A SEMANTICS FOR ∇

Jean Goubault-Larrecq
ENS Paris-Saclay

DaleFest, Dec. 15-16, 2016

HOW I CAME TO KNOW DALE

- ~1990, I knew of Dale's:
  - expansion tree proofs
  - uniform proofs
  - higher-order patterns
  - λ-Prolog
  - etc.
- but we had never got in touch.

375

EXPANSION TREE PROOFS AND THEIR CONVERSION TO NATURAL DEDUCTION PROOFS*

Dale A. Miller
Department of Computer and Information Science
University of Pennsylvania
Philadelphia, PA 19104

Abstract: We present a new form of Herbrand's theorem which is centered around structures called expansion trees. Such trees contains substitution formulas and selected (critical) variables at various non-terminal nodes. These trees encode a shallow formula and a deep formula — the latter containing the formulas which label the terminal nodes of the expansion tree. If a certain relation among the selected variables of an expansion tree is acyclic and if the deep formula of the tree is tautologous, then we say that the expansion tree is a special kind of proof, called an ET-proof, of its shallow formula. Because ET-proofs are sufficiently simple and general (expansion trees are, in a sense, generalized formulas), they can be used in the context of not only first-order logic but also a version of higher-order logic which properly contains first-order logic. Since the computational logic literature has seldomly dealt with the nature of proofs in higher-order logic, our investigation of ET-proofs will be done entirely in this setting. It can be shown that a formula has an ET-proof if and only if that formula is a theorem of higher-order logic. Expansion trees have several pleasing practical and theoretical properties. To demonstrate this fact, we use ET-proofs to extend and complete Andrews' procedure [4] for automatically constructing natural deductions proofs. We shall also show how to use a mating for an ET-proof's tautologous, deep formula to provide this procedure with the "look ahead" needed to determine if certain lines are unnecessary to prove other lines and when and how backchaining can be done. The resulting natural deduction proofs are generally much shorter and more readable than proofs build without using this mating information. This conversion process works without needing any search. Details omitted in this paper can be found in the author's dissertation [16].

Key Words: Higher-order Logic, Expansion Trees, ET-proofs, Natural Deduction, Matings.

1. Introduction

Problem solving in mathematics involves many different kinds of reasoning processes: about propositional connectives, about individual objects in a given domain, about equality and order relations, about sets and functions, and, among a host of others, the more exotic reasoning by example, analogy, etc. Approaches to theorem proving have generally focused on studying the first three of these reasoning processes. Reasoning of the more exotic kinds have also been studied by various artificial intelligence researchers. Although logics based on the ability to reason about sets and functions (higher-order logics) have been studied (see [1, 2, 8, 11, 12, 14, 18, 19, 20, 22]), until very recently few implementations of theorem provers in such logics have been described in the literature.

* This work was supported by NSF grant MCS81-02870.

HOW I CAME TO KNOW DALE

- February 14th, 2002

De Jean Goubault-Larrecq
Sujet **pi-calculus as a theory in linear logic**
Pour dale@cse.psu.edu
Copie à roger@lsv.ens-cachan.fr

```
        Dear Dale,

        I was recently reading your 1992 paper
"The pi-calculus as a theory in linear logic",
and I was wondering whether you had pursued this
line of research, in particular for the pi-calculus.
```

- The same day (we had never met):

```
There are two lines that seems natural to follow here.  One where you
map processes to terms and one where you map them to formulas (of, say,
linear logic).  The former seems to work well.  There is that paper you
mentioned about definitions.  A recent paper on my web site discusses an
approach to doing the pi-calculus in that style as well (Encoding
Generic Judgments: Preliminary results).  I'll assume, however, that
this style encoding is not what you are asking about.

If you map processes to logical formulas directly, you have a lot of
exciting things that can happen.   My original efforts (an experiment,
really) failed, however, for at least two reasons (referring to the
paper "The pi-calculus as a theory in linear logic").
```

THE ∇ QUANTIFIER

THE ∇ QUANTIFIER [MILLER, TIU 2005]

- $\nabla x\ .\ F(x)$ meant to say « $F(x)$ holds for **generic** x »
- Solves similar problem as Gabbay and Pitts' И (« new ») quantifier [1999]
- Distinctive features:
  - ∇x applies **at all types**, Иx only to **names**
  - Various obvious equivalences do not hold, e.g.
    - $\nabla x\ .\ \nabla y\ .\ F(x,y) \not\approx \nabla y\ .\ \nabla x\ .\ F(x,y)$
    - $\nabla x\ .\ F \not\approx F$ where x is not free in F

    (except in Abella [Gacek 2008])
- Semantics by Schöpp [2006]…
  where however ∇x applies only at specific base types.

OUR VARIANT OF ∇

- ∇x will quantify **at all types**
- Our semantics will still verify:
  - $\nabla x\ .\ \nabla y\ .\ F(x,y) \not\approx \nabla y\ .\ \nabla x\ .\ F(x,y)$
  - $\nabla x\ .\ F \not\approx F$ where x is not free in F
- But it will enforce the following, valid in Abella, not in Dale and Alwen's original proposal:
  - $\nabla x\ .\ F \approx \nabla y\ .\ F$
    where x, y not free in F.
- Our logic will be classical, not intuitionistic (out of laziness?).

CLASSICAL $FO\lambda^{\nabla}$

$$\frac{}{\Gamma, (\sigma \triangleright \bot) \longrightarrow \Delta}\,(\bot L) \qquad \frac{}{\Gamma, J \longrightarrow J, \Delta}\,(Ax) \qquad \frac{\Gamma \longrightarrow J, \Delta \quad \Gamma', J \longrightarrow \Delta'}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,(Cut)$$

$$\frac{\Gamma, J, J \rightarrow \Delta}{\Gamma, J \rightarrow \Delta}\,(cL) \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma, J \rightarrow \Delta}\,(wL) \qquad \frac{\Gamma \rightarrow \Delta, J, J}{\Gamma \rightarrow \Delta, J}\,(cR) \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, J}\,(wR)$$

$$\frac{\Gamma, J \rightarrow \Delta}{\Gamma, J' \rightarrow \Delta}\,(J \approx J') \quad (\approx L) \qquad \frac{\Gamma \rightarrow \Delta, J}{\Gamma \rightarrow \Delta, J'}\,(J \approx J') \quad (\approx R)$$

$$\frac{\Gamma \longrightarrow \Delta, (\sigma \triangleright F) \quad \Gamma, (\sigma \triangleright G) \longrightarrow \Delta}{\Gamma, (\sigma \triangleright F \supset G) \longrightarrow \Delta}\,(\supset L) \qquad \frac{\Gamma, (\sigma \triangleright F) \longrightarrow \Delta, (\sigma \triangleright G)}{\Gamma \longrightarrow \Delta, (\sigma \triangleright F \supset G)}\,(\supset R)$$

$$\frac{M : \tau \quad \Gamma, (\sigma \triangleright F[M/x_\tau]) \longrightarrow \Delta}{\Gamma, (\sigma \triangleright \forall x_\tau.F) \longrightarrow \Delta}\,(\forall L) \qquad \frac{\Gamma \longrightarrow \Delta, (\sigma \triangleright F[h\sigma/x_\tau])}{\Gamma \longrightarrow \Delta, (\sigma \triangleright \forall x_\tau.F)}\,(h_{\sigma \rightarrow \tau} \text{ fresh}) \quad (\forall R)$$

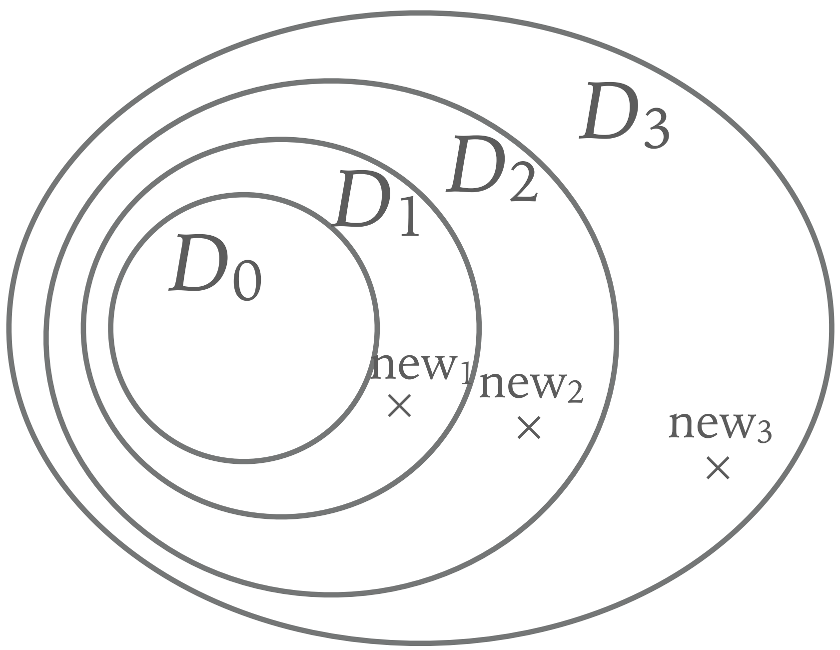
$$\frac{\Gamma, (\sigma, x : \tau \triangleright F) \longrightarrow \Delta}{\Gamma, (\sigma \triangleright \nabla x_\tau.F) \longrightarrow \Delta}\,(\nabla L) \qquad \frac{\Gamma \longrightarrow \Delta, (\sigma, x : \tau \triangleright F)}{\Gamma \longrightarrow \Delta, (\sigma \triangleright \nabla x_\tau.F)}\,(\nabla R)$$

Here σ is a *local signature* $x_1{:}\tau_1, \ldots, x_n{:}\tau_n$

$\sigma \triangleright F$ means «F where $x_1, \ldots, x_n$ are generic»

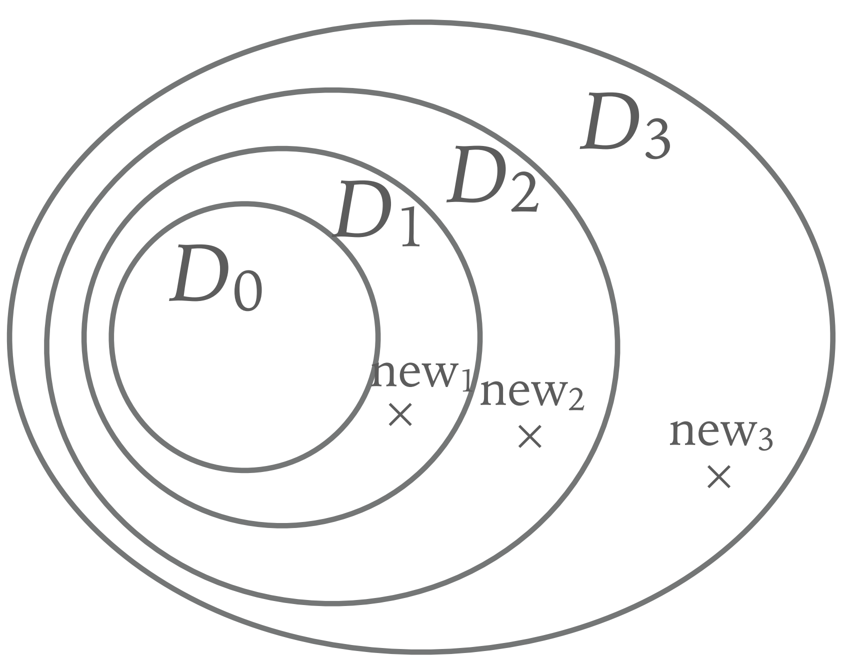
NABLA-SETS

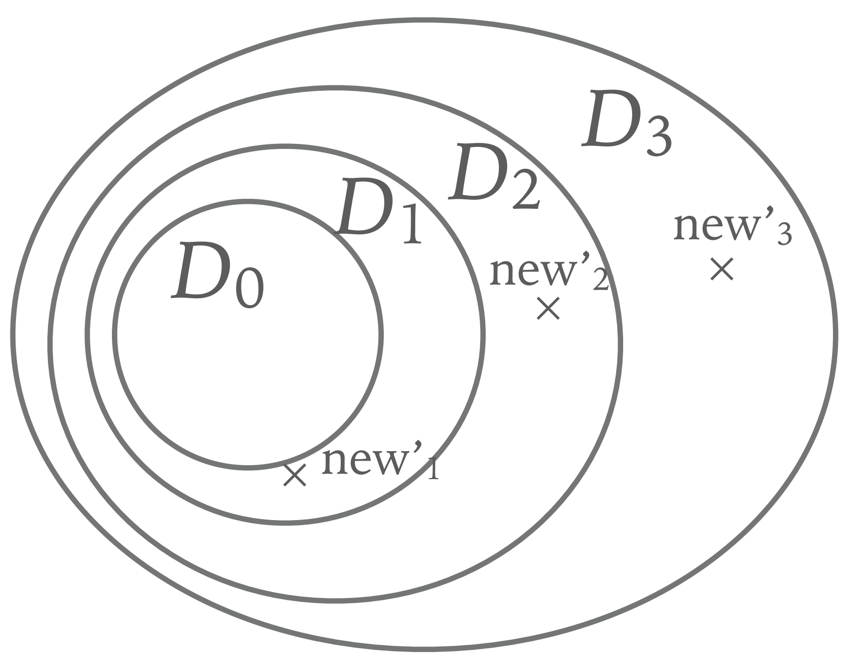
- **Defn.** A nabla-set D is:
  - a family $(D_n)_{n \in \mathbb{N}}$ of non-empty sets
  - injective maps $\mathrm{old}_n : D_n \to D_{n+1}$
  - elements $\mathrm{new}_{n+1} \in D_{n+1}, \notin \mathrm{Im}\ \mathrm{old}_n$.
- Idea: D_n is set of values in D after $\leq n$ calls to ∇.



NABLA-MAPS

- **Defn.** A nabla-map $f : D \to E$ is
  - a collection of maps f_n: $D_n \to E_n$, $n \in \mathbb{N}$
  - such that $\mathrm{old}_n \circ f_n = f_{n+1} \circ \mathrm{old}_n$
- We do **not** require that f_n preserve new_n.
- In particular, the following **variants** are isomorphic (on purpose!)





EXPONENTIAL OBJECTS

- **Prop.** In the category ∇ of nabla-sets:
  - All non-empty products exist.
  - Every object is exponentiable.
- Exponentials $[D \to E]$ can be built by imitating exponentials in the category $\mathbf{Set}^{\mathbb{N}}$ of presheaves over the poset $\mathbb{N}$:
  - $[D \to E]_n$ = families of maps $(f_m)_{m \geq n}$ that commute with old
  - $\mathrm{App} : (f_m)_{m \geq n}, x \in D_n \mapsto f_n(x)$
  - For $f = (f_n)_{n \in \mathbb{N}} : C \times D \to E$, $\mathrm{Lam}(f)_n(c) = (f_m(\mathrm{old}_{n \to m}(c), \_))_{m \geq n}$
  - new_n elements: slightly subtle, needs old to be injective.
- Will serve to interpret simply-typed lambda-calculus.

∇ HAS ENOUGH MAPS

- **Prop.** For every $e \in E_{n+1}$, there is a nabla-map $f : D \to E$ such that $f_{n+1}(\mathrm{new}_{n+1}) = e$.
- This is a curious property for a presheaf-related category
- This will be **required for soundness**.

CHOICE

- Seemingly related, but really strong, and bad news actually:
- **Prop (Choice).** Every epi splits in ∇.
- **Prop (Weak Choice).** Let $R \subseteq D_n \times E_n$.
  Assume that for every d, there is an e such that $(d,e) \in R$.
  Then there is an $(f_m)_{m \geq n} \in [D \to E]_n$ such that
  for every d, $(d, f_n(d)) \in R$.

SOUNDNESS

SEMANTICS

- **Standard structure** S: provide nabla-sets for base types, all others $S[\![\tau]\!]$ are given as exponentials.

- Terms:

$$\begin{array}{rcl} S[\![x_\tau]\!] & = & \pi_{x_\tau} \\ S[\![M\,N]\!] & = & \mathsf{App} \circ \langle S[\![M]\!], S[\![N]\!] \rangle \\ S[\![\lambda x_\varphi . M]\!] & = & \Lambda_{x_\varphi}(S[\![M]\!]) \end{array}$$

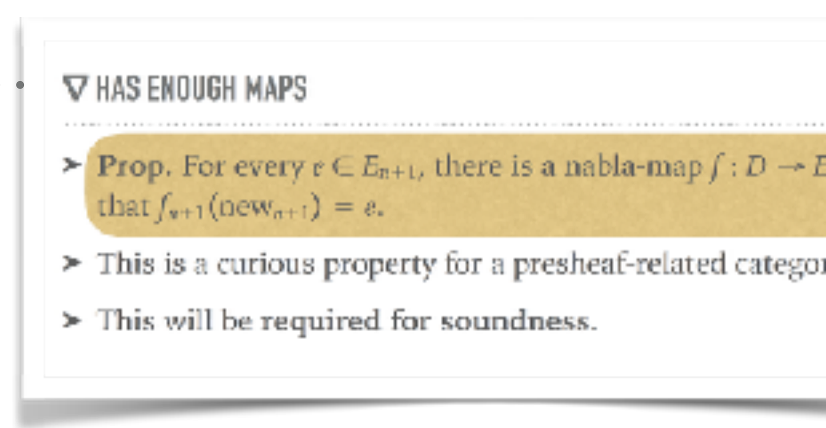
- Formulas:

$$\begin{array}{rcl} S;\rho \models_n P(M_1,\cdots,M_k) & \text{iff} & (S[\![M_1]\!]_n(\rho),\cdots,S[\![M_k]\!]_n(\rho)) \in S[\![P]\!]_n \\ S;\rho \models_n \bot & & \text{never} \\ S;\rho \models_n F \supset G & \text{iff} & (S;\rho \not\models_n F \text{ or } S;\rho \models_n G) \\ S;\rho \models_n \forall x_\tau . F & \text{iff} & (\text{for every } d \in S[\![\tau]\!]_n,\ S;\rho[x \mapsto d] \models_n F) \\ S;\rho \models_n \nabla x_\tau . F & \text{iff} & S;\mathrm{old}_n^{\mathsf{Env}}(\rho)[x \mapsto \mathrm{new}_{n+1}^{S[\![\tau]\!]})] \models_{n+1} F. \end{array}$$

IMPORTANT EQUIVALENCES

- **Lemma.** The following semantic equivalences hold:
  - $\nabla x_\tau \,.\, (F \supset G) \equiv (\nabla x_\tau \,.\, F) \supset (\nabla x_\tau \,.\, G)$
  - $\nabla x_\tau \,.\, F \equiv \nabla y_\varphi \,.\, F$ (x, y not free in F)
  - $\nabla x_\tau \,.\, \forall y_\varphi \,.\, F \equiv \forall h_{\tau \to \varphi} \,.\, \nabla x_\tau \,.\, F[hx/y]$ (« raising »).
- **Note:** raising is valid because ∇ has **enough maps**:

If $\forall h_{\tau \to \varphi} \,.\, \nabla x_\tau \,.\, F$ holds, we want to show $\nabla x_\tau \,.\, \forall y_\varphi \,.\, F$.
Interpret x_τ as new_{n+1}. For every value e for y_φ:
find $f = (f_m)_{m \in \mathbb{N}}$ s.t. $f_{n+1}(\mathrm{new}_{n+1}) = e$
Now interpret $h_{\tau \to \varphi}$ as f_n, satisfying F.
So $\nabla x_\tau \,.\, \forall y_\varphi \,.\, F$ holds.



SOUNDNESS (FOR STANDARD STRUCTURES)

- **Lemma.** The following semantic equivalences hold:
  - $\nabla x_\tau \,.\, (F \supset G) \equiv (\nabla x_\tau \,.\, F) \supset (\nabla x_\tau \,.\, G)$
  - $\nabla x_\tau \,.\, F \equiv \nabla y_\varphi \,.\, F$ $\qquad$ (x, y not free in F)
  - $\nabla x_\tau \,.\, \forall y_\varphi \,.\, F \equiv \forall h_{\tau \to \varphi} \,.\, \nabla x_\tau \,.\, F$ (« raising »).
- **Corollary.** Standard structures are sound for $FO\lambda^\nabla$: Every derivable sequent is valid.

$$\frac{}{\Gamma, (\sigma \triangleright \bot) \longrightarrow \Delta}\,(\bot L) \qquad \frac{}{\Gamma, J \longrightarrow J, \Delta}\,(Ax) \qquad \frac{\Gamma \longrightarrow J, \Delta \quad \Gamma', J \longrightarrow \Delta'}{\Gamma, \Gamma' \longrightarrow \Delta, \Delta'}\,(Cut)$$

$$\frac{\Gamma, J, J \to \Delta}{\Gamma, J \to \Delta}\,(cL) \qquad \frac{\Gamma \to \Delta}{\Gamma, J \to \Delta}\,(wL) \qquad \frac{\Gamma \to \Delta, J, J}{\Gamma \to \Delta, J}\,(cR) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, J}\,(wR)$$

$$\frac{\Gamma, J \to \Delta}{\Gamma, J' \to \Delta}\,(J \approx J') \quad (\approx L) \qquad \frac{\Gamma \to \Delta, J}{\Gamma \to \Delta, J'}\,(J \approx J') \quad (\approx R)$$

$$\frac{\Gamma \longrightarrow \Delta, (\sigma \triangleright F) \quad \Gamma, (\sigma \triangleright G) \longrightarrow \Delta}{\Gamma, (\sigma \triangleright F \supset G) \longrightarrow \Delta}\,(\supset L) \qquad \frac{\Gamma, (\sigma \triangleright F) \longrightarrow \Delta, (\sigma \triangleright G)}{\Gamma \longrightarrow \Delta, (\sigma \triangleright F \supset G)}\,(\supset R)$$

$$\frac{M : \tau \quad \Gamma, (\sigma \triangleright F[M/x_\tau]) \longrightarrow \Delta}{\Gamma, (\sigma \triangleright \forall x_\tau . F) \longrightarrow \Delta}\,(\forall L) \qquad \frac{\Gamma \longrightarrow \Delta, (\sigma \triangleright F[h\sigma/x_\tau])}{\Gamma \longrightarrow \Delta, (\sigma \triangleright \forall x_\tau . F)}\,(h_{\sigma \to \tau} \text{ fresh}) \quad (\forall R)$$

$$\frac{\Gamma, (\sigma, x : \tau \triangleright F) \longrightarrow \Delta}{\Gamma, (\sigma \triangleright \nabla x_\tau . F) \longrightarrow \Delta}\,(\nabla L) \qquad \frac{\Gamma \longrightarrow \Delta, (\sigma, x : \tau \triangleright F)}{\Gamma \longrightarrow \Delta, (\sigma \triangleright \nabla x_\tau . F)}\,(\nabla R)$$

INCOMPLETENESS

… for standard structures

THE AXIOM OF CHOICE

- Consider $(\forall x_\varphi \,.\, \exists y_\tau \,.\, F) \supset (\exists h_{\tau \to \varphi} \,.\, \forall x_\tau \,.\, F[hx/y])$ (AC)
- **Fact.** (AC) is valid.
- This is exactly what Weak Choice says. Remember:

CHOICE

- Seemingly related, but really strong, and bad news actually:
- **Prop (Choice).** Every epi splits in ∇.
- **Prop (Weak Choice).** Let $R \subseteq D_n \times E_n$.
  Assume that for every d, there is an e such that $(d,e) \in R$.
  Then there is an $(f_m)_{m \geq n} \in [D \to E]_n$ such that
  for every d, $(d, f_n(d)) \in R$.

THE AXIOM OF CHOICE

- Consider $(\forall x_\varphi \,.\, \exists y_\tau \,.\, F) \supset (\exists h_{\tau\to\varphi} \,.\, \forall x_\tau \,.\, F[hx/y])$ (AC)
- **Fact.** (AC) is valid.
- **Lemma.** (AC) is not provable.
- **Proof.** Anticipating slightly, there is a Henkin structure
  with enough maps that invalidates (AC),
  built by a diagonal argument.

  (We take $F = P(x,y)$, and build the interpretation of P so that
  for each x there is a y satisfying P,
  but no lambda-term h maps every x to such a y.)

THE AXIOM OF CHOICE

- Consider $(\forall x_\varphi \,.\, \exists y_\tau \,.\, F) \supset (\exists h_{\tau \to \varphi} \,.\, \forall x_\tau \,.\, F[hx/y])$ (AC)
- **Fact.** (AC) is valid.
- **Lemma.** (AC) is not provable.
- **Corollary.** Standard structures are incomplete for $\mathrm{FO}\lambda^{\nabla}$.
- **Note.** This has nothing to do with nabla.
  The same thing happens for any logic of higher-order terms.

COMPLETENESS

… for Henkin structures

HENKIN STRUCTURES

- A standard cure [Henkin 1950]: instead of considering **all** nabla-maps, restrict to **subclasses** of nabla-maps so that everything is still defined.
  (A horrible definition.)
- We must require compatibility with $\beta\eta$, etc.
- We must also require that those subclasses **have enough maps**, i.e.:
  for all types φ, τ, for every $d \in S[\![\tau]\!]_{n+1}$,
  there is an $f \in S[\![\varphi \to \tau]\!]_n$ such that $\mathrm{App}_{n+1}(\mathrm{old}_n(f), \mathrm{new}_{n+1}) = d$.
- All such models are **sound.**
  Standard structures are a particular case.

Definition 4.1 (Henkin Universe). *A Henkin universe S for nabla is the following data:*

- *for each type τ, a nabla-set $S[\![\tau]\!]$;*
  We write Env *for the product $\prod_{x_\tau} S[\![\tau]\!]$, where x_τ ranges over all variables:* Env_n *is the*
  of environments ρ at level n, namely functions mapping each variable x_τ to an elem
  $\rho(x_\tau) \in S[\![\tau]\!]_n$;
- *for each type τ, a set $S(\tau)$ of nabla-maps from* Env *to τ, containing all the projecti*
  π_{x_τ} —where $(\pi_{x_\tau})_n(\rho) = \rho(x_\tau)$ for every environment ρ at level n;
- *for each pair of types φ, τ, a nabla-map* $\mathsf{App}\colon S[\![\varphi \to \tau]\!] \times S[\![\varphi]\!] \to S[\![\tau]\!]$, *with the prope*
  that for every $f \in S(\varphi \to \tau)$ and for every $g \in S(\varphi)$, $\mathsf{App} \circ \langle f, g\rangle$ *is in $S(\tau)$;*
- *for every variable x_φ and each type τ, a function $\Lambda_{x_\varphi} : S(\tau) \to S(\varphi \to \tau)$;*

such that, defining:

$$
\begin{aligned}
S[\![x_\tau]\!] &= \pi_{x_\tau} \\
S[\![MN]\!] &= \mathsf{App} \circ \langle S[\![M]\!], S[\![N]\!]\rangle \\
S[\![\lambda x_\varphi . M]\!] &= \Lambda_{x_\varphi}(S[\![M]\!])
\end{aligned}
$$

then:

1. *for all $\beta\eta$-convertible λ-terms $M, N : \tau$, $S[\![M]\!] = S[\![N]\!]$;*
2. *for every λ-term $M : \tau$, for every $n \in \mathbb{N}$, $S[\![M]\!]_n \rho$ does not depend on $\rho(y)$ if y is not*
   in M, namely: if $\rho(z) = \rho'(z)$ for every $z \neq y$, then $S[\![M]\!]_n \rho = S[\![M]\!]_n \rho'$;
3. *for all λ-terms $N : \tau$ and $M : \varphi$, for every $n \in \mathbb{N}$, for every environment ρ at leve*
   $S[\![N[M/x_\varphi]]\!]_n \rho = S[\![N]\!]_n(\rho[x_\varphi \mapsto S[\![M]\!]_n \rho])$;

TERM (HERBRAND) STRUCTURES

- Assume a unique base type ι.
- Consider **nominal λ-terms**, obtained by adjoining countably many constants $a_i : \iota$ (« **names** »), $i=1, 2, \ldots$, up to $\beta\eta$.
- Let $S[\![\tau]\!]_n$ be $\{M \text{ nominal} : \tau \mid \mathit{FreeNames}(M) \subseteq \{a_1, \ldots, a_n\}\}$
  old_n is identity
  new_{n+1} is $\lambda x_1, \ldots, x_m \,.\, a_{n+1}$, where $\tau = \tau_1 \to \ldots \to \tau_m \to \iota$.
- Application is syntactic application.
- Crucially, this **has enough maps**, meaning that:
  for every $N \in S[\![\tau]\!]_{n+1}$,
  there is an $M \in S[\![\varphi \to \tau]\!]_n$ such that $M(\mathrm{new}_{n+1}) = N$.
  (Take $M = \lambda x{:}\tau . N[x M_1 \ldots M_m / a_{n+1}]$ where $M_i{:}\tau_i$ are dummy terms.)

HINTIKKA THEORIES

- A **signed judgment** is $\pm J = \pm\sigma \rhd F$ (meaning «J is true/false»)
- A **theory** T is a set of signed judgments.
  T is **consistent** iff one cannot derive $J_1, \ldots, J_m \to J'_1, \ldots, J'_n$ by a cut-free proof, for any $+J_1, \ldots, +J_m, -J'_1, \ldots, -J'_n$ in T.
- **Defn.** A **Hintikka theory** is a consistent theory T such that:
  - if $+\sigma \rhd F \supset G$ is in T, then $-\sigma \rhd F$ or $+\sigma \rhd G$ is in T
  - if $-\sigma \rhd F \supset G$ is in T, then $+\sigma \rhd F$ and $-\sigma \rhd G$ is in T
  - if $+\sigma \rhd \forall x_\tau \,.\, F$ is in T, then $+\sigma \rhd F[M/x]$ is in T for every $M{:}\tau$
  - if $-\sigma \rhd \forall x_\tau \,.\, F$ is in T, then $-\sigma \rhd F[h\sigma/x]$ is in T for some variable h that does not occur in σ
  - etc. (Essentially, follow the deduction rules.)

THE HINTIKKA LEMMA

- **Lemma.** Every finite consistent theory is contained in some Hintikka theory.
- **Proof.** Standard: add all missing signed judgments one by one, using an enumeration of those that should be considered that lists each of them infinitely often.

HERBRAND MODELS FROM HINTIKKA THEORIES

- For every local signature $\sigma = x_1{:}\tau_1, \ldots, x_n{:}\tau_n$,
  let $\theta_\sigma = [\mathrm{new}_1/x_1, \ldots, \mathrm{new}_n/x_n]$ (with all new_i's of the right type)
- **Prop.** Assume a unique base type ι.
  - Every Hintikka theory T describes a Herbrand structure H where each relation symbol P holds of those tuples $(M_1\theta_\sigma, \ldots, M_k\theta_\sigma)$ such that $+\sigma \triangleright P(M_1, \ldots, M_k)$ is in T.
  - Then, for every $+J$ in T, J holds in H;
    for every $-J$ in T, J does not hold in H.
- In other words,
  we interpret $x_1{:}\tau_1, \ldots, x_n{:}\tau_n \triangleright F(x_1, \ldots, x_n)$ as $F(\mathrm{new}_1, \ldots, \mathrm{new}_n)$.

COMPLETENESS

- **Thm.** Assume a unique base type ι.
Every sequent that holds in every Herbrand structure
(a fortiori if it holds in every Henkin structure
with enough maps) is provable, by a cut-free proof.
- **Proof.** Assume $J_1, \ldots, J_m \to J'_1, \ldots, J'_n$ is not cut-free provable.
$\{+J_1, \ldots, +J_m, -J'_1, \ldots, -J'_n\}$ is a finite consistent theory.
Extend that to a Hintikka theory T.
T describes a Herbrand structure H.
In H, $J_1, \ldots, J_m$ hold, but $J'_1, \ldots, J'_n$ do not.
Hence H invalidates $J_1, \ldots, J_m \to J'_1, \ldots, J'_n$.
- **Corollary.** Every provable $FO\lambda^{\nabla}$ sequent has a cut-free proof.

Π_1-COMPLETENESS

… for standard structures

RELATING HERBRAND AND STANDARD STRUCTURES

- Let $T[\![\tau]\!]_n$ be the nabla-set of nominal terms of type τ at level n.
- Define a standard structure S_0 by letting $S_0[\![\iota]\!] = T[\![\iota]\!]$ (extended to all types using exponential objects).
- In general, $S_0[\![\tau]\!]$ and $T[\![\tau]\!]$ are very different: e.g., $T[\![\iota \to \iota]\!]$ is countable, $S_0[\![\iota \to \iota]\!]$ is not.
- We can relate $S_0[\![\tau]\!]$ and $T[\![\tau]\!]$ by a **Kripke logical relation**:
- **Defn.** Let $R[\tau]_n \subseteq T[\![\tau]\!] \times S_0[\![\tau]\!]$ be defined by:
  - $R[\iota]_n$ is equality
  - $M\ R[\varphi \to \tau]_n\ f$ iff for every $m \geq n$, for all $N\ R[\varphi]_m\ d$, $MN\ R[\tau]_m\ f(d)$.

PROPERTIES OF THE KRIPKE LOGICAL RELATION

- **Basic Lemma.** If $\theta(x_\varphi) R[\varphi]_n \, \rho(x_\varphi)$ for every variable, then
$$M\theta \quad R[\tau]_n \quad S_0[\![M]\!]_n(\rho), \qquad \text{for every } M{:}\tau.$$
- **Proof.** Standard.

PROPERTIES OF THE KRIPKE LOGICAL RELATION

- **Basic Lemma.** If $\theta(x_\varphi) R[\varphi]_n\, \rho(x_\varphi)$ for every variable, then
  $$M\theta \;\; R[\tau]_n \;\; S_0[\![M]\!]_n(\rho), \qquad \text{for every } M{:}\tau.$$
- **Proof.** Standard.
- **Prop** (Sandwich.) There are nabla-maps
  $$s[\tau] : T[\![\tau]\!] \to S_0[\![\tau]\!] \text{ and } r[\tau] : S_0[\![\tau]\!] \to T[\![\tau]\!]$$
  such that:
  $$s[\tau]_n(M) = d \;\Rightarrow\; M\, R[\tau]_n\, d \;\Rightarrow\; M = r[\tau]_n(d)$$
- **Proof.** (Rough sketch.) By induction on types.
  We build $s[\varphi{\to}\tau]_n$ as $M \mapsto (a \mapsto s[\tau]_m(M\,(r[\varphi]_m(a))))_{m\geq n}$.
  We then note that for every f, there is at most one M
  such that $M\, R[\varphi{\to}\tau]_n\, f$.
  We build $r[\varphi{\to}\tau]_n$ by mapping f to the unique such M if it exists.

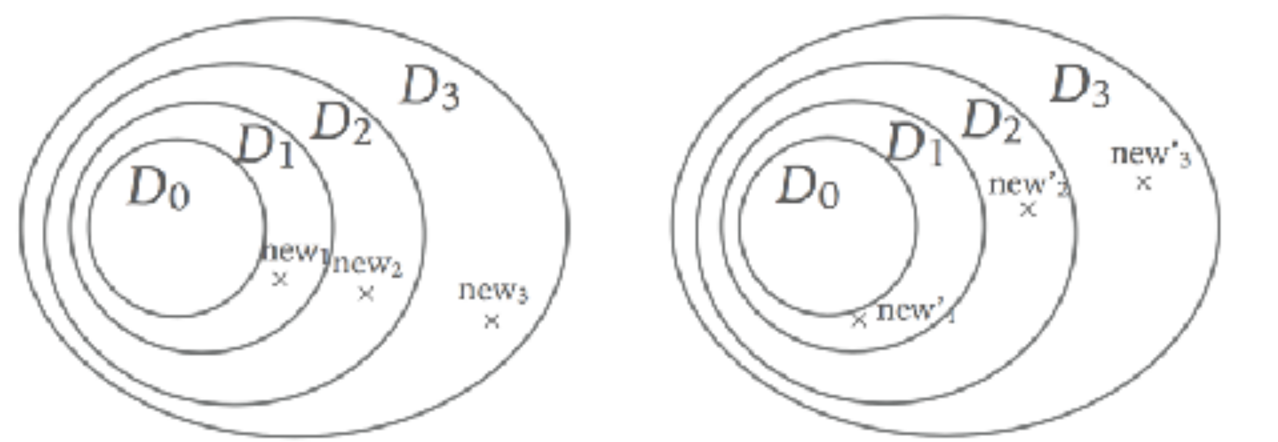
INCIDENTALLY…

- The Sandwich Lemma implies **equational completeness** for nabla-sets, in the manner of [Friedman 1975] for sets:
- **Thm.** There is a standard structure S_0 in which,
for all closed terms $M, N{:}\tau$,
M and N are $\beta\eta$-convertible
if and only if $S_0[\![M]\!]_0 = S_0[\![N]\!]_0$.
- **Proof.** Assume $S_0[\![M]\!]_0 = S_0[\![N]\!]_0$.
By the Basic Lemma, $M\ R[\tau]_0\ S_0[\![M]\!]_0$ and $N\ R[\tau]_0\ S_0[\![N]\!]_0$.
By the Sandwich Lemma,
$M = r[\tau]_0(S_0[\![M]\!]_0)$ and $N = r[\tau]_0(S_0[\![N]\!]_0)$ (up to $\beta\eta$).
So $M = N$ (up to $\beta\eta$).

FROM A HERBRAND STRUCTURE TO A STANDARD STRUCTURE

- Modify S_0 to S_1, obtained by changing the new elements: define new_n of $S_1[\![\tau]\!]$ as $s[\tau]_n(\mathrm{new}_n$ of $T[\![\tau]\!])$.
- Then $S_1[\![\tau]\!]$ is a **variant** of $S_0[\![\tau]\!]$.
- Everything we have said of S_0 holds of S_1.

- We do **not** require that f_n preserve new_n.
- In particular, the following **variants** are isomorphic



- Given a Herbrand structure T, define a standard structure $S_1{}^T$ by letting P be true of $(d_1, \ldots, d_k)$ in $S_1{}^T$ at level n iff

  P is true of $(r[\tau_1]_n(d_1), \ldots, r[\tau_k]_n(d_k))$ in T at level n.

Δ_0-FORMULAE

- **Defn.** A **Δ_0-formula** is one in which the only $\forall$ and $\exists$ quantifications are first-order.
  The ∇ quantifications can be any order you wish.
  The relation symbols can be higher-order as well.
- **Prop.** If $\theta(x_\varphi) R[\varphi]_n\, \rho(x_\varphi)$ for every variable, and F is a Δ_0-formula, then
  $T; \theta \vDash_n F$ if and only if $S_1{}^T; \rho \vDash_n F$
- **Proof.** Structural induction on the formula, using the previous results.

Π_1-FORMULAE

- **Defn.** A **Δ_0-formula** is one in which the only $\forall$ and $\exists$ quantifications are first-order.
  The ∇ quantifications can be any order you wish.

- **Defn.** A Π_1-Formula is a formula $\forall x_1{:}\tau_1, \ldots, x_n{:}\tau_n .\, G$,
  where G is a Δ_0-formula.

- **Prop.** If $\theta(x_\varphi) R[\varphi]_n\, \rho(x_\varphi)$ for every variable,
  and F is a Π_1-formula, then
  $S_1{}^T; \rho \vDash_n F$ implies $T; \theta \vDash_n F$.

Π_1-COMPLETENESS FOR STANDARD STRUCTURES

- **Prop.** If $\theta(x_\varphi) R[\varphi]_n\, \rho(x_\varphi)$ for every variable, and F is a Π_1-formula, then

  $$S_1{}^T; \rho \vDash_n F \text{ implies } T; \theta \vDash_n F.$$

- **Thm.** Assume a unique base type ι.
  Every valid Π_1-formula (in standard structures) is provable, with a cut-free proof.

- **Proof.** If F is valid, the previous proposition shows that it holds in every Herbrand model.
  Now apply Henkin completeness.

- **Note.** That includes every first-order formula, possibly extended with ∇ quantifications at every type.

CONCLUSION

OPEN PROBLEMS

- **Is $FO\lambda^{\nabla}$ + (AC) complete for standard structures?**

  Π_1-completeness is meant to be a stepping stone for that.

- **Can we dispense with the assumption that there is only one base type ι?**

  I don't think so,
  probably requires modifying the notion of nabla-set.

- **What about the intuitionistic cases?**

  Easy exercises, using Kripke models, in my opinion.

- **Can we extend those results to the logic of Abella?**

  Should be doable: nabla-sets with an action of symmetric groups $\mathbf{S}_n$ on D_n, with equivariant maps as morphisms; relations should be required to be equivariant and to satisfy $d \in S[\![P]\!]_n$ iff $old_n(d) \in S[\![P]\!]_{n+1}$.

CONCLUSION

Happy

th, Dale!