WSTS, we mean one where:
  — opens U are **representable** by some data structure
  — the inclusion test $U \subseteq V$ is **decidable**
  — one-step predecessors $\delta^{-1}(U)$ of open sets are **computable**.

- When the Noetherian state space is Alexandroff (a wqo), there is a standard representation of open (upwards-closed) sets:

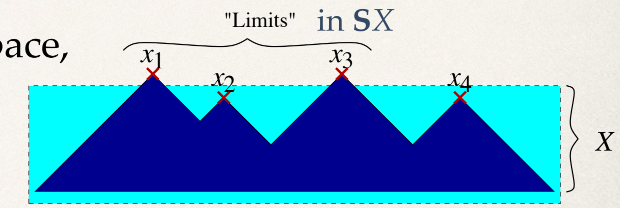
- **Prop.** In a wqo, every upwards-closed subset is the upward closure $\uparrow\{x_1, \ldots, x_n\}$ of finitely many points.



- No longer true in more general Noetherian spaces.

Sobrification and closed sets

- Recall that, in a sober Noetherian space, every closed set C is **finitary**:
  $C = \downarrow \{x_1, \ldots, x_n\}$.



- Provides alternate representation of open sets U in Noetherian X:
  — U is also open in larger space **S**X
  — represent U by its closed complement in **S**X,
  — i.e., by finite sets of **points** in **S**X.

- ⇨ Find computable **representations** of **points** of **S**X.

Outline

- Characterizations of Noetherian spaces (half of them well-known)
- Transfering results from wqo theory to topology
- Applications in software verification
- Representations
- Conclusion

A simple case

- Consider $\mathbb{N}$ with the (Alexandroff topology of) its ordering $\leq$.
- Its closed subsets are $\emptyset$, $[n]=\{0,1,\ldots,n\}$ and the whole of $\mathbb{N}$.
- All except $\emptyset$ are irreducible. Hence:
- **Prop.** A representation for $\mathbf{S}\mathbb{N}$ is $\mathbb{N}_\omega$, i.e., $\mathbb{N}$ plus a top element ω.
- This is an effective representation.
- I'm not giving the topology on $\mathbb{N}_\omega$:
  this must be the upper topology of its ordering.

Products

- Another simple case.
  We know that $\mathbf{S}(\prod_i X_i) = \prod_i \mathbf{S}X_i$, up to iso (R.-E. Hoffmann 1979).
  Hence:

- **Prop.** A representation of $\mathbf{S}(X_1 \times \ldots \times X_n)$ is
  the Cartesian product of representations for $\mathbf{S}X_i$.

- This is effective,
  provided the representations for $\mathbf{S}X_i$ are.

Words and regular expressions

- **Thm** (Finkel&JGL 2009). A representation for $\mathbf{S}(X^*)$ is the space of **word products**, i.e., **regular expressions** of the form:

$$R_1\ R_2\ \dots\ R_n$$

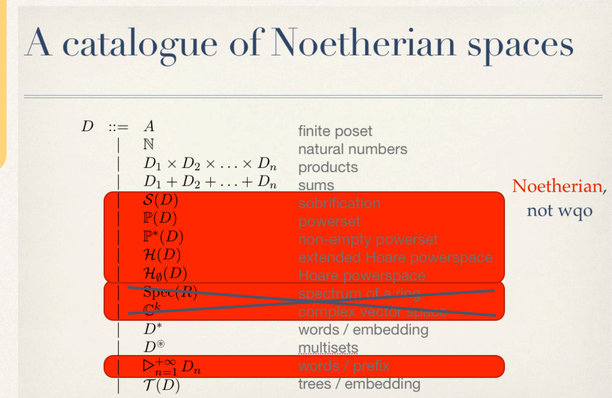
where each R_i is of the form:
— $(\downarrow a)^?$ with $a \in \mathbf{S}(X)$, or
— $(\downarrow \{a_1, \dots, a_k\})^*$ with $a_1, \dots, a_k \in \mathbf{S}(X)$

- Proof omitted. Again, effectivity is preserved. Was already known for wqos (Kabil&Pouzet 1992).

- Embedding $X^* \to \mathbf{S}(X^*)$ maps $a_1 a_2 \dots a_n$ to $(\downarrow a_1)^?\ (\downarrow a_2)^?\ \dots\ (\downarrow a_n)^?$
Limit elements include $(\downarrow a_1)^?\ (\downarrow \{a_2, a_3\})^*\ (\downarrow a_1)^?$, for example.

Representations

- **Thm** (Finkel&JGL, unpublished).
  For **all** the spaces X in our catalogue of Noetherian spaces,
  $\mathbf{S}X$ has effective representations.



- ...

Representations

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  For **all** the spaces X in our catalogue of Noetherian spaces,
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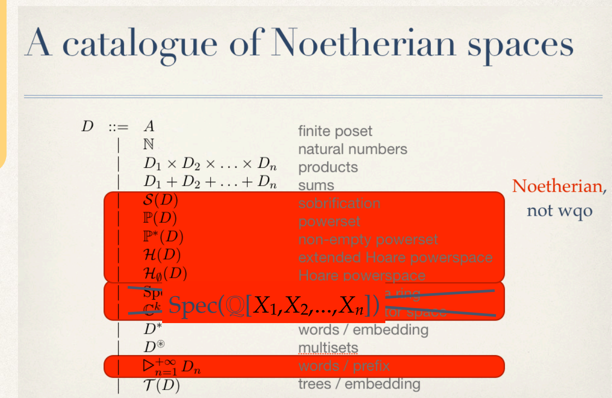
- ... up to a small change: replace Spec(R) and $\mathbb{C}^k$ (Zariski) with some concrete spectrum, say, Spec($\mathbb{Q}[X_1,X_2,\ldots,X_n]$).



Representations

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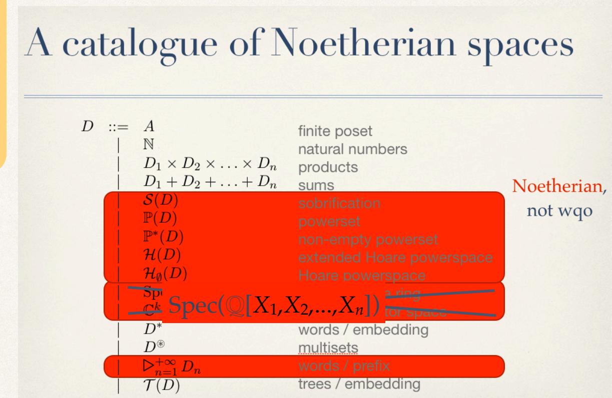
- Including (infinite) powersets!



Representing powersets?

- Recall $\mathbb{P}(X)$ has subbase $\Diamond U = \{A \mid A \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.
  Let $\mathbb{F}(Y)$ be the set of **finitary** subsets of Y, with subspace topology.

- **Prop.** $\mathbf{S}(\mathbb{P}(X)) = \mathbb{F}(\mathbf{S}X)$, up to iso.
  Hence a representation for $\mathbf{S}(\mathbb{P}(X))$ is given by finite sets [antichains] of elements from a representation of $\mathbf{S}X$.

- *Proof*. Let $\mathbf{H}(X)$ be subspace of closed subsets of X.
  $\mathrm{cl} : \mathbb{P}(X) \to \mathbf{H}(X)$ is a quasi-iso, i.e. $\mathbf{O}(\mathrm{cl}) : \mathbf{OH}(X) \to \mathbf{O}\mathbb{P}(X)$ is iso.
  (Exercise: use $A \cap U \neq \emptyset$ iff $\mathrm{cl}(A) \cap U \neq \emptyset$.)
  Hence $\mathbf{S}(\mathbb{P}(X)) = \mathbf{S}(\mathbf{H}(X))$.
  But $\mathbf{H}(X)$ is always sober (Schalk93): so $\mathbf{S}(\mathbb{P}(X)) = \mathbf{H}(X)$.
  Since $\mathbf{O}X = \mathbf{OS}X$, $\mathbf{H}(X) = \mathbf{H}(\mathbf{S}(X))$: so $\mathbf{S}(\mathbb{P}(X)) = \mathbf{H}(\mathbf{S}(X))$.
  All elements of $\mathbf{H}(\mathbf{S}(X))$ are finitary: so $\mathbf{S}(\mathbb{P}(X)) = \mathbb{F}(\mathbf{S}X)$. $\square$

Outline

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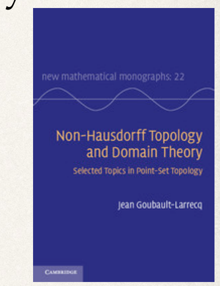
Conclusion

- There is more to Noetherian spaces than algebraic geometry.
  A deep connection with **wqos**.
  Basic theory in (JGL 2013), Section 9.7.



- Any topological analogue of **better** quasi-orderings?

- Any topological analogue of the Robertson-Seymour theorem (for undirected finite graphs with labels in a Noetherian space)?

- Any topological analogue of the theory of **maximal order types** of wqos? (Hint: ordinal height of $\mathbf{H}(X)$, of $\mathbf{S}X$.) Application to complexity of WSTS verification tasks (à la Schnoebelen, Schmitz, Halfon).