An introduction to Asymmetric Topology and Domain Theory: Why, What and How

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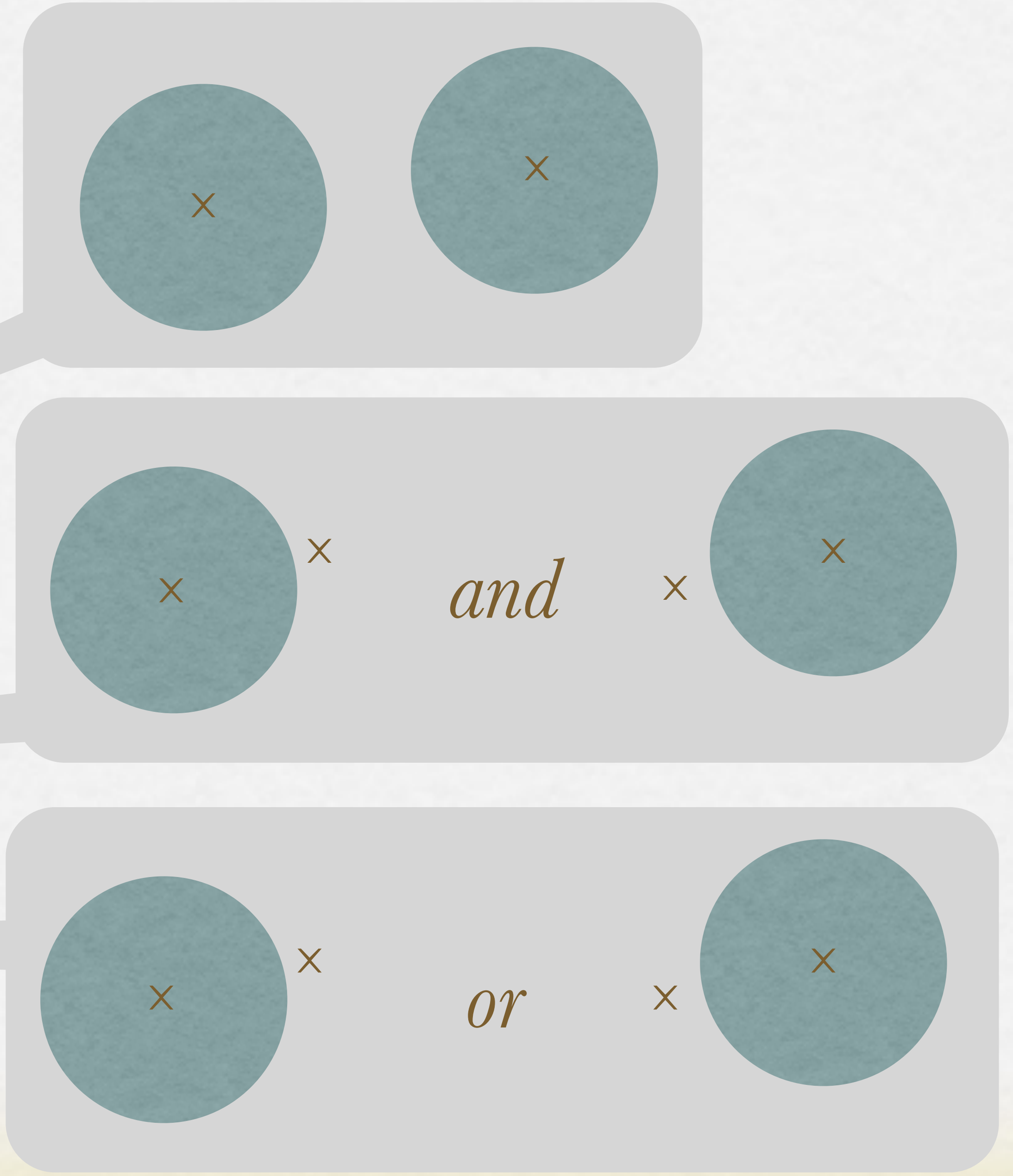
SEPARATION AXIOMS

- T_4 (normal + T_1)
- $T_{3\frac{1}{2}}$ (completely regular + T_1)
- T_3 (regular + T_1)
- T_2 (Hausdorff)
- T_1
- T_0
- None



T_0 SPACES: WHY

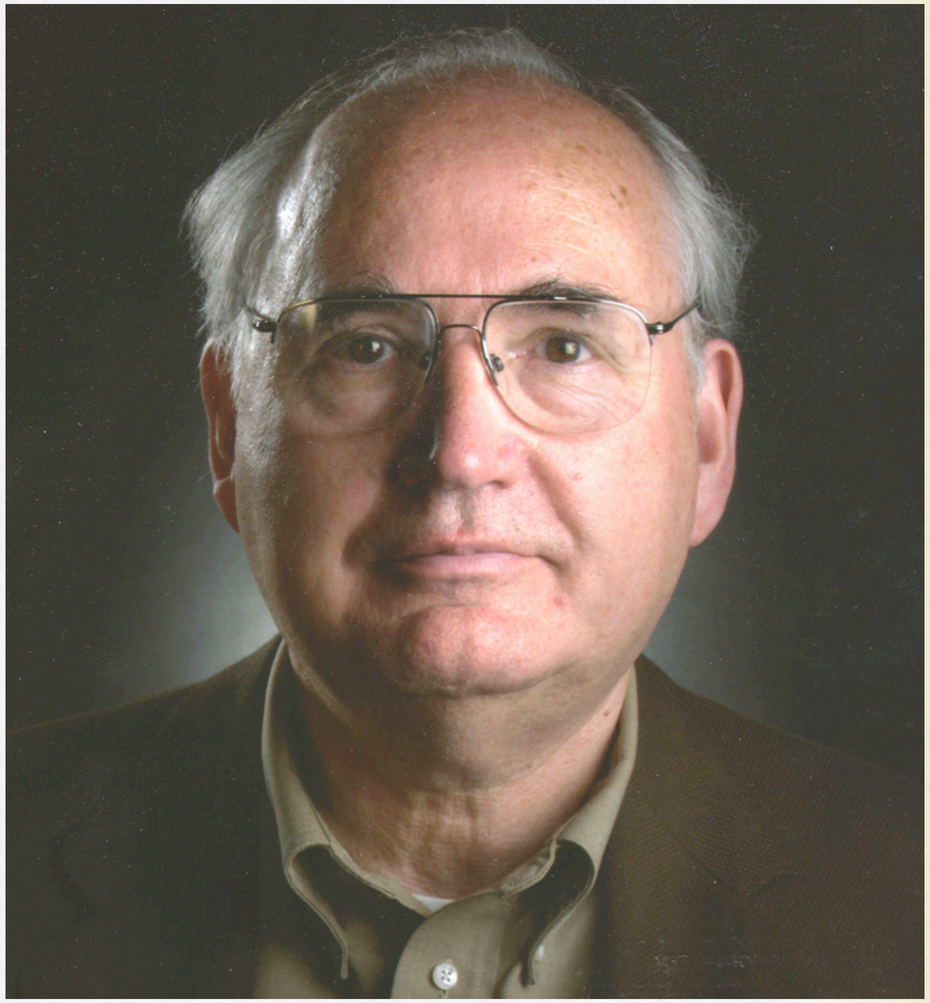
- Spectrum of rings, with their Zariski topology
(algebraic geometry)
- Stone duals of various kinds of posets
(fundamental link between topology and order theory)
- Domain theory
(order theory? computer science)

T_0 SPACES: WHAT

- Although many earlier results apply to T_0 or even general topological spaces,
  I would like to start with the birth of **domain theory** in logic and computer science.
- The purpose was to give **meaning to programs**, but I won't talk about that.
- Domain theory is concerned with (apparently) very simple T_0 spaces (certain posets), but:
  — this is deceptive, and
  — domain theory vastly helped us organize T_0 topology.

DOMAIN THEORY 101

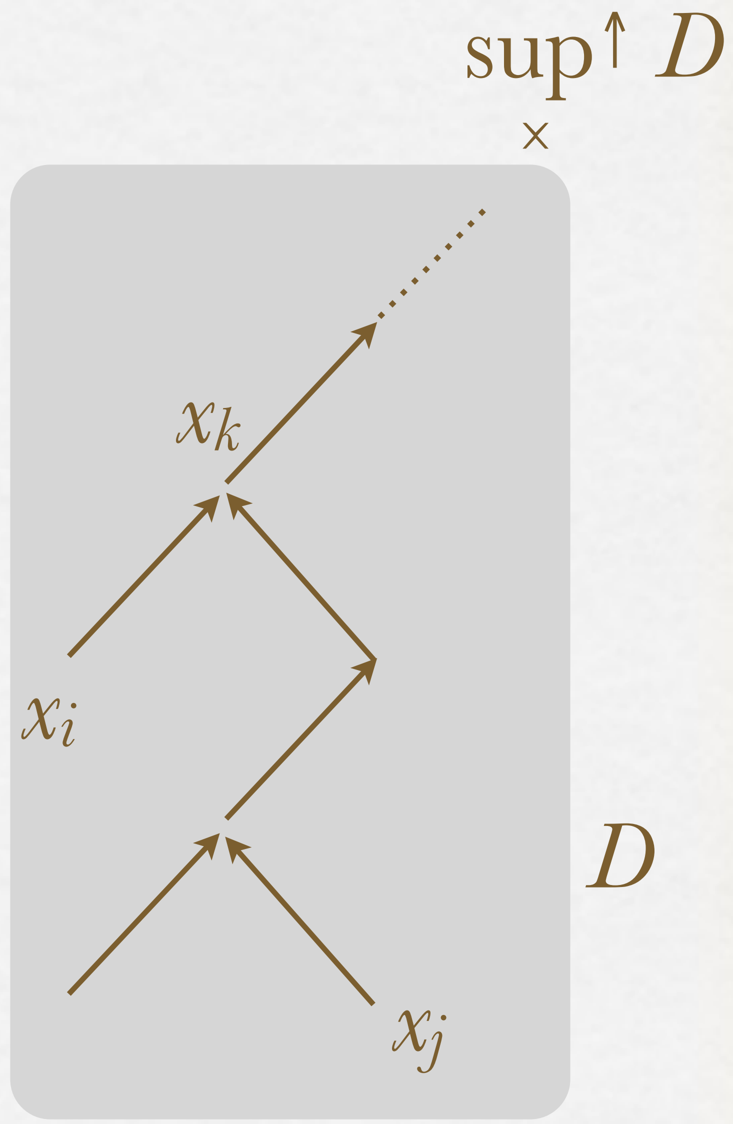
- Dana S. Scott, *A type-theoretic alternative to ISWIM, CUCH and OWHY*. Unpublished, 1969. Founding paper. «Re»printed, TCS 1993.



- From Wikipedia: «His research career involved computer science, mathematics, and philosophy. His work on automata theory earned him the ACM Turing Award in 1976, while his collaborative work with Christopher Strachey in the 1970s laid the foundations of modern approaches to the semantics of programming languages.»

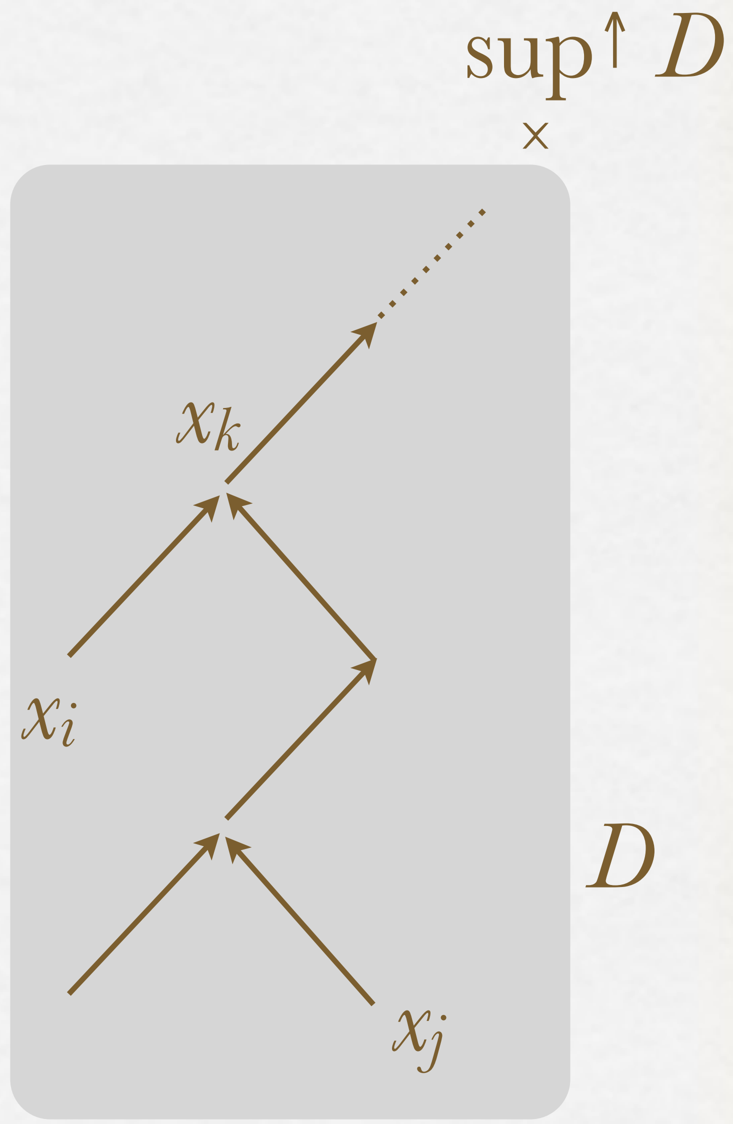
DCPOS

- A **directed complete** partial order (**dcpo**) is one where every directed family D has a least upper bound $\sup^{\uparrow} D$.
- $D = (x_i)_{i \in I}$ is directed iff non-empty, and for all i, j in I there is a k in I such that x_i , $x_j \leq x_k$.



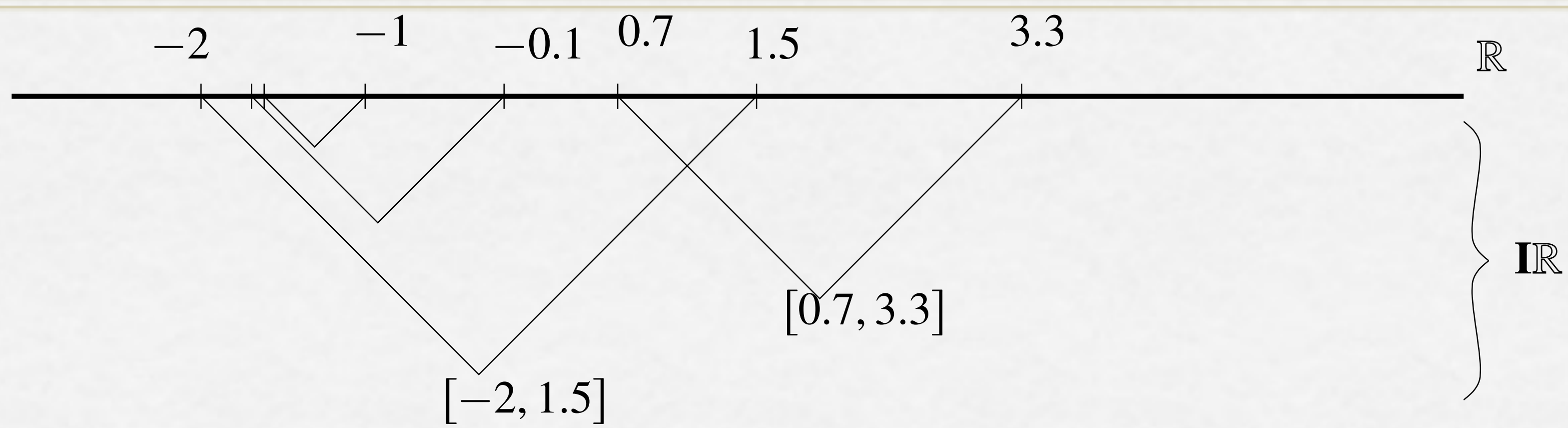
DCPOS

- Every chain is directed.
- Directed families are easier to work with than chains.
- Points are **partial values** ~ what partial information you get by typing ctrl-C
  max points are **total values**
  $\leq$ is order of **information**

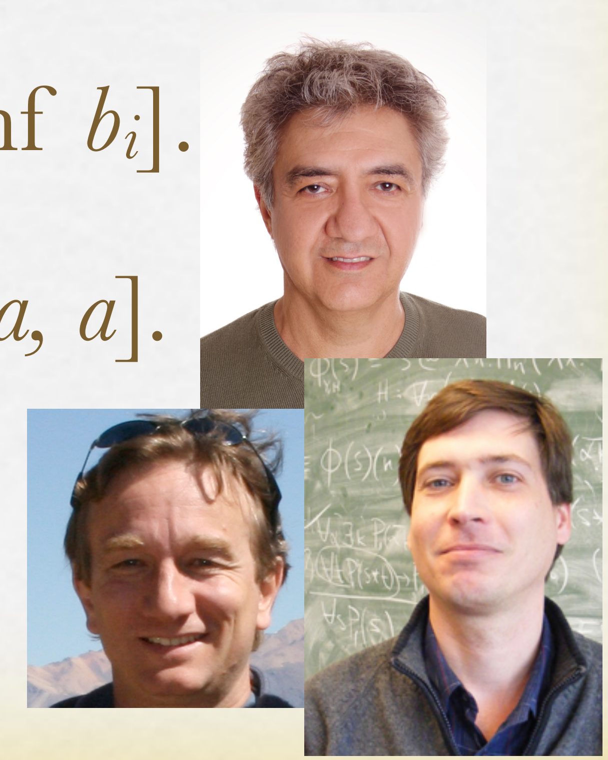


(1pt) Every point x in a dcpo is $\leq$ some maximal point. Why?

A SIMPLE EXAMPLE

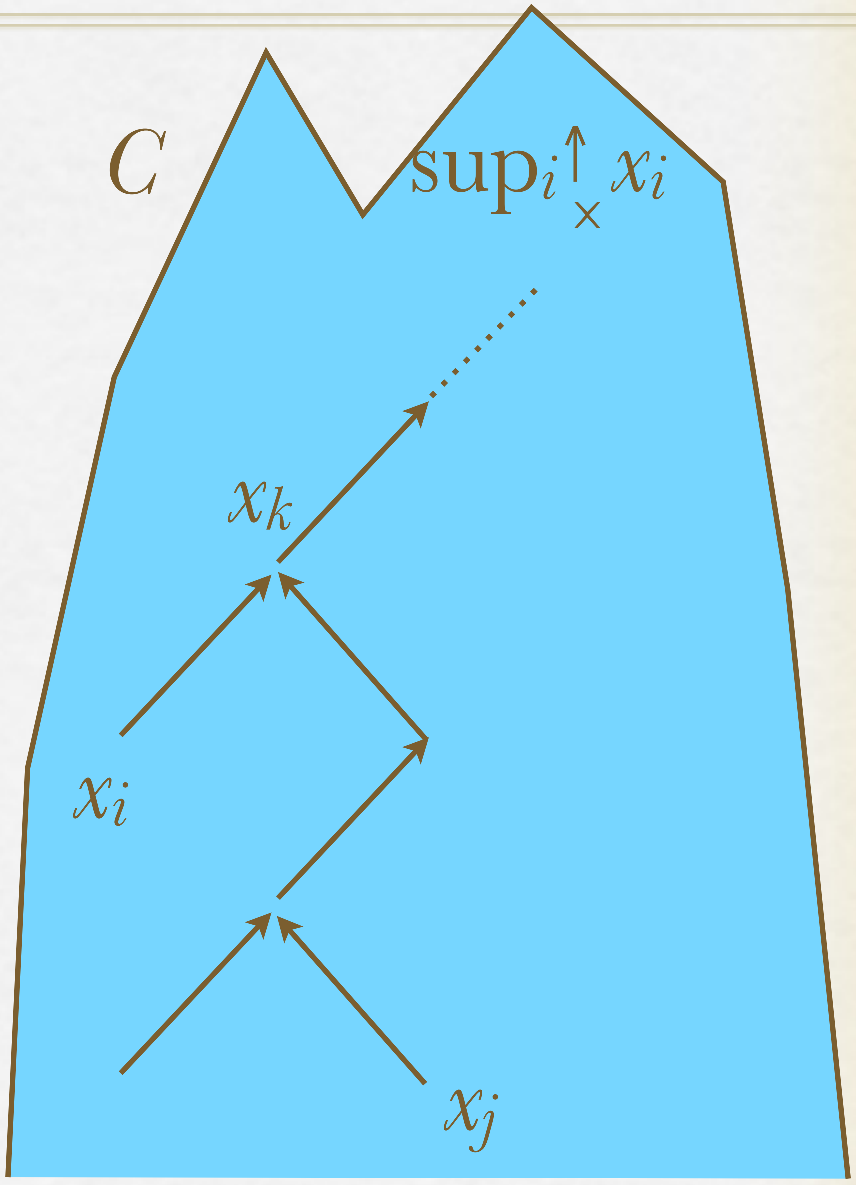


- $\mathbf{I}\mathbb{R}$={closed intervals $[a,b]$ of reals}, ordered by $\supseteq$.
- $\sup^{\uparrow}_{i \in I} [a_i, b_i] = \bigcap_{i \in I} [a_i, b_i] = [\sup a_i, \inf b_i]$.
- Total values are… just reals a, coded as $[a, a]$.
- This dcpo is useful in modeling **exact real arithmetic** in computers [Edalat, Potts, Sünderhauf, Escardó].



THE SCOTT TOPOLOGY

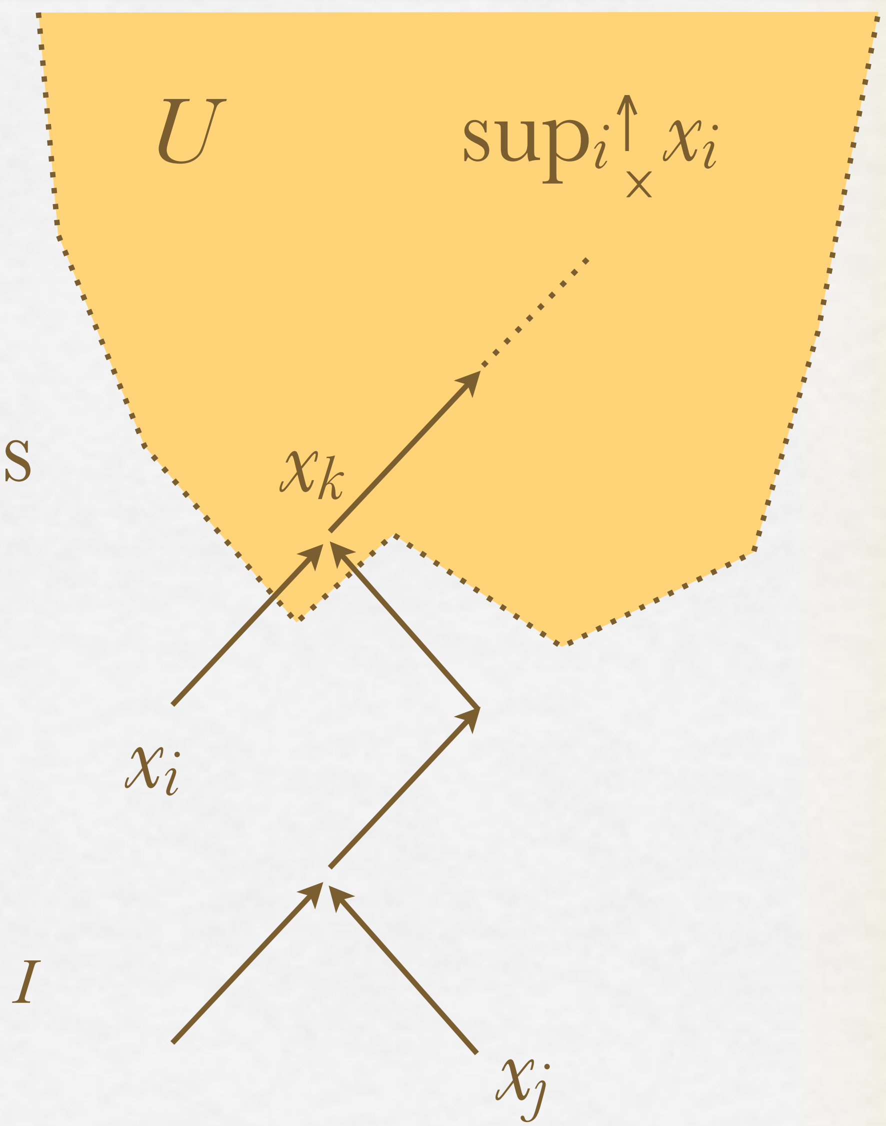
- A subset C of a dcpo is **Scott-closed** iff:
  — C is downwards-closed, and
  — C is closed under directed sups



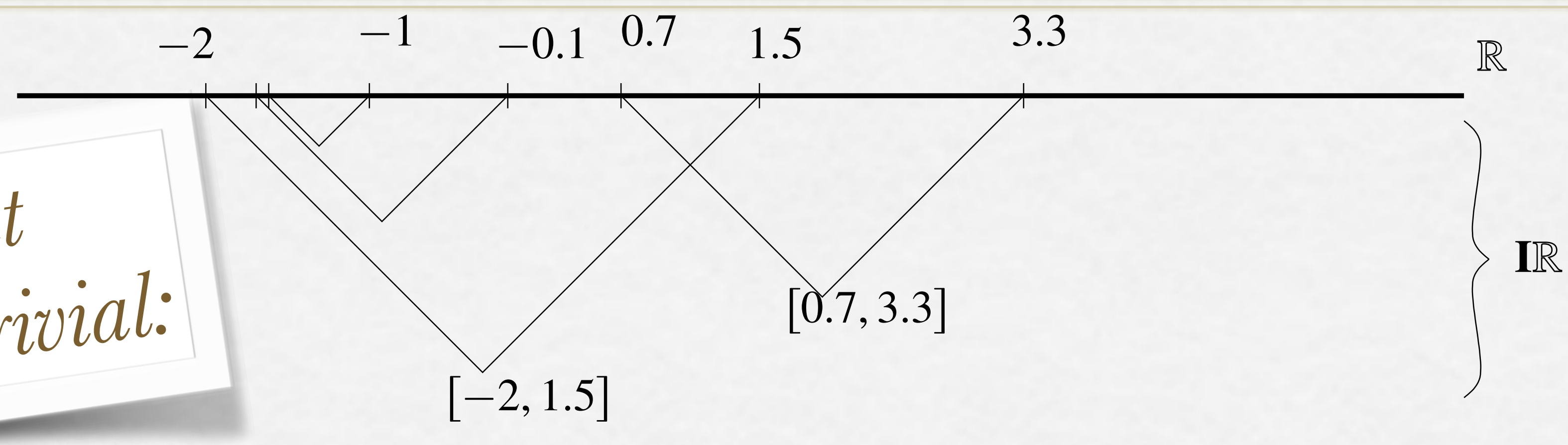
THE SCOTT TOPOLOGY

- A subset C of a dcpo is **Scott-closed** iff:
  — C is downwards-closed, and
  — C is closed under directed sups

- A subset U of a dcpo is **Scott-open** iff:
  — U is upwards-closed, and
  — for every directed family $(x_i)_{i \in I}$ with sup in U, some x_i is in U.



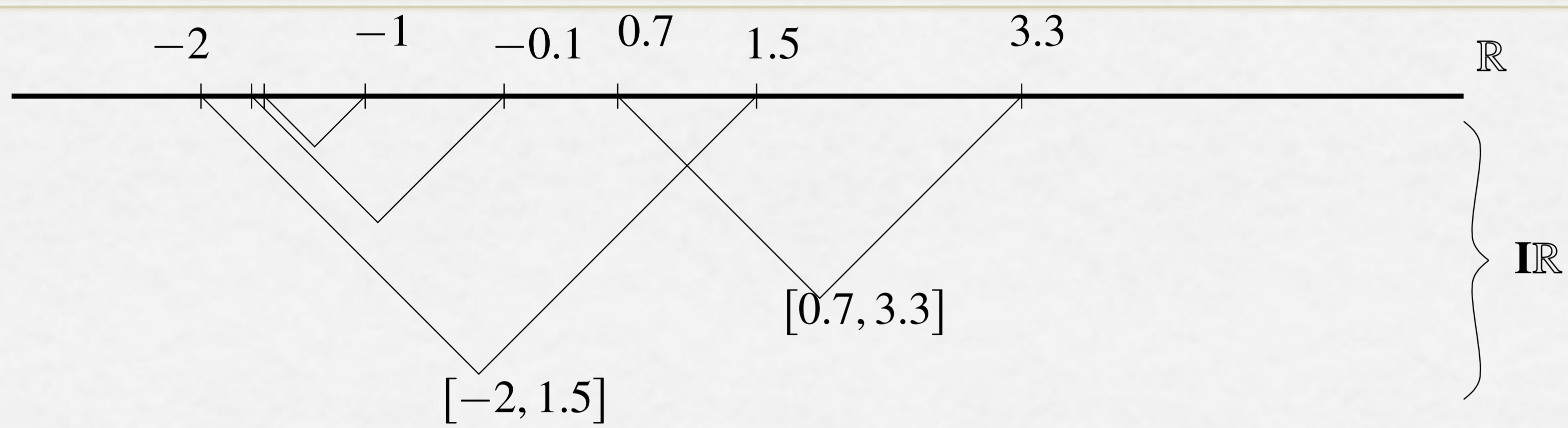
THE $\mathbf{I}\mathbb{R}$ MODEL



In case you thought Scott topologies were trivial:

- The Scott topology on $\mathbf{I}\mathbb{R}$ induces a topology on $\mathbb{R}$.
- **Fact:** That topology is the usual one on $\mathbb{R}$.
- *Proof.* ($\supseteq$) Check that $\uparrow[a,b]=\{[c,d] \mid a<c\leq d<b\}$ is Scott-open. Its trace on $\mathbb{R}$ is (a,b).
  ($\subseteq$) If $U=V\cap\mathbb{R}$, V Scott-open, let x in U: $x=\sup^{\uparrow}_{\varepsilon}[x-\varepsilon, x+\varepsilon]$, so some $[x-\varepsilon, x+\varepsilon]\in V$. Then $(x-\varepsilon, x+\varepsilon)\subseteq U$. □

MODELS



- A **model** of a (T_1) space X is any dcpo that embeds X as its subspace of maximal elements.
A vast subject! [Lawson, Martin] E.g.:

- **Thm** (Martin, 2003): The T_3 spaces that have an ω-continuous model are exactly the **Polish spaces**.

- But I won't talk about that... Let's get back to basics.

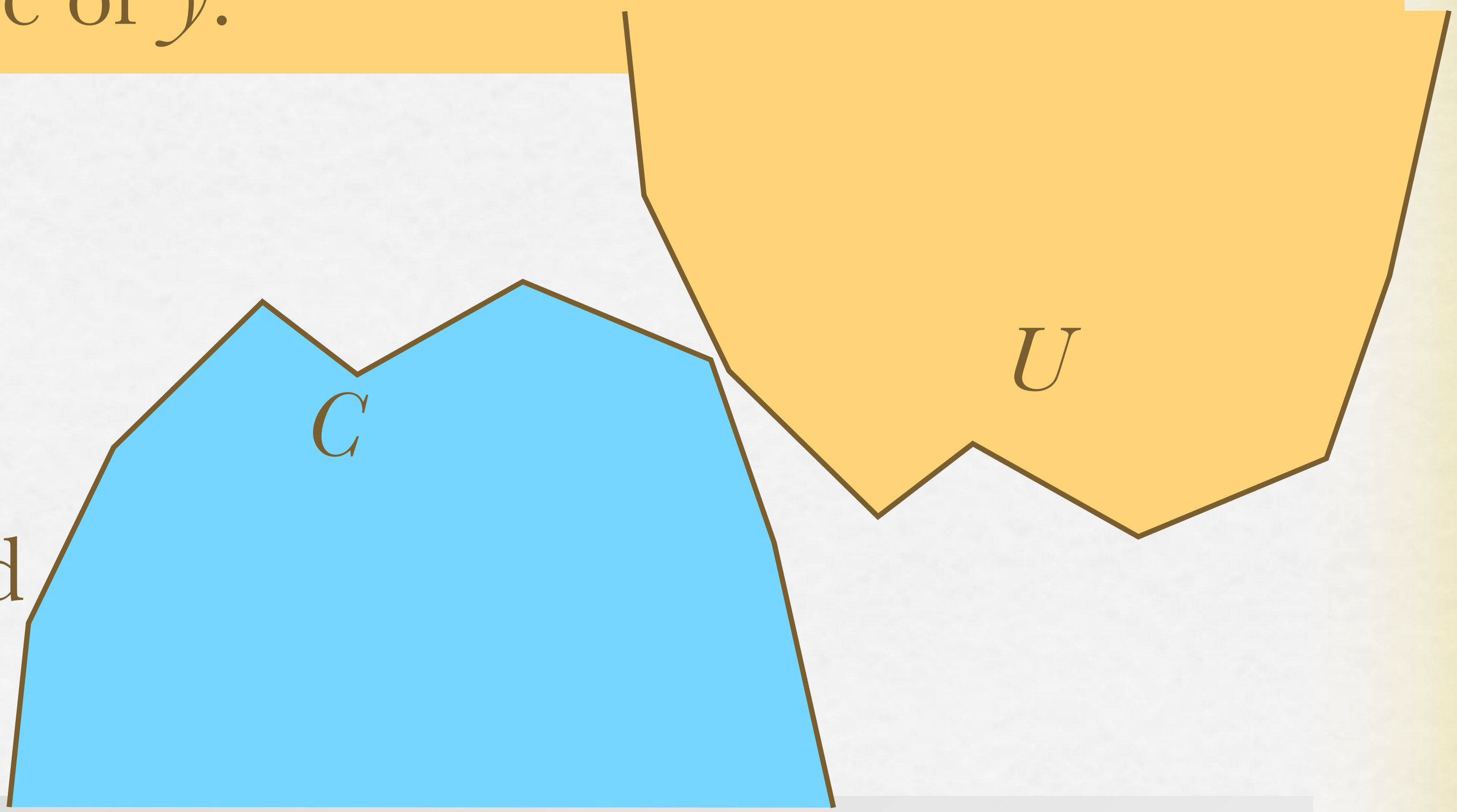
THE SPECIALIZATION ORDER

- A fundamental notion for T_0 spaces (not just dcpos!)
- **Defn** (specialization, $\leq$): In a topological space X, $x \leq y$ iff every open U that contains x also contains y.
- X is T_0 iff $\leq$ is antisymmetric (an ordering).
- **Ex:** For a dcpo $(X, \sqsubseteq)$ in its Scott topology, $\leq$ is just $\sqsubseteq$.

(2pt) Prove this. Note that $\downarrow x$ is always Scott-closed.

THE SPECIALIZATION ORDER

- **Defn** (specialization, $\leq$): In a topological space X, $x \leq y$ iff every open U that contains x also contains y iff x is in the closure of y.
- Every open is upwards-closed
- Every closed set is downward-closed



(Not just in dcpos)

(1pt) Show that $\downarrow x = \{y \mid y \leq x\}$ is the closure of x in any space X.

MONOTONICITY

- **Prop:** A continuous map $f{:}X \to Y$ is always monotonic (w.r.t. the specialization orderings).

- *Proof*. Assume $x \leq x'$.
  We must show $f(x) \leq f(x')$, namely that
  every open neighborhood V of $f(x)$ contains $f(x')$.

Any ideas?

SCOTT-CONTINUITY

- In the special case of dcpos:
- **Prop:** A map $f: X \to Y$ between dcpos is continuous iff it is monotonic, and preserves directed sups.
- *Proof.* Every continuous map f is monotonic.
  Let $x = \sup_i^\uparrow x_i$. $\sup_i^\uparrow f(x_i) \leq f(x)$ by monotonicity.
  To show $f(x) \leq \sup_i^\uparrow f(x_i)$, let V be an open nbd of $f(x)$.
  Hence $x \in f^{-1}(V)$, so some x_i is in $f^{-1}(V)$: $f(x_i)$ is in V, so $\sup_i^\uparrow f(x_i)$ is in V, too. Since $\leq = \sqsubseteq$, $f(x) \leq \sup_i^\uparrow f(x_i)$.
  Conversely, ... *Exercise.*

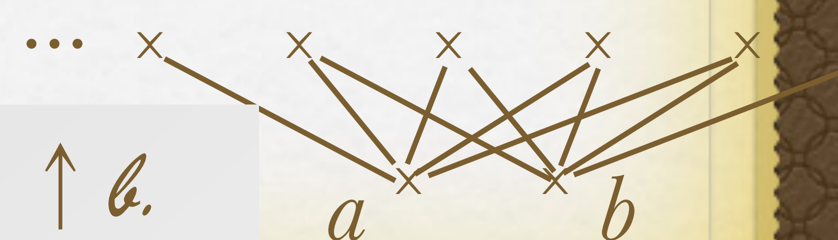
- Let us return to general T_0 spaces.
- Let me give a few words of warning.

THINGS YOU SHOULD FORGET

- *Limits are unique*: **no.** [unless space is Hausdorff.]
  In fact, any point $\leq$ a limit is also a limit.
  In dcpos, $\sup_i^{\uparrow} x_i$ is the largest limit of $(x_i)_{i \in I}$.

- *Compact subsets are closed*: **no.** [Note: no separation assumed in compactness.]
  E.g., any finite subset is compact,
  but closed sets are downwards-closed.

- *Intersections of compact subsets are compact*: **no.**



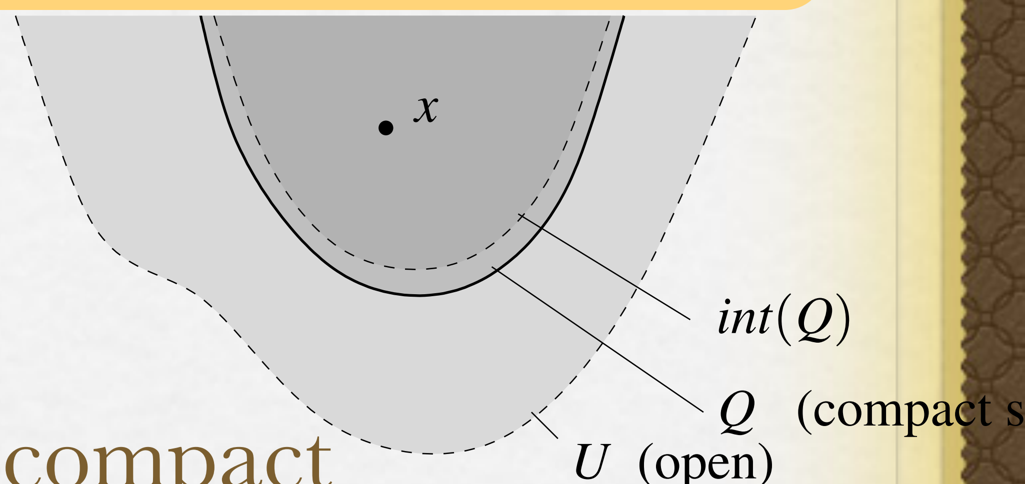
(1pt) Show that $\uparrow a$, $\uparrow b$ are compact, but not $\uparrow a \cap \uparrow b$.

THINGS YOU SHOULD **NOT** FORGET

- **Everything else** works in the expected way.
- A closed subset of a compact space is compact.
- Closure of A=set of limits of nets of points of A.
- Continuous images of compact sets are compact.
- If a filtered intersection of closed sets intersects a compact set then one of them intersects it too.
- Etc.

THINGS YOU SHOULD PAY ATTENTION TO

- Local compactness has to be redefined.
- X is **locally compact** iff every point x has a base of compact neighborhoods, i.e.,
  for every open U containing x,
  there is a compact Q such that
  $x \in \mathrm{int}(Q) \subseteq Q \subseteq U$.



- Usual definition (every point has a compact neighborhood) equivalent in Hausdorff spaces, but too weak in general.

T_0 SPACES: HOW

- Let me guide you through a case study: D. S. Scott's characterization of the **injective T_0 spaces** as the continuous lattices.
- This will let us go through some of the important notions in the field.

INJECTIVE SPACES

$$\begin{tikzcd} Y \arrow[r, dotted] & Z \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

- A standard problem in topology:
  Let $f : X \to Z$ be continuous,
  and $i : X \to Y$ be an embedding.
  Show (under some conditions) that f extends to a continuous map from Y to Z.

- **Ex:** If Y normal, X closed in Y, $Z=\mathbb{R}$ (Tietze-Urysohn)

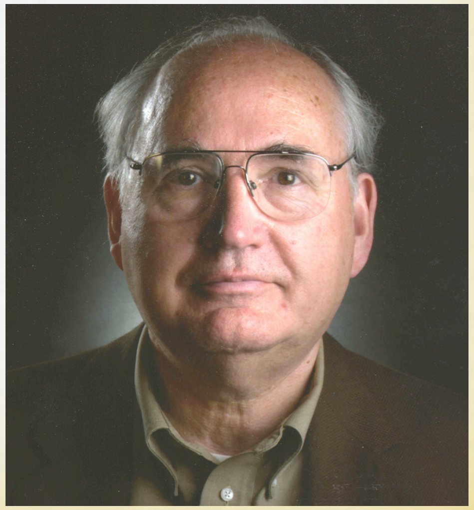
- See also Dugundji, Lavrentiev, etc.

INJECTIVE SPACES

- **Defn:** The T_0 space Z is **injective** iff
  for all T_0 spaces X and Y,
  for every continuous $f : X \to Z$,
  for every embedding $i : X \to Y$,
  f extends to a continuous map from Y to Z.

$$\begin{tikzcd} Y \arrow[r, dotted] & Z \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

- Note: X, Y are *arbitrary* (among T_0 spaces).
- What are the injective spaces?
- Solved by Dana S. Scott, *Continuous lattices*, Springer LNM 274, 97-136, 1972.



SIERPIŃSKI SPACE

- $\mathbb{S} = \{0 < 1\}$, Scott topology
- Opens= $\varnothing$, $\{1\}$, $\{0, 1\}$ - not $\{0\}$
- T_0, not T_1
- Trivial, but important:

(1pt) Show that $\mathcal{U} \mapsto \chi_{\mathcal{U}}$ is a one-to-one correspondence between <u>opens</u> of X and <u>continuous maps</u> from X to $\mathbb{S}$.

SIERPIŃSKI SPACE

- $\mathbb{S} = \{0 < 1\}$, only non-trivial open $\{1\}$.
- **Fact:** $\mathbb{S}$ is injective.
- *Proof.* Take a continuous map $f: X \to \mathbb{S}$.
  f is equal to χ_U, where $U = f^{-1}(\{1\})$.
  Since X embeds into Y through i, U is the trace on X of an open subset V of Y. (Formally, $U = i^{-1}(V)$.)
  Then f extends to $\chi_V : Y \to \mathbb{S}$, as $\chi_V(i(x)) = \chi_U(x)$. $\square$

THE ČECH EMBEDDING

- Let $\mathbf{O}X$ be the complete lattice of open sets of X.
- **Thm** (Čech 1966) Let $\eta: X \to \mathbb{S}^{\mathbf{O}X} : x \mapsto (\chi_U(x))_{U \in \mathbf{O}X}$.
  For every T_0 space X, η is a topological embedding.
- *Proof*: later. The point is that $\mathbb{S}^{\mathbf{O}X}$ has a wealth of good properties. E.g., it is (stably) compact.

(1pt) Compact… but not Hausdorff! Show that any space with a least element w.r.t. $\leq$ is compact. Hence compactness is not much to ask without Hausdorffness.

THE ČECH EMBEDDING

- **Thm.** Let $\eta : X \to \mathbb{S}^{\mathbf{O}X}$ map x to $(\chi_U(x))_{U \in \mathbf{O}X}$.
  For every T_0 space X, η is a topological embedding.

- *Proof*: A subbase of $\mathbb{S}^{\mathbf{O}X}$ is given by
  $\pi_U^{-1}(\{1\})=\{$tuples that have a 1 at position $U\}$.
  — $\eta^{-1}(\pi_U^{-1}(\{1\})) = U$ is open, so η is continuous.
  — η is **almost open**, i.e., for every open U of X,
  $U=\eta^{-1}(V)$ for some open V of $\mathbb{S}^{\mathbf{O}X}$ [take $V= \pi_U^{-1}(\{1\})$]
  — η is injective, because X is T_0.
  Such a map is a homeomorphism onto its image. □

PRODUCTS

- **Fact:** Every product of injectives is injective.
- *Proof.* Let $\mathcal{Z}_j$ be injective, $j \in \mathcal{J}$, $\mathcal{Z}$ be their product, and π_j be the projections : $\mathcal{Z} \to \mathcal{Z}_j$.
  Let $f{:}X \to \mathcal{Z}$ be continuous, $i{:}X \to \mathcal{Y}$ be an embedding.
  For each j, $\pi_j \circ f$ extends to $f'_j{:}\ \mathcal{Y} \to \mathcal{Z}_j$.
  So f itself extends to $y \mapsto (f'_j(y))_{j \in \mathcal{J}}$. $\square$
- **Corl:** $\mathbb{S}^{\mathbf{O}X}$ is injective.

RETRACTS

$$\begin{tikzcd} Y \arrow[r, dotted] & Z \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

- Now assume Z is injective.
- Let X=Z, Y= $\mathbb{S}^{\mathbf{O}Z}$, i=η, f=id.
- Then Z arises as a **retract** of $\mathbb{S}^{\mathbf{O}Z}$: there is a continuous map $r : \mathbb{S}^{\mathbf{O}Z} \to Z$ such that $r \circ \eta = \mathrm{id}$.

$$\begin{tikzcd} \mathbb{S}^{\mathbf{O}Z} \arrow[r, dotted, "r"] & Z \\ Z \arrow[u, "\eta"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

RETRACTS

- **Fact:** a retract of an injective is injective.
- *Proof.* Let Z be retract of Z' injective.
  $s \circ f$ extends to the dotted arrow.
  Post-compose with r and use $r \circ s = \mathrm{id}$. $\square$

$$\begin{tikzcd}
 & Z' \arrow[d, shift left, "r"] \\
Y \arrow[ur, dotted] & Z \arrow[u, shift left, "s"] \\
X \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}$$

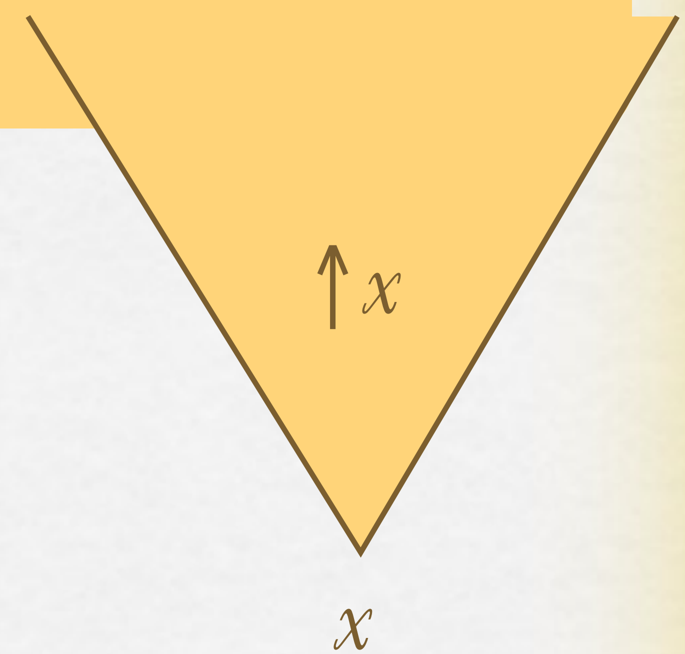
- **Corl:** The following are equivalent:
  (1) Z is injective
  (2) Z is a retract of $\mathbb{S}^{\mathbf{O}Z}$
  (3) Z is a retract of some power of $\mathbb{S}$.

INTERMISSION

- We now know that the injective spaces are the retracts of powers of $\mathbb{S}$.
- To characterize these, let us spend some time doing basic domain theory:
  — algebraic dcpos
  — continuous dcpos
  That will be useful later.
- (I told you the study of the theorem would be an excuse!)

FINITE ELEMENTS

- **Defn:** An element x of a poset X is **finite** iff for every directed family $(y_i)_{i \in I}$ whose sup y exists and is $\geq x$, some y_i is already $\geq x$.



- Equivalently, iff $\uparrow x$ is Scott-open.
- **Ex:** every finite poset (in particular, $\mathbb{S}$) is a dcpo where every element is finite.
- **Ex:** The powerset $\mathbb{P}(A)$, $\subseteq$ is a dcpo. Its finite elements are... the finite subsets of A.

(1pt) Show this.

ALGEBRAIC POSETS

- **Defn:** An element x of a poset X is **finite** iff for every directed family $(y_i)_{i \in I}$ whose sup y exists and is $\geq x$, some y_i is already $\geq x$.

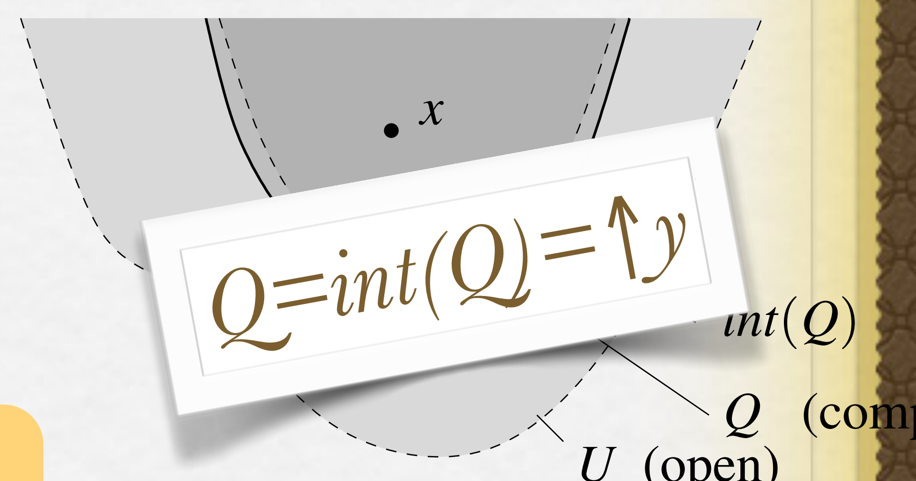
- **Defn:** A poset X is **algebraic** iff every point x is a directed sup of finite elements below x.

- **Ex:** The powerset $\mathbb{P}(A)$, $\subseteq$ is algebraic. Each $B \subseteq A$ is $\sup^{\uparrow}$ {finite subsets of B}. (Sup=union.)

(2pt) Show that if A is uncountable, then A itself is not the sup of a chain of finite subsets. This is why we took directed families, not chains.

B-SPACES

- Marcel Erné, *The ABC of order and topology*, 1991.
- A **b-space** is a space with a base of (compact) opens of the form $\uparrow y$.
- A strong form of local compactness:



$Q = int(Q) = \uparrow y$

- **Thm:** The posets that are b-spaces in their Scott topology are exactly the algebraic posets.

(2pt) Show half of this: any algebraic poset must be a b-space.

POWERSETS

- $\mathbb{S}^A \cong \mathbb{P}(A)$: $(b_a)_{a \in A} \mapsto \{a \in A \mid b_a = 1\}$
- **Prop:** this is a homeomorphism: the product topology (on $\mathbb{S}^A$) is the Scott topology (on $\mathbb{P}(A)$).
- Mod $\cong$, the product topology on $\mathbb{P}(A)$ has subbasic sets $\pi_a^{-1}(\{1\})=\{B \subseteq A \mid a \in B\}$. Note $\pi_a^{-1}(\{1\})=\uparrow\{a\}$. Take finite intersections: basic sets $\uparrow F$, F finite $\subseteq A$.
- Let $B \in U$ Scott-open in $\mathbb{P}(A)$. $B=\sup{\uparrow}_i F_i$, F_i finite. So some F_i is in U. Hence $\uparrow F_i$ open nbd of B inside U. ➪ Scott topology has basic sets $\uparrow F$, F finite $\subseteq A$, too. □

THE ČECH EMBEDDING REVISITED

- **Thm.** Let $\eta : X \to \mathbb{P}(\mathbf{O}X)$ map x to $N_x = \{U \in \mathbf{O}X \mid x \in U\}$. For every T_0 space X, η is a topological embedding.
- X is injective iff X is a retract of $\mathbb{P}(\mathbf{O}X)$
  iff X is a retract of some powerset.
- (But let us proceed with our intermission.)

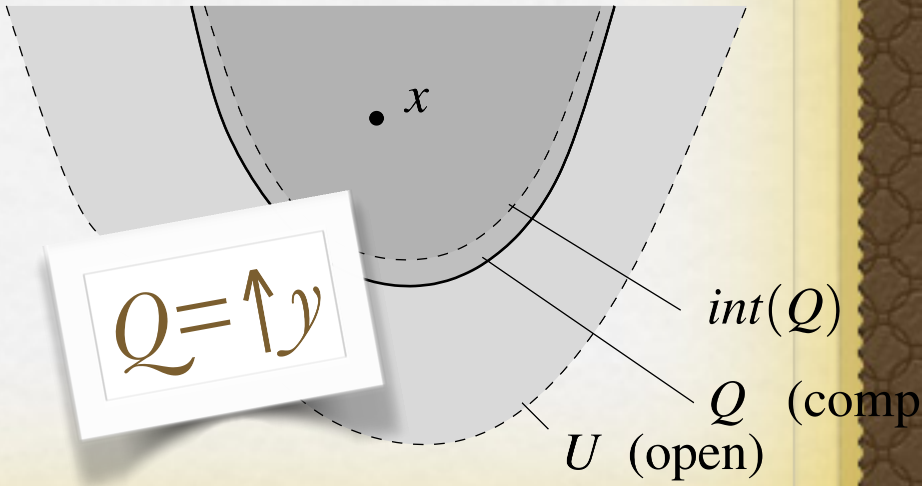
C-SPACES

- Yuri L. Ershov, *The theory of A-spaces*, Algebra and Logic 12(4), 1973. Marcel Erné, *The ABC of order and topology*, 1991.

- A **c-space** is a space where every point x has a base of (compact) neighborhoods of the form $\uparrow y$.

- A strong form of local compactness:

- Compared to b-spaces, we do not require $\uparrow y$ to be open.



B- AND C-SPACES

- **Prop:** A retract of a b-space is a c-space.
- *Proof*: first, every b-space is trivially a c-space.
  Let $r : C \to X$, $s : X \to C$ be a retraction, C a c-space.
  Let us show that X is a c-space, too.

  Let x be in X, U be an open neighborhood of x.

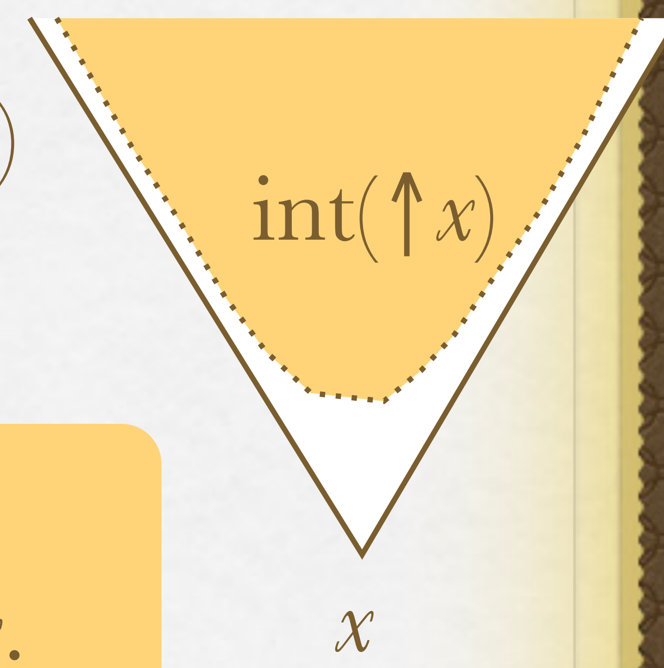
Any ideas?

CONTINUOUS POSETS

- I might take that as a definition:
- **Thm** (Erné, 2005): The posets that are c-spaces in their Scott topology are exactly the continuous posets.
- Let us unknit that, and try to reconstruct what a continuous poset might be, with an eye to that theorem.

THE WAY-BELOW RELATION

- **Defn:** Let $x \ll x'$ iff, for every directed family $(y_i)_{i \in I}$ whose sup y is $\geq x'$, some y_i is already $\geq x$.
- Note: $x \ll x'$ if x' is in the Scott interior of $\uparrow x$. (Iff in continuous posets = c-spaces.)
- Note: x is finite iff $x \ll x$.
- **Defn:** A continuous poset X is one where every point x is a directed sup of points $\ll x$.
- **Ex:** [0, 1] is a continuous dcpo, $x \ll x'$ iff $x=0$ or $x<x'$.



(1pt) Show this.

BASES

- I have said that $x \ll x'$ if x' is in the Scott interior of $\uparrow x$. The converse holds in continuous posets, (admitted). We shall prove Erné's theorem later, too.
- **Prop:** Let $\twoheaduparrow x = \{x' \mid x \ll x'\}$. In a continuous poset, $\twoheaduparrow x = \mathrm{int}(\uparrow x)$, and those sets form a base of the Scott topology.
- **Prop:** In an algebraic poset, $x \ll x'$ iff $x \leq w \leq x'$ for some finite w. The sets $\uparrow w$, w finite, are (compact and) open and form a base of the Scott topology.

A CONUNDRUM

- A retract Y of a b-space X is a c-space
- For a poset, algebraic $\Leftrightarrow$ b-space in its Scott topology
- For a poset, continuous $\Leftrightarrow$ c-space in its Scott topology
- Is a retract Y of an algebraic dcpo X continuous?
  Difficulty: ?

THE SOPHISTICATED WAY OUT

- Invoke sobriety (see later, if we've got time).
- **Thm:** Algebraic **dcpo** = **sober** b-space.
- **Thm:** Continuous **dcpo** = **sober** c-space.
- (In particular, a sober b- or c-space has the Scott topology of its specialization ordering.)
- Retracts of sober spaces are sober.
- We conclude: all retracts of algebraics dcpos are continuous dcpos.

CONVERSELY?

- We know that every retract of an algebraic dcpo is a continuous dcpo.
(Modulo the sobriety thing.)
- We wish to establish the converse: every continuous dcpo X arises as the retract of some algebraic dcpo.
- That algebraic dcpo is the ideal completion $\mathbf{I}(X)$ of X.

IDEALS

- An (order-) **ideal** D of a poset X is a directed, downwards-closed subset of X.
- Let $\mathbf{I}(X) = \{\text{ideals of } X\}$, ordered by $\subseteq$.
- **Prop:** $\mathbf{I}(X)$ is a dcpo, and directed sups are unions.
- *Proof*: *Exercise.*

IDEAL COMPLETION

- An (order-) **ideal** D of a poset X is a directed, downwards-closed subset of X.
  Let $\mathbf{I}(X)$ = {ideals of X}, ordered by $\subseteq$.

- **Prop:** $\mathbf{I}(X)$ is an algebraic dcpo, and its finite elements are the ideals of the form $\downarrow x$, x in X.

- *Proof*: $\downarrow x$ is finite: if $\downarrow x \subseteq \sup\uparrow_i D_i$, then $x \in \sup\uparrow_i D_i$, so x is in some D_i, i.e., $\downarrow x \subseteq D_i$.
  Clearly, (*) $D = \sup\uparrow_{x \in D} \downarrow x$.
  If D finite, by (*) $D \subseteq \downarrow x$ for some $x \in D$, so $D = \downarrow x$.
  Finally, by (*) every D is a $\sup\uparrow$ of finite elements. $\square$

IDEAL COMPLETION

- The ideal completion has many properties:
- There is an order-embedding $i{:}x \mapsto {\downarrow}x$ of X into $\mathbf{I}(X)$.
- $\mathbf{I}(X)$ is the the **free dcpo** over X: every monotonic map f from X to a dcpo Z extends to a unique Scott-continuous map f' from $\mathbf{I}(X)$ to Z.

$$
\begin{tikzcd}
\mathbf{I}(X) \arrow[r, dotted, "f'"] & Z \\
X \arrow[u, "i"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

- Every algebraic dcpo X is isomorphic to the ideal completion $\mathbf{I}(B)$ of its poset B of finite elements.

(2pt) Exercise.

IDEAL COMPLETION

- Let $\mathord{\downarrow\!\!\!\downarrow} x = \{x' \mid x' \ll x\}$, when X is continuous.
  There is another embedding $s{:}x \mapsto \mathord{\downarrow\!\!\!\downarrow} x$ of X into $\mathbf{I}(X)$, and a map $r{:}D \mapsto \sup D$ from $\mathbf{I}(X)$ to X.

- **Prop:** If X is a continuous dcpo, then r, s exhibit X as a retract of $\mathbf{I}(X)$.

- *Proof*: - $r \circ s = \mathrm{id}$... by the def. of continuous posets.
  - r is monotonic and preserves $\sup^\uparrow$, so is continuous.
  - A basic open subset of $\mathbf{I}(X)$ is $\uparrow_{\mathbf{I}(X)} \downarrow_X x$ (upwards-closure of a finite element). Its inverse image by s is $\mathord{\uparrow\!\!\!\uparrow} x$, which is open. So s is continuous. $\square$

CONTINUOUS VS. ALGEBRAIC

- We therefore obtain (modulo the sobriety thing):
- **Thm:** The continuous dcpos are exactly the retracts of algebraic dcpos.
- That is too much for our purpose (characterizing injective spaces), but nice anyway.

INJECTIVE⇒ CONTINUOUS LATTICE

- One checks easily that an order-retract of a complete lattice is a complete lattice. So:
- **Thm:** A retract of an algebraic complete lattice is a continuous complete lattice.
- Recall that an injective space $\mathcal{Z}$ is a retract of the algebraic dcpo $\mathbb{S}^{\mathbf{O}\mathcal{Z}} \cong \mathbb{P}(\mathbf{O}\mathcal{Z})$, also a complete lattice.
- **Corl:** Every injective space is a continuous complete lattice, in its Scott topology.

AN EXTENSION FORMULA

- **Prop:** Let Z be a continuous complete lattice. Every continuous map $f:X \to Z$ extends to a continuous map f' from $\mathbb{P}(\mathbf{O}X)$ to Z.
  (I.e., $f' \mathrm{o}\, \eta = f$, or equivalently, $f'(N_x) = f(x)$ for every x.)

- *Proof.* For A in $\mathbb{P}(\mathbf{O}X)$, let $f'(A) = \sup \{z \mid f^{-1}(\uparrow z) \in A\}$.
  - f' preserves (all) unions, hence is Scott-continuous.
  - $f'(N_x) = \sup \{z \mid x \in f^{-1}(\uparrow z)\}$
  $= \sup^{\uparrow} \{z \mid z \ll f(x)\} = f(x)$ (continuous dcpo). □

CONTINUOUS LATTICE ⇒INJECTIVE

- **Prop:** Let Z be a continuous complete lattice. Every continuous map $f{:}X \to Z$ extends to a continuous map f' from $\mathbb{P}(\mathbf{O}X)$ to Z. [I.e., $f' \circ \eta = f$.]
- Let $X=Z$, $f=\mathrm{id}$:
- **Corl:** Let Z be a continuous complete lattice. There is a continuous map $\rho=\mathrm{id}'$ from $\mathbb{P}(\mathbf{O}Z)$ to Z such that $\rho \circ \eta = \mathrm{id}$ [i.e., $\rho(N_z)=z$ for every z in Z.]
- That is, Z is a **retract** of $\mathbb{P}(\mathbf{O}Z)$. So Z is **injective**.

SCOTT'S THEOREM

- **Thm** (Scott, 1972)**:** The following are equivalent:
  (1) $\mathcal{Z}$ is injective
  (2) $\mathcal{Z}$ is a retract of $\mathbb{S}^{\mathbf{O}\mathcal{Z}}=\mathbb{P}(\mathbf{O}X)$
  (3) $\mathcal{Z}$ is a retract of some power of $\mathbb{S}$ (=some powerset)
  (4) $\mathcal{Z}$ is a continuous complete lattice in its Scott topology.
- (Modulo the sobriety thing... we are coming to it.)

STONE DUALITY

- (I'll be quicker here.)
  Let a **frame** be a complete lattice where
  $$u \wedge \bigvee_{i\in I} v_i = \bigvee_{i\in I}(u \wedge v_i)$$
  Frame morphisms preserve finite $\wedge$ and arbitrary $\vee$.
  Together they form a category **Frm**.

- There is a functor $\mathbf{O} : \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$:
  — mapping every topological space X to $\mathbf{O}X$
  — and every continuous map $f\colon X \to Y$ to the frame morphism $\mathbf{O}f\colon \mathbf{O}Y \to \mathbf{O}X : V \mapsto f^{-1}(V)$.

- Can we retrieve X from its frame of opens?

POINTS IN A FRAME

- Let L be a frame.
  If L=**O**X, where X is T_0, we can equate x with N_x.
  N_x is a **completely prime filter**:
  — it is non-empty
  — it is upwards-closed
  — it is closed under $\wedge$
  — (c.p.) if $\vee_{i \in I} v_i$ is in it, then some v_i is in it.

- Let **pt** L be the set of c.p. filters of L, (a.k.a., **points**) with the **hull-kernel** topology, whose opens are $O_u = \{x \text{ point} \mid u \in x\}$.

THE STONE ADJUNCTION

- **pt** defines a functor : $\mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$, right-adjoint to $\mathbf{O}$.
- The unit $\eta : X \to \mathbf{pt}\,\mathbf{O}X$ maps x to N_x, and is injective iff X is T_0. (Then it is an embedding.)
- **Defn:** X is **sober** iff η is bijective

  iff η is a homeomorphism.
- In other words, the T_0 space X is sober iff every c.p. filter of opens is N_x for some point x.

IRREDUCIBLE CLOSED SETS

- For a c.p. filter $\boldsymbol{F}$, the union V of all opens not in $\boldsymbol{F}$ is not in $\boldsymbol{F}$ — by c.p. This is the largest open not in $\boldsymbol{F}$.
- Let C be the complement of the largest open not in $\boldsymbol{F}$. Note: for U open, C intersects U iff $U \not\subseteq V$ iff $U \in \boldsymbol{F}$.
- **Lemma:** C is **irreducible closed**, namely: if C intersects finitely many opens $U_1, \ldots, U_n$, then it intersects $\bigcap_{i=1}^n U_i$.
- *Proof.* Each U_i is in $\boldsymbol{F}$, so $\bigcap_{i=1}^n U_i$ is in $\boldsymbol{F}$ (filter), too. □

IRREDUCIBLE CLOSED SETS

- **Lemma:** C is **irreducible closed**, namely: if C intersects finitely many opens U_1, ..., U_n, then it intersects $\bigcap_{i=1}^n U_i$.
- Equivalently: if C is included in a union of finitely many closed sets C_1, ..., C_n, then it is included in some C_i — when the name *irreducible closed*.
- Conversely, let C be irreducible closed.
  Let $\boldsymbol{F}$ be the set of all opens U that intersect C.
  Then $\boldsymbol{F}$ is a c.p. filter.

(1pt) Show this.

SOBER SPACES

- Because of the one-to-one-correspondence between c.p. filters and irreducible closed subsets, we have:

- **Prop:** Up to iso, **pt O**X is the sobrification **S**X of X, whose points are the irreducible closed subsets of X. Its opens are $\Diamond U = \{C \mid C \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.
  If X is T_0, $\eta : X \to \mathbf{pt}\ \mathbf{O}X : x \mapsto \downarrow x$ is an embedding.

- **Corl:** The T_0 space X is sober iff every irreducible closed subset is the closure $\downarrow x$ of a (unique) point x.

T2 ⇒ SOBER

- C is irreducible closed iff: if C intersects finitely many opens $U_1, \ldots, U_n$, then it intersects $\bigcap_{i=1}^n U_i$.
  Note that irreducible implies non-empty (take $n=0$).
- **Thm:** Every Hausdorff space is sober.

Exercise!

- $T_2 \Rightarrow$ sober $\Rightarrow T_0$, sober incomparable with T_1.

CONTINUOUS DCPO $\Rightarrow$ SOBER

- **Thm:** Every continuous dcpo is sober.

- *Proof.* Let C be irreducible closed.
  — Let $D = \{x \mid \Uparrow x \text{ intersects } C\}$. I claim D is directed.

  Exercise

  — $\sup D$ is in C since C Scott-closed, so $\downarrow \sup D \subseteq C$.
  — For every y in C, write y as a sup of $x \ll y$. Each such x is in D, so $y \leq \sup D$. Hence $C = \downarrow \sup D$. $\square$

OPERATIONS ON SOBER SPACES

- Sober spaces are closed under coproducts, (T_0 quotients of) quotients, products; but not subspaces.
- **Prop:** Sober spaces are closed under retracts.
- *Proof*. Let $r : Z \to X$, $s : X \to Z$ be a retraction, Z sober. Let C be irreducible closed in X. (1) We check that $\mathrm{cl}(s(C))$ is irreducible closed.

Your turn.

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- *Proof*. Let $r : Z \to X$, $s : X \to Z$ be a retraction, Z sober.
  Let C be irreducible closed in X.
  (1) We check that $\mathrm{cl}(s(C))$ is irreducible closed.
  (2) So $\mathrm{cl}(s(C)) = {\downarrow} z$ for some z in Z. We claim $C = {\downarrow} r(z)$.

Your turn.

SOBER ⇒ MONOTONE CONVERGENCE

- **Thm** (O. Wyler, 1977): Let X be sober. Then:
  (1) $\leq$ is directed complete
  (2) all opens are Scott-open.
  (A space satisfying those is a *monotone convergence* space. All T_1 spaces are, too.)

- *Proof*. Let D be directed. Then $\mathrm{cl}(D)$ is irreducible closed.
  (1) By sobriety, $\mathrm{cl}(D)={\downarrow}x$ for some x: we show $x=\sup^{\uparrow} D$.

Your turn.

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  (1) By sobriety, $\mathrm{cl}(D)={\downarrow}x$ for some x: we show $x=\sup^{\uparrow} D$.
  (2) If U open contains $\sup^{\uparrow} D=x$, U intersects ${\downarrow}x=\mathrm{cl}(D)$, hence also D. So U is Scott-open. □

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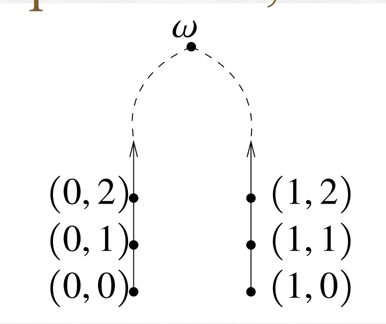
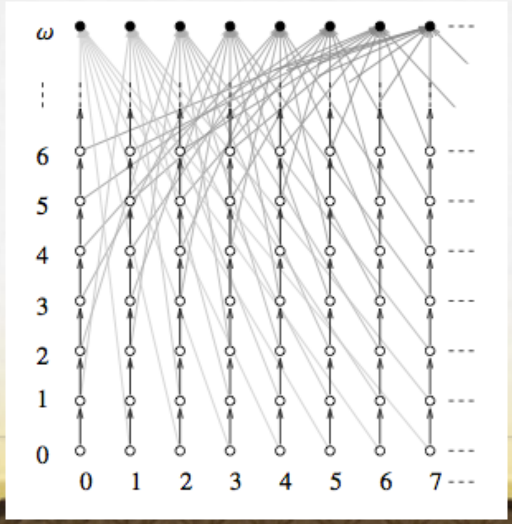
- Beware: X not continuous in general

(here, a non-continuous, sober dcpo)

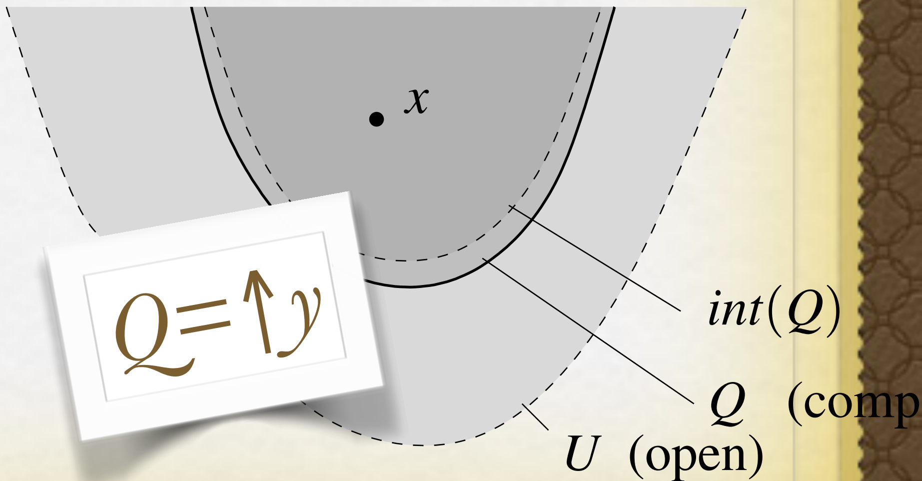


- Beware: Johnstone's non-sober dcpo:



SOBER C-SPACES

- We know that a continuous dcpo is:
(1) sober
(2) a c-space in its Scott topology.
- I claimed the converse, earlier.
- Let us prove this.
- Recall that a c-space is a space with a very strong local compactness property: if $x \in U$ open, then there is a point y such that $x \in \mathrm{int}(\uparrow y) \subseteq \uparrow y \subseteq U$.



SOBER C-SPACES

- **Prop** (Erné)**:** A sober c-space X is a continuous dcpo, and its topology is the Scott topology.

- *Proof*. Define $y < y'$ iff $y' \in \mathrm{int}(\uparrow y)$.
  (1) We first show that $y < y'$ implies $y \ll y'$.

Your turn.

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  (2) Now show: $D = \{y \mid y < y'\}$ is directed and $\sup^{\uparrow} D = y'$.

Your turn.

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- *Proof*. Define $y < y'$ iff $y' \in \text{int}(\uparrow y)$.
(1) We first show that $y < y'$ implies $y \ll y'$.
(2) Now show: $D=\{y \mid y < y'\}$ is directed and $\sup{\uparrow} D=y'$.
(3) So, with $\leq$, X is a continuous dcpo.
... Every y' is the $\sup{\uparrow}$ of a family D of elements $\ll y'$.

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(2) Now show: $D=\{y \mid y < y'\}$ is directed and $\sup^{\uparrow} D=y'$.
(3) So, with $\leq$, X is a continuous dcpo.
(4) $y \ll y'$ implies $y < y'$.

Your turn.

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  (2) Now show: $D = \{y \mid y < y'\}$ is directed and $\sup^{\uparrow} D = y'$.
  (3) So, with $\leq$, X is a continuous dcpo.
  (4) $y \ll y'$ implies $y < y'$.
  (5) Every Scott-open is open (in the original topology).

Your turn.

SOBER C-SPACES

- **Prop** (Erné)**:** A sober c-space X is a continuous dcpo, and its topology is the Scott topology.

- *Proof*. Define $y \prec y'$ iff $y' \in \mathrm{int}(\uparrow y)$.
  (1) We first show that $y \prec y'$ implies $y \ll y'$.
  (2) Now show: $D = \{y \mid y \prec y'\}$ is directed and $\sup^{\uparrow} D = y'$.
  (3) So, with $\leq$, X is a continuous dcpo.
  (4) $y \ll y'$ implies $y \prec y'$.
  (5) Every Scott-open is open (in the original topology).
  (6) Every open is Scott-open.
  ... because a sober space is monotone convergence. ☐

CONCLUSION

- This fills the last gap in our proof.
- There would be many things more to say.
  — The Hofmann-Mislove theorem
  — The theory of stably compact spaces
  — Quasi-metric spaces
  — Etc. (but I had to make choices.)
- Read the book, follow the blog!
  http://projects.lsv.ens-cachan.fr/topology/