Musings Around the Geometry of Interaction, and Coherence

Jean Goubault-Larrecq

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Outline

Motivation
Linear Inverse Monoids
The Danos-Regnier Category
  Inductive Groupoids
  Introducing $DR(M)$
  The Weakly Cantorian Case
  Miserable Failures (Ignoring Great Successes)
Coherence Completions
  The Modified Hu-Joyal Construction
  Lifting Monoidal Structure, Creating the Rest
  Relation to Syntax?
  Towards Fibrations?
Conclusion
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A famous hard problem, first described in 1966, solved by G. Sénizergues (1997);

- Shown primitive recursive by C. Stirling (2002);

- Basically: given two Deterministic PushDown Automata, do they generate the same language?
DPDA Equivalence

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- Shown primitive recursive by C. Stirling (2002);
- Basically: given two Deterministic PushDown Automata, do they generate the same language?
- Note: Determinism is essential: NPDA equivalence is undecidable.
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Some constructions are constructions in coherence spaces.

Can we make a bridge with linear logic? Can we extend this?
A pushdown automaton is a prefix-rewrite system consisting of rules:

\[ pX \xrightarrow{a} q\alpha \]

- \( p \) is current control state,
- \( q \) is new control state,
- \( X \) is top stack symbol,
- \( \alpha \) is new sequence of stack symbols,
- \( a \) is letter read or \( \epsilon \).
A *pushdown automaton* is a prefix-rewrite system consisting of rules:

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- $X$ is top stack symbol,
- $\alpha$ is new sequence of stack symbols,
- $a$ is letter read or $\epsilon$.

Build a *strict deterministic grammar* (Harrison and Havel, 1973). From a coherence relation $\sqsubseteq$ on letters, define

- $w \sqsubseteq^* w'$ iff $w = w'$,
- or $w = w_0 aw_1$, $w' = w_0 bw_1'$, $a \neq b$, $a \sqsubseteq b$
DPDA and Coherence, cont’d

- If you build formal sums $\sum_i w_i$ of coherent words (cliques),
If you build formal sums $\sum_i w_i$ of coherent words (cliques), and look at the action of each letter $a$:

$$\hat{a} \left( \sum_i w_i \right) = \sum_j w'_j$$

where $w'_j$ ranges over all words such that $w_i \xrightarrow{a} w'_j$ for some $i$. 
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$$\hat{a} \left( \sum_i w_i \right) = \sum_j w'_j$$

where $w'_j$ ranges over all words such that $w_i \xrightarrow{a} w'_j$ for some $i$,

then $\sum_j w'_j$ is well-defined (a clique again),

and $\hat{a}$ is linear:

$$w_i \succeq^* w_k, \quad \begin{cases} \quad w_i \xrightarrow{a} w'_j \\ \quad w_k \xrightarrow{a} w'_\ell \end{cases} \quad \Rightarrow \quad w'_j \succeq^* w'_\ell$$

and if $w'_j = w'_\ell$ then $w_i = w_k$.
Coherence of Paths, and Trees

Let $\Sigma$ be a first-order signature. For each $n$-ary symbol $f$ ($n \geq 1$), define letters $f/i$.

Let $f/i \close g/j$ iff $f = g$ (whatever $i$, $j$).

- Every tree (first-order term) defines a coherence space over paths, with coherence $\close^*$. 
Coherence of Paths, and Trees

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\[
\begin{array}{c}
\text{f} \\
\text{g} \\
\text{f} \\
\text{a} \\
\text{a} \\
\text{c} \\
\text{f} \\
\text{b} \\
\text{b} \\
\text{f} \\
\text{f} \\
\text{a} \\
\text{a} \\
\text{c} \\
\end{array}
\]
Coherence of Paths, and Trees

Let \( \Sigma \) be a first-order signature. For each \( n \)-ary symbol \( f \) \((n \geq 1)\), define letters \( f/i \).

Let \( f/i \supseteq g/j \) iff \( f = g \) (whatever \( i, j \)).

Every tree (first-order term) defines a coherence space over paths, with coherence \( \supseteq^* \).

\[
\begin{array}{c}
\text{a} \\
f \\
g \\
f/1 \\
g/1 \\
f/1 a
\end{array}
\]
Coherence of Paths, and Trees

Let $\Sigma$ be a first-order signature. For each $n$-ary symbol $f$ ($n \geq 1$), define letters $f/i$.

$\triangleright$ Every tree (first-order term) defines a coherence space over paths, with coherence $\triangleright$.

\begin{align*}
\text{Let } f/i \triangleright g/j \text{ iff } f = g \text{ (whatever } i, j). \\
\end{align*}
Let $\Sigma$ be a first-order signature. For each $n$-ary symbol $f$ ($n \geq 1$), define letters $f/i$.

Let $f/i \circ g/j$ iff $f = g$ (whatever $i, j$).

- Every tree (first-order term) defines a coherence space over paths, with coherence $\circ^*$.
- Conversely, every clique of paths embeds into some tree (possibly infinite).
Coherence of Paths, and Trees

Let $f/i \sqsubseteq g/j$ iff $f = g$ (whatever $i, j$).

- Every tree (first-order term) defines a coherence space over paths, with coherence $\sqsubseteq^*$.  
- Conversely, every clique of paths embeds into some tree (possibly infinite).
Let $f/i \odot g/j$ iff $f = g$ (whatever $i, j$).

- Every tree (first-order term) defines a coherence space over paths, with coherence $\odot^*$.
- Conversely, every clique of paths embeds into some tree (possibly infinite).

\[
\begin{array}{c}
  f \\
  \downarrow \\
  g \\
  \downarrow \\
  f \\
\end{array}
\quad
\begin{array}{c}
  f/2 \ g/1 \\
  f/2 \ g/2 \ f/2
\end{array}
\]
Coherence of Paths, and Trees

Let $f/i \sim g/j$ iff $f = g$ (whatever $i, j$).

- Every tree (first-order term) defines a coherence space over paths, with coherence $\sim^*$.
- Conversely, every clique of paths embeds into some tree (possibly infinite).

\[
\begin{align*}
&f/2 \ g/1 \\
&f/2 \ g/2 \ f/2 \\
&f/2 \ g/2 \ f/1 \ g/3
\end{align*}
\]
Coherence of Paths, and Trees

Let $f/i \sqsubseteq g/j$ iff $f = g$ (whatever $i$, $j$).

- Every tree (first-order term) defines a coherence space over paths, with coherence $\sqsubseteq^*$.  
- Conversely, every clique of paths embeds into some tree (possibly infinite).

Motto: Cliques = chunks of infinite trees.
Paths in the $\lambda$-Calculus

Can we describe $\lambda$-terms, or rather Böhm(-like) trees, by the sets of (finite) paths through them?
Paths in the $\lambda$-Calculus

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- Yes, at least for linear $\lambda\mu Y$-terms (MLL+fixpoint),
Paths in the $\lambda$-Calculus

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- Recipe: take the **weights** from the dynamic algebra, forget about the underlying $\lambda$-term.
Paths in the \( \lambda \)-Calculus

- Can we describe \( \lambda \)-terms, or rather Böhm(-like) trees, by the sets of (finite) paths through them?
- Yes, at least for linear \( \lambda \mu Y \)-terms (MLL+fixpoint),
- Recipe: take the weights from the dynamic algebra, forget about the underlying \( \lambda \)-term.
- This will give us a nice \( * \)-autonomous category \( \mathcal{DR}(M) \), for any linear inverse monoid \( M \).
An example: $\lambda x, y, z \cdot zxy$
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\[ pppp^*p^*q^* \]
An example: $\lambda x, y, z \cdot zxy$
An example: $\lambda x, y, z \cdot zxy$

$\lambda x^4 p^4 q^4 p^4 p^4$
An example: $\lambda x, y, z \cdot zxy$

$pppp^* p^* q^* p^* p^*$

$+ ppqpp p^* p^* p^*$
An example: $\lambda x, y, z \cdot zxy$
An example: $\lambda x, y, z \cdot zxy$

$$\text{pppp}^*p^*q^*p^*p^*$$
$$+ \text{ppqpp}^*p^*p^*$$
$$+ \text{ppqq}^*q^*p^*$$
$$+ \text{ppqq}^*q^*$$
An example: $\lambda x, y, z \cdot zxy$
An example: $\lambda x, y, z \cdot zx y$

$$
\begin{align*}
pppp^*p^*q^*p^*p^* \\
+ ppqpp^*p^*p^* \\
+ ppqppq q^*p^* \\
+ ppqq q^* \\
+ pqq^*p^*q^*p^*p^* \\
+ q q^*q^*p^*p^* 
\end{align*}
$$
Static Denotations vs. Dynamic Behaviors

- GoI is usually described as some kind of dynamic process (stress on computation);
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- My interest is into denotational (static) models,
- and ultimately deciding equality of Böhm(-like) trees for a large class of programs,
Static Denotations vs. Dynamic Behaviors

- GoI is usually described as some kind of dynamic process (stress on computation);
- I don’t care about computation here.
- My interest is into denotational (static) models,
- and ultimately deciding equality of Böhm(-like) trees for a large class of programs,
- including deterministic higher-order program schemes (still an open problem).
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Inverse Monoids

In the example above, paths were words over $p, p^*, q, q^*$, plus some relations;
Inverse Monoids

- In the example above, paths were words over $p$, $p^*$, $q$, $q^*$, plus some relations;
- Think of $p : n \in \mathbb{N} \mapsto 2n$, $q : n \in \mathbb{N} \mapsto 2n + 1$;
- $p^*$, $q^*$ their inverses;
- They all live in $\text{PI}(\mathbb{N})$ (partial injections over $\mathbb{N}$), the small model of GoI;
Inverse Monoids

Think of $p : n \in \mathbb{N} \mapsto 2n$, $q : n \in \mathbb{N} \mapsto 2n + 1$;
$p^*$, $q^*$ their inverses;
They all live in $\text{PI}(\mathbb{N})$ (partial injections over $\mathbb{N}$), the small model of GoI;
In general, let $M$ be an inverse monoid (e.g., $\text{PI}(\mathbb{N})$):

\[
\begin{align*}
(u^*)^* &= u \\
(uv)^* &= v^*u^* \\
uu^*u &= u \\
uu^*vv^* &= vv^*uu^*
\end{align*}
\]

The idempotents are all of the form $\langle u \rangle = uu^*$ (codomain). Domain is $\langle u^* \rangle = u^*u$.
4th equation: idempotents commute.
(In $\text{PI}(\mathbb{N})$, they are [identities on] sets of integers.)
The Natural Ordering $\leq$

- Every inverse monoid can be equipped with an *natural ordering* $\leq$: $u \leq v$ iff any of the following equivalent conditions hold:
  
  \begin{align*}
  (i) & \quad vu^* = uu^* \\
  (ii) & \quad uv^* = uu^* \\
  (iii) & \quad \langle u \rangle v = u \\
  (iv) & \quad v \langle u^* \rangle = u \\
  (v) & \quad u^* v = u^* u \\
  (vi) & \quad v^* u = u^* u
  \end{align*}

  (This is standard, and well-known.)

- In $\Pi(\mathbb{N})$, $\leq$ is *graph inclusion*:

```
0 0 0
1 1 1
2 2 2
3 3 3
4 4 4
5 5 5
6 6 6
7 7 7
```

$\leq$
The Natural Coherence ⊇

- This one seems to be new.
- Let $u \supseteq v$ iff

$$u \langle v^* \rangle = v \langle u^* \rangle \text{ and } \langle v \rangle u = \langle u \rangle v$$

- In $\text{PI}(\mathbb{N})$, means that $u$ and $v$ agree on their domain and on their codomain:

\[
\begin{array}{c|cccc|c|cccc|c}
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
1 & & & & 1 & & & 1 & & \\
2 & & & & 2 & & & 2 & & \\
3 & \ldots & \ldots & \ldots & 3 & \ldots & \ldots & 3 & \ldots & \ldots \\
4 & \ldots & \ldots & \ldots & 4 & \ldots & \ldots & 4 & \ldots & \ldots \\
5 & \ldots & \ldots & \ldots & 5 & \ldots & \ldots & 5 & \ldots & \ldots \\
6 & \ldots & \ldots & \ldots & 6 & \ldots & \ldots & 6 & \ldots & \ldots \\
7 & \ldots & \ldots & \ldots & 7 & \ldots & \ldots & 7 & \ldots & \ldots \\
\hline
& & \supseteq & & & & \supseteq & & & \\
\end{array}
\]

agree on common intersection:

\[
\begin{array}{c|cccc|c|cccc|c}
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
1 & & & & 1 & & & 1 & & \\
2 & & & & 2 & & & 2 & & \\
3 & \ldots & \ldots & \ldots & 3 & \ldots & \ldots & 3 & \ldots & \ldots \\
4 & \ldots & \ldots & \ldots & 4 & \ldots & \ldots & 4 & \ldots & \ldots \\
5 & \ldots & \ldots & \ldots & 5 & \ldots & \ldots & 5 & \ldots & \ldots \\
6 & \ldots & \ldots & \ldots & 6 & \ldots & \ldots & 6 & \ldots & \ldots \\
7 & \ldots & \ldots & \ldots & 7 & \ldots & \ldots & 7 & \ldots & \ldots \\
\hline
& & \supseteq & & & & \supseteq & & & \\
\end{array}
\]
Linear Inverse Monoids

Definition

An inverse monoid $M$ is linear iff:
- Every clique $(u_i)_{i \in I}$ has a sup $\sum_{i \in I} u_i$;
- Sups distribute over product: $v \left( \sum_{i \in I} u_i \right) w = \sum_{i \in I} vu_i w$. 
Definition

An inverse monoid \( M \) is linear iff:

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This is certainly the case in \( \text{PI}(\mathbb{N}) \), where sups are unions:
Bi-unambiguous Automata

\[ b \xrightarrow{a} q_I \xrightarrow{b} q_F \xrightarrow{a^*} a^* \]

is notation for \( b^* b + \sum_{n \in \mathbb{N}} a^* (bb^*)^n a. \)
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One can write whole words over arrows: no confusion can arise, because of distributivity.
Bi-unambiguous Automata

\[ b \rightarrow b^* \]
\[ a \rightarrow a^* \]
\[ q_I \rightarrow q_F \]
\[ b^* b + \sum_{n \in \mathbb{N}} a^* (bb^*)^n a. \]

One can write whole words over arrows: no confusion can arise, because of distributivity.

**Bi-unambiguous** as soon as makes sense: whichever way you travel from \( q_I \) to \( q_F \) or conversely, you get the same result.
Bi-unambiguous Automata

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\[ a \rightarrow a^* \]
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\[ b^* b + \sum_{n \in \mathbb{N}} a^*(bb^*)^n a. \]

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**Bi-unambiguous** as soon as makes sense: whichever way you travel from \( q_I \) to \( q^F \) or conversely, you get the same result.

Special case: **Bideterminism** (here, when \( ab^* = a^* b = 0 \)): at most one path forward, at most one path backward.
A Few Useful Theorems

Let $M$ be an inverse linear monoid.

**Theorem**
Define $\#$ (conflict) as the negation of $\sqcap$; Then $(M, \leq, \#)$ is an event structure.
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Let $M$ be an inverse linear monoid.

**Theorem**
Define $\# \text{ (conflict)}$ as the negation of $\sqcap$; Then $(M, \leq, \#)$ is an event structure.

**Theorem [Preston-Wagner]**
Every inverse monoid $M$ embeds into some linear inverse monoid (i.e., $\text{PI}(M)$), preserving product, inverse, unit, and preserving and reflecting order.
Constructions of Linear Inverse Monoids

- $\mathcal{F}_{isg}(M) =$ the free linear inverse monoid over an inverse monoid $M$:
  - Elements are down-closed cliques of $M$;
  - $UV = \{uv | u \in U, v \in V\}$;
  - $U^* = \{u^* | u \in U\}$. 

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GoI and Coherence
Constructions of Linear Inverse Monoids

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- Inf-semi-lattices (product is inf), groups, semi-direct products of groups and lattices;
Constructions of Linear Inverse Monoids

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  - $UV = \{uv | u \in U, v \in V\}$;
  - $U^* = \{u^* | u \in U\}$.
- Inf-semi-lattices (product is inf), groups, semi-direct products of groups and lattices;
- Bimonotonic partial injections on some partially ordered space;
- Bicontinuous partial injections on some topological space;
- Coherence-preserving-and-reflecting partial injections on some coherence space $E$;
Constructions of Linear Inverse Monoids, cont’d

- $M_\Sigma$: Girard’s (1995) rudimentary clauses $s \leftarrow t$, $s$ and $t$ having same free variables.
  - $(s \leftarrow t) \cdot (s' \leftarrow t') = \begin{cases} 
    s\sigma \leftarrow t'\sigma & \text{if } t, s' \text{ unify, with mgu } \sigma \\
    0 & \text{otherwise}
  \end{cases}$
  - $(s \leftarrow t)^* = t \leftarrow s$
Constructions of Linear Inverse Monoids, cont’d

- $M_\Sigma$: Girard’s (1995) rudimentary clauses $s \leftarrow t$, $s$ and $t$ having same free variables.
  - $(s \leftarrow t) \cdot (s' \leftarrow t') = \begin{cases} 
    s\sigma \leftarrow t'\sigma & \text{if } t, s' \text{ unify, with mgu } \sigma \\
    0 & \text{otherwise}
  \end{cases}
  
- $(s \leftarrow t)^* = t \leftarrow s$

- $\mathcal{F}_{isg}(M_\Sigma) =$ cliques of rudimentary clauses = bideterministic automata on terms.
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Conclusion
The Inductive Groupoid of an Inverse Monoid

- A well-known construction. $J_G(M)$ has:
  - Objects: all idempotents $\langle u \rangle$ of $M$;
  - Morphisms: $A \xrightarrow{u} B$ provided $\langle u^* \rangle = A$ and $\langle u \rangle = B$. 
The Inductive Groupoid of an Inverse Monoid

- A well-known construction. $\mathcal{IG}(M)$ has:
  - Objects: all idempotents $\langle u \rangle$ of $M$;
  - Morphisms: $A \xrightarrow{u} B$ provided $\langle u^* \rangle = A$ and $\langle u \rangle = B$.
- $\mathcal{IG}(M)$ is an inductive groupoid, a groupoid such that:
  - $\leq$ on objects is an inf-semi-lattice ($A \wedge B = AB = BA$);
  - $\leq$ on morphisms makes $\mathcal{IG}(M)$ order-enriched;
- If $\forall i$, then $A_1 \leq A_2$ and $B_1 \leq B_2$. 

\[ A_2 \xrightarrow{u_2} B_2 \]
\[ A_1 \xrightarrow{u_1} B_1 \]
The Inductive Groupoid of an Inverse Monoid

- $\mathcal{IG}(M)$ is an *inductive groupoid*, a groupoid such that:
  - $\leq$ on objects is an inf-semi-lattice ($A \wedge B = AB = BA$);
  - $\leq$ on morphisms makes $\mathcal{IG}(M)$ order-enriched;
    \[ A_2 \xrightarrow{u_2} B_2 \]
  - If $A_1 \leq A_2$ and $B_1 \leq B_2$.
    \[ A_1 \xrightarrow{u_1} B_1 \]
    \[ A_2 \xrightarrow{u_2} B_2 \]
  - If $A_1 \leq A_2$ then $A_1 \leq A_2$ and $B_1 \leq B_2$.
    \[ A_1 \xrightarrow{(u_2|A_1)} B_1 \]

for a unique $B_1$ and a unique $(u_2|A_1)$. 
The Inverse Monoid of an Inductive Groupoid

- Elements are morphisms (whatever source and target);
- Product of $A_2 \xrightarrow{u_2} B_2$ by $A_1 \xrightarrow{u_1} B_1$ is

$$
\begin{align*}
A_1 \xrightarrow{u_1} B_1 \\
B_1 \wedge A_2 \xrightarrow{(B_1 \wedge A_2 \mid u_1)} B_1 \wedge A_2 \\
\leq (u_2 \mid B_1 \wedge A_2) \\
\leq A_2 \xrightarrow{u_2} B_2
\end{align*}
$$

- Inverse is... inverse.
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Conclusion
The Danos-Regnier Category of $M$

$\mathcal{DR}(M)$ is defined similarly... but morphisms are more complex.

- Objects: idempotents of $M$ again;

- Morphisms from $A$ to $B$: diagrams $\begin{array}{ccc}
A & \xrightarrow{\beta} & A \\
\downarrow{a} & & \uparrow{a^*} \\
B & \xleftarrow{\gamma} & B
\end{array}$

  where:

  - (co)domains of $a$, $\beta$, $\gamma$ are smaller than $A$, resp. $B$;
  - $\beta$ and $\gamma$ are self-inverses: $\beta^* = \beta$, $\gamma^* = \gamma$;
  - Bideterminism: $a\beta = 0$, $\gamma a = 0$. 

Morphisms from $A$ to $B$: diagrams

- (co)domains of $a$, $\beta$, $\gamma$ are smaller than $A$, resp. $B$;
- $\beta$ and $\gamma$ are self-inverses: $\beta^* = \beta$, $\gamma^* = \gamma$;
- Bideterminism: $a\beta = 0$, $\gamma a = 0$.

Intuition:

- $\gamma$ is the sum of paths from output ($B$) to output (see the $\lambda x, y, z.zxy$ example);
- $a$ is the sum of paths from input ($A$) to output ($B$);
- $\beta$ is the sum of paths from input to input.
Identity, Composition

- Identity on $A = A \circ A = A$ (Zero arrows omitted.)
Identity, Composition

- Identity on $A = A \circ A$
- Composing:

(Zero arrows omitted.)
Identity, Composition

- Identity on $A = A$
- Composing:

\[
\begin{align*}
\text{Identity on } A &= A \\
\text{Composing:} &
\end{align*}
\]

Note non-trivial loops: this is the feedback equation.

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GoI and Coherence
This is a Category

Proof

\[
\begin{array}{c}
A \\
B \\
C \\
D
\end{array} \quad \xrightarrow{\beta} \quad \begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\]

\[
\begin{array}{c}
a \\
a' \\
a'' \\
\circ
\end{array} \quad \xleftarrow{\beta'} \quad \begin{array}{c}
a* \\
a' * \\
a'' * \\
\circ
\end{array}
\]

\[
\begin{array}{c}
\beta' \quad \beta'' \quad \gamma \quad \gamma' \quad \gamma''
\end{array}
\]
Isos

Theorem

The groupoid of $\mathcal{DR}(M)$ is $\mathcal{IG}(M)$. I.e., the isos in $\mathcal{DR}(M)$ are

$\bullet A \xrightarrow{a} \circ B \xleftarrow{a^*} A$

with $\langle a^* \rangle = A$ and $\langle a \rangle = B$ (totality).
Epis, Monos

Theorem

Epis in $\mathcal{DR}(M)$ are $a$ with $\langle a \rangle = B$ (right totality).

Monos in $\mathcal{DR}(M)$ are $a^*$ with $\langle a^* \rangle = A$ (left totality)
Epis, Monos

Theorem

Epis in $\mathcal{DR}(M)$ are $a \beta a^* \text{ with } \langle a \rangle = B$ (right totality).

Monos in $\mathcal{DR}(M)$ are $a \alpha a^* \text{ with } \langle a^* \rangle = A$ (left totality).

Theorem

$\mathcal{DR}(M)$ has an epi-mono factorization. Epis, monos are split.

$Iso = Epi \cap Mono.$
Theorem

$\mathcal{DR}(M)$ is enriched over CPO.
CPO Enrichment

**Theorem**

$\text{DR}(M)$ is enriched over CPO.

Curiously, not over COH or even STAB.
**Theorem**

$\mathcal{DR}(M)$ is enriched over CPO.

In particular, every $f : A \rightarrow A$ has a least fixpoint $\mu f = \sum_{n \in \mathbb{N}} f^n \circ \bot$ (where $I = 0$, $\bot = $ all-zero morphism):
Duality

Theorem

There is a dualizing functor, inducing an isomorphism of categories, \( \perp : DR(M) \rightarrow DR(M)^{op} \):

- Objects: \( A_\perp = A \);
- Morphisms:

\[
\begin{align*}
\begin{array}{c}
\bullet A \\
ar & \gamma & a^* \\
\circ B & \beta & \circ B
\end{array}
\end{align*}
\]

\( \perp \)

\[
\begin{align*}
\begin{array}{c}
\bullet B \\
a^* & \gamma & a \\
\circ A & \beta & \circ A
\end{array}
\end{align*}
\]

Jean Goubault-Larrecq

GoI and Coherence
Outline

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Linear Inverse Monoids

The Danos-Regnier Category
  Inductive Groupoids
  Introducing $DR(M)$

The Weakly Cantorian Case
Miserable Failures (Ignoring Great Successes)

Coherence Completions
  The Modified Hu-Joyal Construction
  Lifting Monoidal Structure, Creating the Rest
  Relation to Syntax?
  Towards Fibrations?

Conclusion
Remember Our Example?

\[
pppp^*p^*q^*p^*p^* \\
+ ppqpp \ p^*p^*p^* \\
+ ppqpq \ q^*p^* \\
+ ppqqq \ q^* \\
+ qq^*p^*q^*p^*p^* \\
+ \ q^*q^*q^*p^*p^* \\
\]
(Weakly) Cantorian Linear Inverse Monoids

This can be encoded provided $M$ is weakly Cantorian, i.e., contains two elements $p, q$ such that:

$$p^* q = 0 \quad \langle p^* \rangle = \langle q^* \rangle = 1$$

(Cantorian if additionally, $\langle p \rangle + \langle q \rangle = 1$.)
(Weakly) Cantorian Linear Inverse Monoids

This can be encoded provided $M$ is weakly Cantorian, i.e., contains two elements $p, q$ such that:

$$p^* q = 0 \quad \langle p^* \rangle = \langle q^* \rangle = 1$$

(Cantorian if additionally, $\langle p \rangle + \langle q \rangle = 1$.)

This provides a symmetric monoidal structure:

$$u \otimes v \text{ abbreviates } p u p^* + q v q^*.$$
**DR(M) is Compact-Closed**

- In other words, it is a model of (classical) MLL.
- We already have $\otimes$ and $(\_)^\perp$.
- Let us define:
  - $A \rightarrow B = qAq^* + pBp^* \ (\equiv B \otimes A)$;
  - Linear application $\text{app}_{A,B} : (A \rightarrow B) \otimes A \rightarrow B$;
  - Linear abstraction $\lambda^C_{A,B}$,
    so that $\lambda^C_{A,B}(f) : C \rightarrow (A \rightarrow B)$ for each $f : C \otimes A \rightarrow B$;
  - The $\sim_{A} : \sim \rightarrow A \rightarrow A$ operator, where $\sim A = A \rightarrow 0$;
- Of course, all usual equations should be satisfied.
Linear Application

\[
\text{app}_{A,B} = \text{(A} \otimes \text{B)} \otimes A
\]

Jean Goubault-Larrecq

GoI and Coherence
Linear Abstraction

\[ \lambda_{A,B}^{C} \left( \begin{array}{c} C \otimes A \\ a \\ B \\ B \\ a^* \end{array} \rightarrow \begin{array}{c} C \otimes A \\ \beta \\ B \\ B \\ a^* \end{array} \right) = \]

\begin{array}{c}
\bullet C \\
pC \\
q A \\
pB \\
A \rightarrow B
\end{array} \rightarrow \begin{array}{c}
\bullet C \\
q^* \\
pB \\
A \rightarrow B
\end{array}
Let $\sim A$ be $A \rightarrow 0$ (linear negation, $\cong A^\perp = A$).

\[ C_A = A \rightarrow q^2 A \rightarrow A \rightarrow q^2 A \rightarrow A \]
Compact Closure

Theorem
$\mathcal{DR}(M)$ is compact-closed.
Compact Closure

**Theorem**

$\mathcal{DR}(M)$ is compact-closed.

**Proof**

Funny calculations, repeatedly replacing crossings

\[ p \quad p^* \]

\[ q \quad q^* \]

by double connections

\[ \quad \quad \quad \quad \quad \quad \]
My Idea of Fun?
Every compact-closed category has a canonical trace. In $\mathcal{DR}(M)$,

$$
\text{Tr}^X_{A,B} \left( \begin{array}{c}
\bullet A \otimes X \xrightarrow{\beta} A \otimes X \\
\downarrow a \\
\circ B \otimes X \xleftarrow{\gamma} B \otimes X \circ
\end{array} \right) =
\begin{array}{c}
\bullet A \\
pA
\end{array} \rightarrow
\begin{array}{c}
A \otimes X \\
a
\end{array} \rightarrow
\begin{array}{c}
A \otimes X \\
a^* \circ
\end{array}
$$

Jean Goubault-Larrecq
Gol and Coherence
Relation to GoI Situations

- Start from $\mathcal{P}\mathcal{G}(M)$, whose morphisms are $A \xrightarrow{u} B$ where $\langle u^* \rangle \leq A$ and $\langle u \rangle \leq B$ (instead of equality as in $\mathcal{G}(M)$).
- Apply the $\mathcal{G}$ construction. Composition by symmetric feedback:

\[
\begin{array}{c}
C^- & \xrightarrow{g} & C^+ \\
B^+ & \xrightarrow{f} & B^- \\
B^- & \xrightarrow{g} & B^+ \\
A^+ & \xrightarrow{f} & A^- \\
\end{array}
\]
Relation to GoI Situations, cont’d

Each \( f \) in \( \mathcal{PJG}(M) \) is, equivalently:

\[
\begin{align*}
A^+ & \xrightarrow{f^{++}} B^+ \\
A^+ & \xrightarrow{f^{+-}} A^-
\end{align*}
\]

\[
\begin{align*}
B^- & \xrightarrow{f^{-+}} B^+ \\
B^- & \xrightarrow{f^{--}} A^-
\end{align*}
\]
Each $f$ in $\mathcal{PJM}(M)$ is, equivalently:

\[
\begin{align*}
A^+ & \xrightarrow{f++} B^+ & A^+ & \xrightarrow{f+-} A^- \\
B^- & \xrightarrow{f-+} B^+ & B^- & \xrightarrow{f--} A^- \\
A^+ & \xrightarrow{f+-} A^- \\
B^+ & \xrightarrow{f--} B^- \\
B^- & \xrightarrow{f-+} B^+
\end{align*}
\]

Write this $f++$, by imitation...
Theorem

$\mathcal{DR}(M)$ is a subcategory of $\mathcal{G}(\mathcal{PIG}(M))$.

Proof (essentially)
Relation to GoI Situations, cont’d (2)

**Theorem**
\( \mathcal{DR}(M) \) is a subcategory of \( \mathcal{G}(P\mathcal{I}G(M)) \).

**Proof (essentially)**

Only difference: in \( \mathcal{DR}(M) \), you require some symmetry:

\[
\begin{align*}
  f^{+++} &= f^{--}, & f^{+--} &= f^{+++}, & f^{---} &= f^{++}.
\end{align*}
\]

Intuitively, if you can go from \( A \) to \( B \) by \( u \), you can come back from \( B \) to \( A \) by \( u^* \).
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Great Successes

- New insights into implementations of the \( \lambda \)-calculus;
- Wadsworth-Lévy labels and sharing graphs;
- Optimal \( \lambda \)-reduction.
Great Successes

- New insights into implementations of the $\lambda$-calculus;
- Wadsworth-Lévy labels and sharing graphs;
- Optimal $\lambda$-reduction.

This being said, let me gently recall that I am not interested in computation here, but in equality of Böhm(-like) trees. I would therefore like to have an implementation of full $\lambda$-calculus, at least, and why not a categorical model of linear logic?
Additives

Equipping $\mathcal{DR}(M)$ with additives means finding enough limits:

- Terminal, initial objects;
- Products, coproducts.
No Limits (1): Terminal, Initial Objects

**Theorem**

The following statements are equivalent:

1. \(DR(M)\) has a terminal object;
2. 0 is terminal in \(DR(M)\);
3. \(DR(M)\) has an initial object;
4. 0 is initial in \(DR(M)\);
5. \(M = \{0\}\).

**Proof**

Calculation. Uniqueness of morphisms is essential!
No Limits (2): Products, Coproducts

**Theorem**
Let $A$ and $B$ be any two objects of $\mathcal{DR}(M)$. Then the following conditions are equivalent:

1. $A \times B$ exists;
2. $A + B$ exists;
3. $M = \{0\}$.

**Proof**
Even more boring calculation.
Theorem
Let $A$ and $B$ be any two objects of $\mathcal{DR}(M)$. Then the following conditions are equivalent:

1. $A \times B$ exists;
2. $A + B$ exists;
3. $M = \{0\}$.

Proof
Even more boring calculation.

Consequence: $\mathcal{DR}(M)$ hates additives.
Theorem
Let $A$ and $B$ be any two objects of $\mathcal{DR}(M)$. Then the following conditions are equivalent:

1. $A \times B$ exists;
2. $A + B$ exists;
3. $M = \{0\}$.

Proof
Even more boring calculation.

Consequence: $\mathcal{DR}(M)$ hates additives. (But do we care?)
Exponentials, Anyone?

Try to turn $\mathcal{DR}(M)$ into a model of MELL. Several possibilities (essentially equivalent, see Melliès 2002):

- as a new-Lafont category, i.e.:
  - a $*$-autonomous category $\mathcal{C}$ ($= \mathcal{DR}(M)$?),
  - a full sub-monoidal category $\mathcal{M}$ of the category of cocommutative comonoids over $\mathcal{C}$,
  - $U : \mathcal{M} \to \mathcal{C}$ has a right adjoint $F$. 
Try to turn $\mathcal{DR}(M)$ into a model of MELL. Several possibilities (essentially equivalent, see Melliès 2002):

- as a *new-Lafont category*, i.e.:
  - a $\ast$-autonomous category $\mathcal{C}$ ($= \mathcal{DR}(M)$?),
  - a full sub-monoidal category $\mathcal{M}$ of the category of cocommutative comonoids over $\mathcal{C},$
  - $U : \mathcal{M} \to \mathcal{C}$ has a right adjoint $F.$

- as an *LNL category* (Benton, 1995), i.e.:
  - a $\ast$-autonomous category $\mathcal{C}$ ($= \mathcal{DR}(M)$?),
  - a category $\mathcal{M}$ with finite products,
  - a symmetric monoidal adjunction $U \dashv F$ between some $U : \mathcal{M} \to \mathcal{C}$ and some $F : \mathcal{C} \to \mathcal{M}.$
Exponentials, Anyone?

Try to turn $\mathcal{DR}(M)$ into a model of MELL. Several possibilities (essentially equivalent, see Melliès 2002):

- as a *new-Lafont category*, i.e.:
  - a $\ast$-autonomous category $\mathcal{C}$ ($= \mathcal{DR}(M)$?),
  - a full sub-monoidal category $\mathcal{M}$ of the category of cocommutative comonoids over $\mathcal{C}$,
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  - a category $\mathcal{M}$ with finite products,
  - a symmetric monoidal adjunction $U \dashv F$ between some $U : \mathcal{M} \rightarrow \mathcal{C}$ and some $F : \mathcal{C} \rightarrow \mathcal{M}$.

- as a *linear category* (Bierman, 1995) [definition omitted].

In any case, studying cocommutative comonoids over $\mathcal{DR}(M)$ must be the key.
Comonoids in $\mathcal{DR}(M)$ ($M$ Weakly Cantorian)

A comonoid $(A, d_A, e_A)$ verifies 1. left unit, 2. right unit, 3. associativity.

**Theorem**
If $(A, d_A, e_A)$ verifies 1 and 2, there is a partition $A_p, A_q$ of $A$ (i.e., $A = A_p + A_q, A_pA_q = 0$) such that

$$d_A = \begin{array}{ccc}
\bullet & A & \leftarrow \\
\circ & A \otimes A & \\
\begin{array}{c}
q \beta_0 p^* + p \beta_0 q^*
\end{array}
\end{array}$$

where $\beta_0$ is iso between $A_p$ and $A_q$.

Then $d_A$ is associative, so $(A, d_A, e_A)$ is a comonoid in $\mathcal{DR}(M)$.
(Almost) No Comonoid is Cocommutative

Theorem
The only cocommutative comonoid of $DR(M)$ is

\[
\begin{array}{cc}
0 & 0 \\
0 & 0 \\
\end{array}
\]

\[
A = 0 \quad d_0 = \quad e_0 =
\begin{array}{cc}
0 \otimes 0 & 0 \otimes 0 \\
0 \otimes 0 & 0 \\
\end{array}
\]

Proof
Cocommutativity implies $A_p = A_q$. But $A_p$ and $A_q$ do not intersect, so $A_p = A_q = 0$. So $A = A_p + A_q = 0$. 
(Almost) No Comonoid is Cocommutative

**Theorem**
The only cocommutative comonoid of $\mathcal{DR}(M)$ is

$$
\begin{array}{ccc}
\circ & 0 & \circ \\
0 & 0 & 0
\end{array}
$$

$$
A = 0 \quad d_0 =
\begin{array}{ccc}
0 \otimes 0 & 0 \otimes 0 & 0 \\
0 & 0 & 0
\end{array}
$$

$$
e_0 =
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0
\end{array}
$$

**Proof**
Cocommutativity implies $A_p = A_q$. But $A_p$ and $A_q$ do not intersect, so $A_p = A_q = 0$. So $A = A_p + A_q = 0$.

Note that this is the only cocommutative comonoid of $\mathcal{DR}(M)$ which is guaranteed to exist in any symmetric monoidal category...
No Exponentials

Corollary

- \( \mathcal{DR}(M) \) is the \( \mathcal{C} \) (linear) component of a new-Lafont category if and only if \( M = \{0\} \);
- \( \mathcal{DR}(M) \) is a linear category if and only if \( M = \{0\} \);
- \( \mathcal{DR}(M) \) is the \( \mathcal{C} \) (linear) component of an LNL category if and only if \( M = \{0\} \).
No Exponentials

Corollary

- $\mathcal{DR}(M)$ is the C (linear) component of a new-Lafont category if and only if $M = \{0\}$;
- $\mathcal{DR}(M)$ is a linear category if and only if $M = \{0\}$;
- $\mathcal{DR}(M)$ is the C (linear) component of an LNL category if and only if $M = \{0\}$.

Consequence: $\mathcal{DR}(M)$ hates exponentials, too.
Corollary

- $\mathcal{DR}(M)$ is the $\mathcal{C}$ (linear) component of a new-Lafont category if and only if $M = \{0\}$;
- $\mathcal{DR}(M)$ is a linear category if and only if $M = \{0\}$;
- $\mathcal{DR}(M)$ is the $\mathcal{C}$ (linear) component of an LNL category if and only if $M = \{0\}$.

Consequence: $\mathcal{DR}(M)$ hates exponentials, too. (We do care.)
\( \mathcal{DR}(M) \) is a Weak Linear Category

- Every linear inverse semigroup endomorphism on \( M \) defines a traced sym. mon. endofunctor \( ! \) on \( \mathcal{DR}(M) \); (And there are not that many others.)
- The retractions \( \theta : A \triangleleft B : \theta' \) of \( A \) into \( B \) in \( \mathcal{PIG}(M) \) are the morphisms \( \theta \) such that \( \langle \theta^* \rangle = A, \langle \theta \rangle \leq B \), and \( \theta' = \theta^* \).
- The natural monoidal retractions are of the form \( uF(A) : F(A) \triangleleft G(A) : F(A)u^* \) for some fixed \( u \in M \) such that \( \langle u^* \rangle = F(1) \).
- \( \ldots \) Essentially allows one to get back the Danos-Régnier interpretation over \( \text{PI}(\mathbb{N}) \), or a Girard-like interpretation over rudimentary clauses.
- Note that \( \delta, \epsilon \) fail all the comonad equations. For computation, this is enough, though.
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There is a way out!
Modify a construction by Hu and Joyal (1997):

- The *coherence completion* of $\mathcal{C}$ is $\ast$-autonomous when $\mathcal{C}$ is.
- Any other structure (comonad, product) lifts from $\mathcal{C}$ to its coherence completion.
There is a way out!
Modify a construction by Hu and Joyal (1997):

- The coherence completion of $\mathcal{C}$ is $\ast$-autonomous when $\mathcal{C}$ is.
- Any other structure (comonad, product) lifts from $\mathcal{C}$ to its coherence completion.
- New: if you take the identity comonad on $\mathcal{C}$, and the multiclique exponential on coherence spaces, then this construction creates exponentials and additives in the coherence completion.
Coherence Completion of $\mathcal{C}$: Objects

Pairs $(X, (A_i)_{i \in X})$ where each $A_i$ is an object of $\mathcal{C}$. 

\[ \begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5 \\
A_6 \\
\end{array} \]

\[ \begin{array}{c}
|X| \\
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array} \]
Coherence Completion of $\mathcal{C}$: Morphisms

Pairs of a linear map between the bases, and morphisms in $\mathcal{C}$ for each connected elements:
Coherence Completion of $\mathcal{C}$: Morphisms

Pairs of a linear map between the bases, and morphisms in $\mathcal{C}$ for each connected elements:
This is a Category!

The following never happens:
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Monoidal Structure

Inherited from \( \mathcal{C} (= DR(M)) \) and COH:

- \((X, (A_i)_{i \in |X|} \otimes (Y, (B_j)_{j \in |Y|})) = (X \otimes Y, (X_i \otimes Y_j)_{(i,j) \in |X\otimes Y|})\);

- Remember that in coherence spaces, \(|X \otimes Y| = |X| \times |Y|\), with pairwise coherence.
Monoidal Structure

Inherited from $\mathcal{C} (= \mathcal{DR}(M))$ and COH:

- $(X, (A_i)_{i \in |X|} \otimes (Y, (B_j)_{j \in |Y|})) = (X \otimes Y, (X_i \otimes Y_j)_{(i,j) \in |X| \otimes |Y|});$
- Remember that in coherence spaces, $|X \otimes Y| = |X| \times |Y|$, with pairwise coherence.
- $(X, (A_i)_{i \in |X|} \circ (Y, (B_j)_{j \in |Y|})) = (X \circ Y, (X_i \circ Y_j)_{(i,j) \in |X| \circ |Y|});$
- Remember that in coherence spaces, $|X \circ Y| = |X| \times |Y|$, with $(x, y) \sqsubseteq (x', y')$ iff $x \sqsubseteq x'$ implies $(y \sqsubseteq y'$ and $y = y'$ implies $x = x'$) (linear maps);
Coprodcts, Products

Coproducts inherited purely from COH:

\[ |X| \]

\[ A_1, A_2, A_3, A_4, A_5, A_6 \]

\[ |Y| \]

\[ B_1, B_2, B_3, B_4, B_5, B_6 \]
Coproducts, Products

Coproducts inherited purely from COH:

Products: similar, except all dots from $X$ are coherent with all dots from $Y$. 
The ! Comonad

Inherited from COH, using the symmetric monoidal structure from $\mathcal{C} = \mathcal{DR}(M)$:

- $!(X, (A_i)_{i \in I}) = (X, (\bigotimes_{i \in e} A_i)_{e \in |!X|});$

- Requires some fine, boring, formal points (e.g., equipping the $X$ part with a total ordering to make $\bigotimes_{i \in e} A_i$ well-defined. . . )

- $|!X|$ is set of *multicliques* of $X$—Would not work with just cliques.
The **shriek** (well, part of it) of

\[ |X| \]

is

\[ \emptyset \]
And Fixpoints!

- Again inherited purely from COH:
  - (Recursion game.) For any (multi)clique $e$ of $!X \rightarrow X$, define $\text{rec} \mathcal{D}_e \subseteq |X|$ as the smallest subset such that whenever $(e', x) \in e$ and $e' \subseteq \text{rec} \mathcal{D}_e$, then $x \in \text{rec} \mathcal{D}_e$.
  - For every coherence space $A = (|A|, \sqcup_A)$,
    $$Y_A = \{ (0, (e, x)) | x \in \text{rec} \mathcal{D}_e, e \text{ multiclique of } !A \rightarrow A \}$$
    is a fixpoint operator $!:(!A \rightarrow A) \rightarrow A$.
  - Application to the coherence completion: exercice!
Again inherited purely from COH:

- (Recursion game.) For any (multi)clique $e$ of $!X \rhd X$, define $\text{rec}e \subseteq |X|$ as the smallest subset such that whenever $(e', x) \in e$ and $e' \subseteq \text{rec}e$, then $x \in \text{rec}e$.
- For every coherence space $A = (|A|, \sqcup_A)$,

$$Y_A = \{(0, (e, x)) \mid x \in \text{rec}e, \text{e multiclique of } !A \rhd A\}$$

is a fixpoint operator $!:(!A \rhd A) \rhd A$.
- Application to the coherence completion: exercice!

This is slightly strange: the fact that $\mathcal{DR}(M)$ itself has (linear) fixpoints does not play any role in the construction.
The coherence completion of any *-autonomous category (including $\mathcal{DR}(M)$ when $M$ is weakly Cantorian) is a model of full linear logic (a linear category) with fixpoints.
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Proof Nets for Full Linear Logic: LLPA

Inspired from Abramsky’s (1993) Linear CHAM:

Terms

\[ s, t, \ldots = x \]

\[
\begin{align*}
& \text{name} \\
& \text{stop} \\
& \text{abort} \\
& \text{\(\lambda\)-abstraction} \\
& \text{application} \\
& \text{empty tuple} \\
& \text{ext. choice, pairing} \\
& \text{first projection} \\
& \text{second projection} \\
& \text{dissolve} \\
& \text{box}
\end{align*}
\]

Solutions

\[ S ::= m_1, \ldots, m_n \]

Matches

\[ m ::= N \iff s \]

Name sets

\[ N ::= \{ x_1, \ldots, x_n \} \]
Well-Formedness, Types

- Unusual notion of free/bound name:
  - $x$ is *free* in $t \ (S, m)$ iff it occurs once in it;
  - $x$ is *bound* in $t \ (S, m)$ iff it occurs twice in it;
  - No name can occur more than twice.
Well-Formedness, Types

- Unusual notion of free/bound name:
  - $x$ is *free* in $t (S, m)$ iff it occurs *once* in it;
  - $x$ is *bound* in $t (S, m)$ iff it occurs *twice* in it;
  - No name can occur more than twice.
  - Caveat: “once” and “twice” have a different meaning in the presence of additive boxes $(S \cdot x \boxplus T \cdot y)$, where $S \cdot x$ and $T \cdot y$ are constrained to have exactly the same free names.
Well-Formedness, Types

- Unusual notion of free/bound name:
  - \( x \) is *free* in \( t \ (S, m) \) iff it occurs *once* in it;
  - \( x \) is *bound* in \( t \ (S, m) \) iff it occurs *twice* in it;
  - No name can occur more than twice.
  - Caveat: “once” and “twice” have a different meaning in the presence of additive boxes \( (S \cdot x \llparenthesis T \cdot y) \), where \( S \cdot x \) and \( T \cdot y \) are constrained to have exactly the same free names.

- Judgments \( \Gamma \vdash S \dashv \Delta \), where each name occurs *twice*.

- Use a minimal set of connectives:

\[
F, G, H, \ldots \ ::= \ A \quad \text{atomic type} \\
\mid \bot \quad \text{false (multiplicativ}\mathcal{e}) \\
\mid T \quad \text{true (additive)} \\
\mid F \multimap G \quad \text{linear implication} \\
\mid F \& G \quad \text{with (Cartesian product)} \\
\mid !F \quad \text{of course}
\]
Typing Rules (1): Identity Group

(Ax)

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(S₂)

(S₁)
Typing Rules (2): Multiplicatives, $\rightarrow^\circ$ Right

\[
\frac{\Gamma \vdash - \circ \Delta}{\Gamma \vdash S \rightarrow^\circ \Delta}
\]
Typing Rules (3): Multiplicatives, $\circ$ Left

\[ \Gamma_1 \vdash x_1 : G, \Gamma_2 \vdash x_2 : F \]

\[ \Gamma \vdash (x_1 \circ x_2) : \circ F \]

Jean Goubault-Larrecq
Typing Rules (4): Multiplicative Unit ⊥
Typing Rules (5): Exponential Promotion

\[ \Gamma \vdash S \vdash x : F \]

\[ \Gamma \vdash !S \vdash z : !F \]
Typing Rules (6): Exponential Dereliction

\[
\frac{x: F}{(\downarrow \vdash) \quad \frac{z: \forall F}{\Delta}}
\]
Typing Rules (7): Exponential Fan

\[ \Gamma \vdash x_1 : !F.S \Delta \\
\vdash x_2 : !F.S \Delta \\
\vdash x_n : !F.S \Delta \\

\text{(Fan)} \]
Typing Rules (8): Additives

- Omitted... hard to draw!
The Free Linear Category

**Theorem**

**LLPA** with the right reduction rules defines the free linear category over the given set of atomic formulae.

**Proof**

Free is obvious. What is harder is that the reduction rules terminate and obey all the right laws. Currently under way.
Fixpoints

One can also add typed fixpoints. No need for a new operator.

\[
\Gamma, x_1 :! F, \ldots, x_n :! F \vdash S \dashv y :! F, \Delta
\]

\[
\Gamma \vdash S, \{z, x_1, \ldots, x_n\} \triangleleft y \vdash z :! F, \Delta
\]

(FanRec)

Then the following is \( Y \):

Of course, termination fails in the presence of (FanRec).
Outline

Motivation
Linear Inverse Monoids
The Danos-Regnier Category
   Inductive Groupoids
   Introducing $DR(M)$
   The Weakly Cantorian Case
   Miserable Failures (Ignoring Great Successes)

Coherence Completions
The Modified Hu-Joyal Construction
Lifting Monoidal Structure, Creating the Rest
Relation to Syntax?
Towards Fibrations?

Conclusion
Towards Fibrations

- Coherence completions are categories of trivial fibrations over webs of coherence spaces.
Towards Fibrations

- Coherence completions are categories of trivial fibrations over webs of coherence spaces.

- Conjecture

  LLPA solutions define (non-trivial) fibrations over webs of coherence spaces.

  Necessarily non-trivial: the construction of $X \rightarrow Y (\exists X$, too?) is too big for syntax.
Towards Fibrations

- Coherence completions are categories of trivial fibrations over webs of coherence spaces.

- **Conjecture**
  - **LLPA** solutions define (non-trivial) fibrations over webs of coherence spaces.

  Necessarily non-trivial: the construction of $X \multimap Y$ (!$X$, too?) is too big for syntax.

- **Idea**:
  - total space of the fibration=solution;
  - get fibers by erasing boxes and selecting one input per exponential fan;
  - base space will then be space of fibers.
Reconstructing Solutions From Fibrations

- Think of each fiber $A_i$ as a chunk of a linear $\lambda$-term drawn on *tracing paper*.
- Then superpose each layer; *coherence* in the base space should ensure this makes sense.
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- Think of each fiber $A_f$ as a chunk of a linear $\lambda$-term drawn on *tracing paper*.
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Conclusion
Conclusion

In a nutshell: **Coherence**!

- In the base space: coherence spaces;
- In each fiber: coherence in linear inverse monoids.