FORMAL BALLS

Jean Goubault-Larrecq

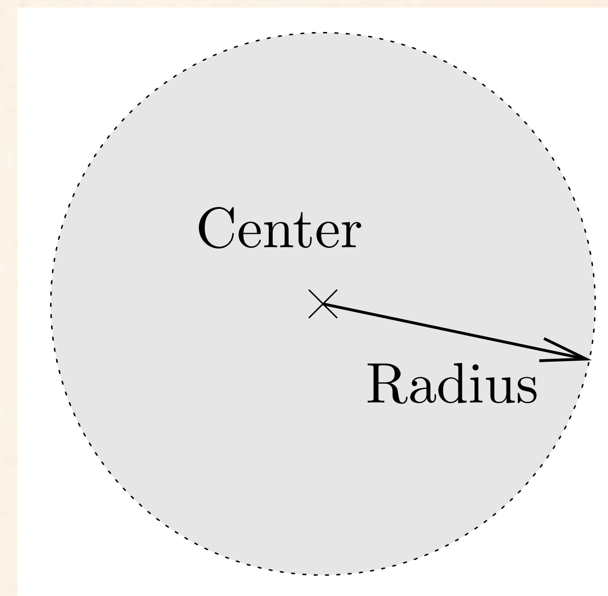
Domains XII

Cork

QUASI-METRICS

A quasi-metric d on a set X:

- $d(x, x)=0$
- if $d(x, y)=0$ and $d(y, x)=0$ then $x=y$
- $d(x, z) \leq d(x, y) + d(y, z)$
- $d(x, y)= d(y, x)$



An open ball

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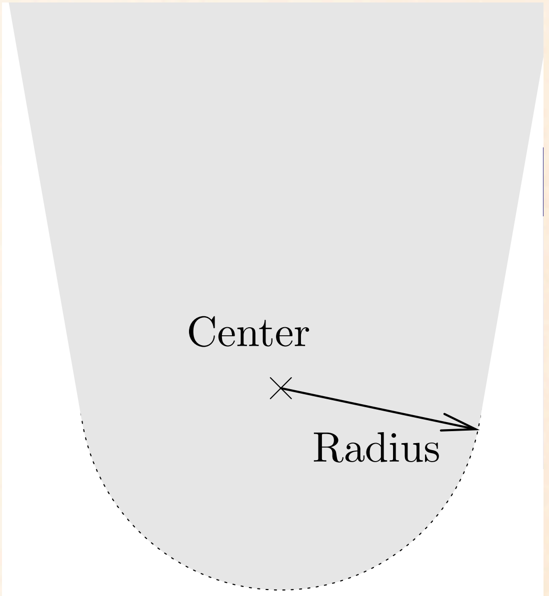

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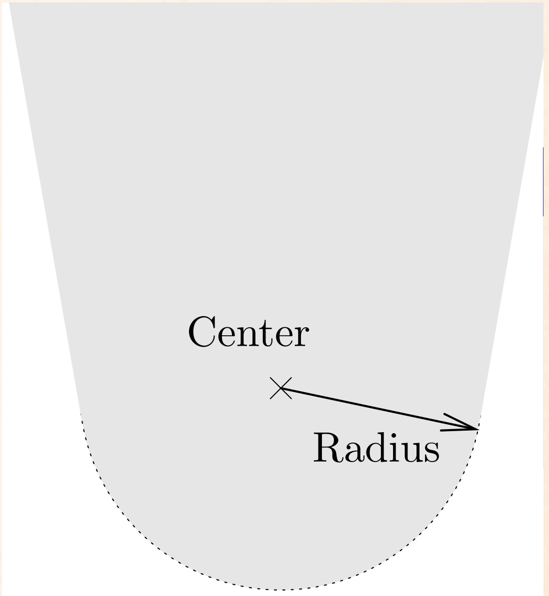


An open ball

Specialization ordering of the open ball topology:

$x \leq y$ iff $d(x, y) = 0$

QUASI-METRICS

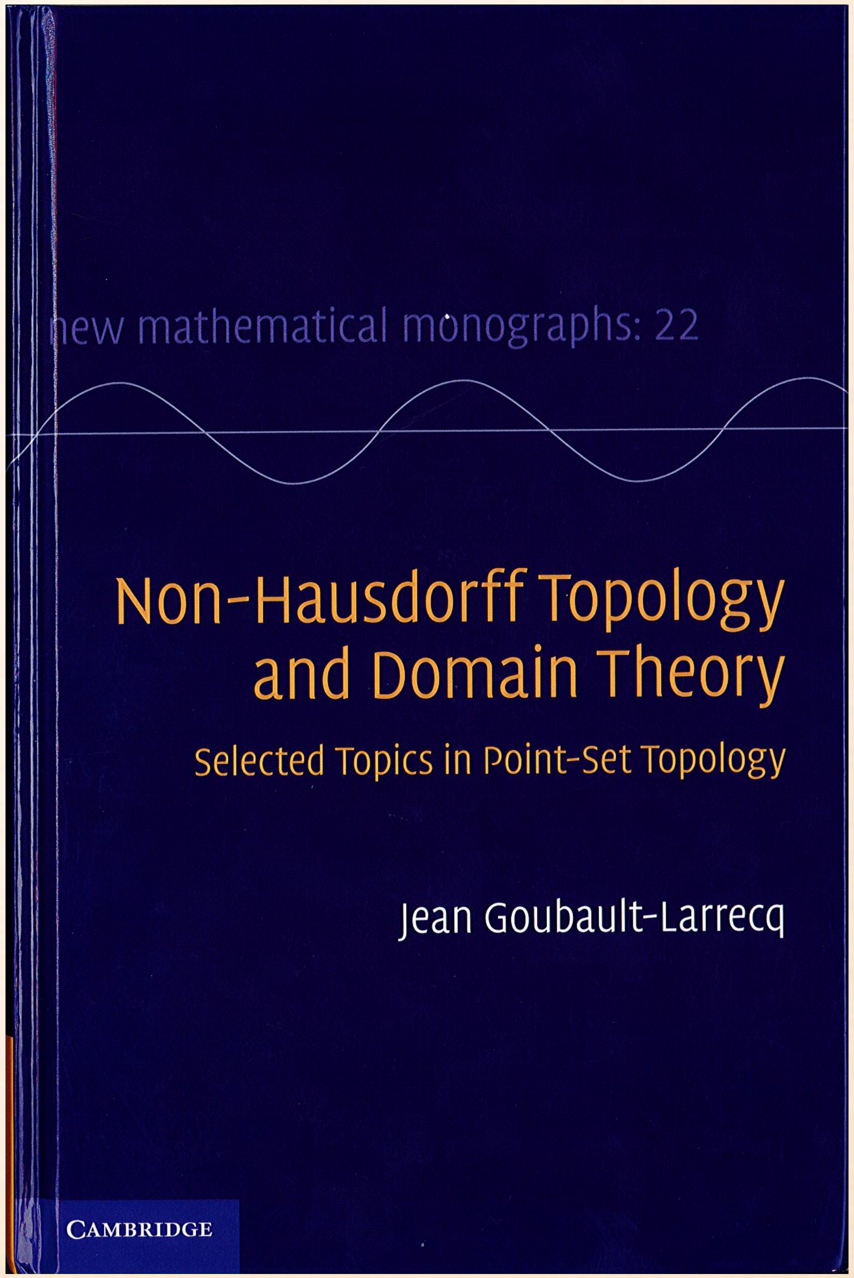
- «Classical theory»: R. **Wilson** (1931), H.-P. **Künzi** (1983), M. **Smyth** (1989), M. **Schellekens** (1995), Ph. **Sünderhauf** (1993)
- «As enriched category theory»: F.W. **Lawvere** (1973), J.J.M.M. **Rutten** (1996), M.M. **Bonsangue**, F. **van Breugel** (1998)
- and...

FORMAL BALLS

- «Formal Balls»:
  - discovered by K. **Weihrauch** and U. **Schreiber** (1981), *embedding metric spaces in cpo's*.
  - studied further by A. **Edalat** and R. **Heckmann** (1998)
- New important developments (2009-2010) by P. **Waszkiewicz**, M. **Kostanek**, S. **Romaguera**, O. **Valero**, M. **Ali-Akbari**, B. **Honari**, M. **Pourmahdian**, M. M. **Rezaii**.
- **Formal balls provide a unifying view, via domain theory**

A SHAMELESS AD

- In this talk: an excerpt of the Book



OUTLINE

- Notions of completeness
- Low-hanging fruit: fixed point theorems
- The formal ball completion
- Low-handing fruit: miscellanea

OUTLINE

- Notions of completeness
- Low-hanging fruit: fixed point theorems
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- Low-handing fruit: miscellanea

CAUCHY NETS

- A net $(x_i)_{i \in I}$ is **Cauchy** iff for every $\varepsilon>0$,
  $d(x_i, x_j)<\varepsilon$ for all $i \leq j$ large enough
- **Example 1**: for d metric, usual notion ($i \leq j$ unimportant)
- **Example 2**: d is an **ordering** iff $d(x, y) = 0$ or $+\infty$ for all x, y
  ($x \leq y$ iff $d(x, y) = 0$)
  Cauchy = eventually monotonic ($x_i \leq x_j$ for all $i \leq j$ large enough)

D-LIMITS

- A ***d*-limit** of a net $(x_i)_{i \in I}$ is a point x such that,

  for every y, $d(x, y) = \limsup d(x_i, y)$

- unique if it exists (contrarily to limits)
  but **not** a limit (for the open ball topology)

- **Example 1**: if d is metric, d-limit = ordinary limit

- **Example 2**: if d ordering, d-lim of Cauchy(≈mono) net=**sup**

- **Example 3**: on $\mathbb{R}$, let $d_{\mathbb{R}}(x, y) = x - y$ if $x \geq y$, 0 otherwise
  Then $d_{\mathbb{R}}$-limit $(x_i)_{i \in I}$ = **limsup** x_i

YONEDA-COMPLETENESS

- X, d is **Yoneda-complete** iff every Cauchy net has a d-limit
- **Example 1**: =usual notion of completeness for metrics
- **Example 2**: an ordering ($x \leq y$ iff $d(x, y) = 0$) is Yoneda-complete iff a **dcpo**
- **Example 3**: $\mathbb{R}, d_{\mathbb{R}}$ is not Yoneda-complete

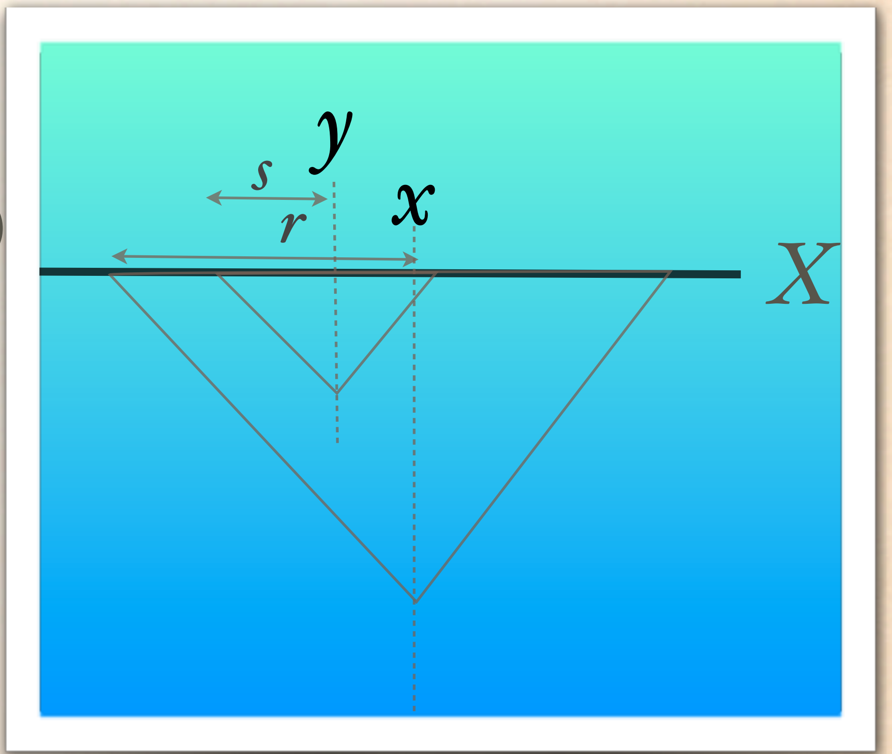
  ... but $\mathbb{R} \cup \{+\infty\}$ is, or $\mathbb{R}^+ \cup \{+\infty\}$

THE K-W THEOREM

- **Theorem** (Kostanek, Waszkiewicz 2010):
  X, d is Yoneda-complete iff $\mathbf{B}(X)$ is a **dcpo**.

- $\mathbf{B}(X)$ is the poset of **formal balls** (x, r) ($x \in X, r \in \mathbb{R}^+$) with:

  $(x, r) \leq (y, s)$ iff $d(x, y) \leq r - s$
  (in particular, $r \geq s$)



PROOF OF K-W (1)

- A net $(x_i, r_i)_{i \in I}$ is **Cauchy-weighted** iff

  $\inf^{\downarrow} r_i = 0$

  $d(x_i, x_j) \leq r_i - r_j$ for all $i \leq j$

  Then we say that $(x_i)_{i \in I}$ is **Cauchy-weightable**

- Cauchy-weightable implies Cauchy

  Every Cauchy net has a Cauchy-weightable subnet [EH98]

  Hence X, d Yoneda-complete iff

  every Cauchy-**weightable** net has a d-limit

PROOF OF K-W (2)

- A net $(x_i, r_i)_{i \in I}$ is **Cauchy-weighted** iff

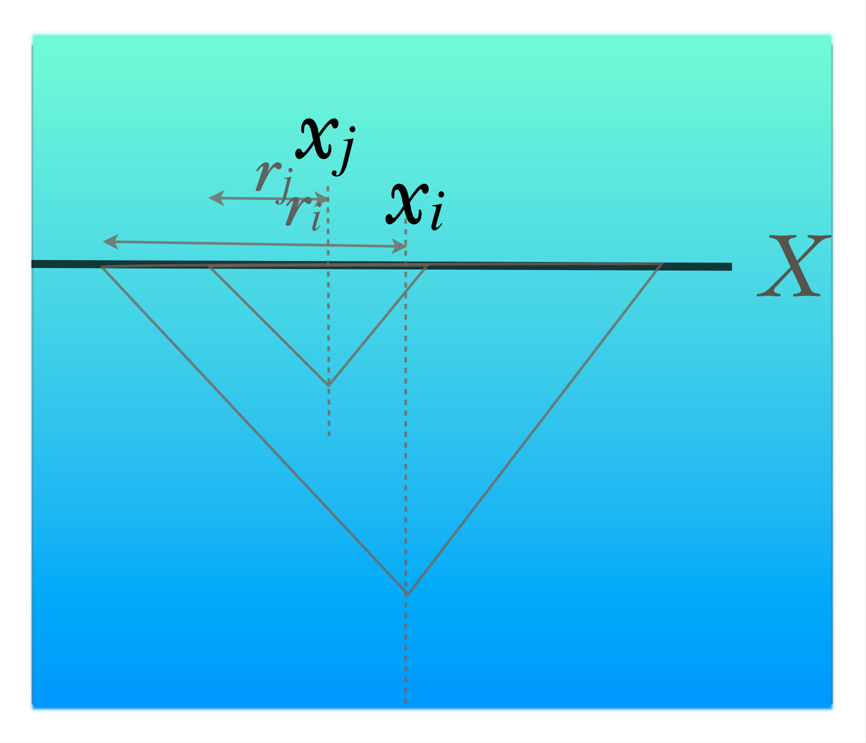
  $\inf^{\downarrow} r_i = 0$

  $d(x_i, x_j) \leq r_i - r_j$ for all $i \leq j$

  Then we say that $(x_i)_{i \in I}$ is **Cauchy-weightable**

- **Note**: a Cauchy-weighted net is
  - a **monotone** net $(x_i, r_i)_{i \in I}$ in $\mathbf{B}(X)$

    $(x_i, r_i) \leq (x_j, r_j)$ for all $i \leq j$
  - with **aperture** $\inf r_i$ equal to 0



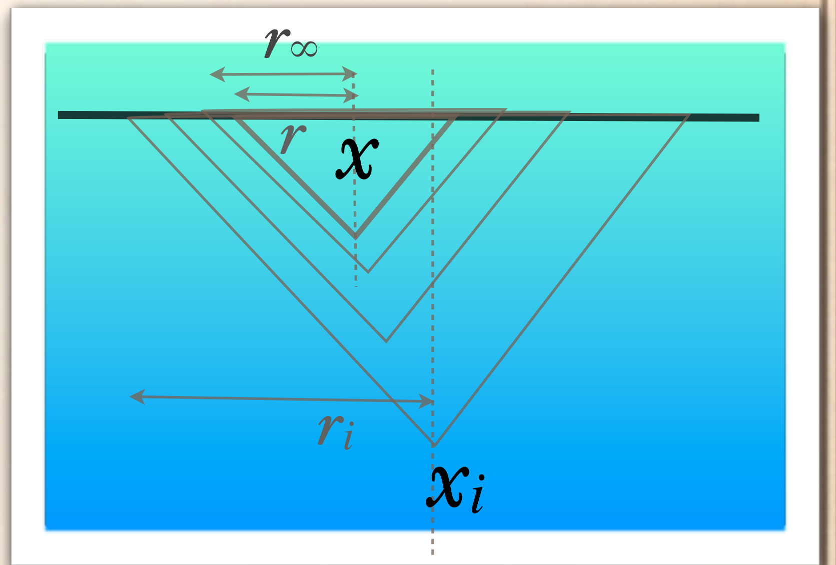
PROOF OF K-W (3)

- **Lemma 1**. Let $(x_i, r_i)_{i \in I}$ be a monotone net in $\mathbf{B}(X)$.
  If $(x_i)_{i \in I}$ has a d-limit x and r=inf r_i, then (x, r) = sup $(x_i, r_i)_{i \in I}$

- **Lemma 2**. If $\mathbf{B}(X)$ is a dcpo, the converse holds.

- **Proof**: We first show $r = \inf r_i$.

- Let $r_\infty = \inf r_i$.
  Since $r_i \geq r$ for every i, $r_\infty \geq r$.
  Translate by r_∞: the net $(x_i, r_i - r_\infty)_{i \in I}$ has a **sup** (x', r').
  By the same argument, $\inf (r_i - r_\infty) \geq r'$. So r'=0.
  $(x', r') \geq (x_i, r_i - r_\infty) \Rightarrow (x', r_\infty) \geq (x_i, r_i)$, so $(x', r_\infty) \geq (x, r)$: $r_\infty \leq r$.



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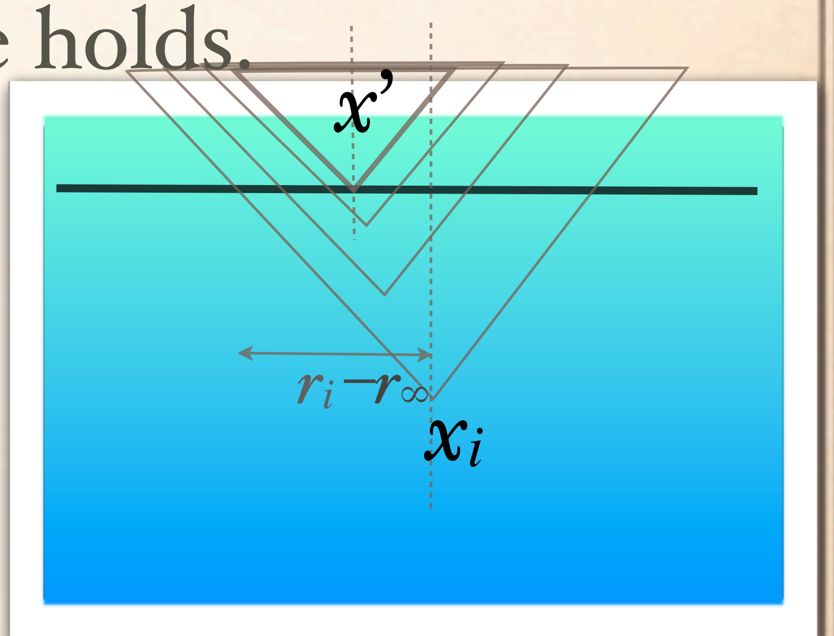
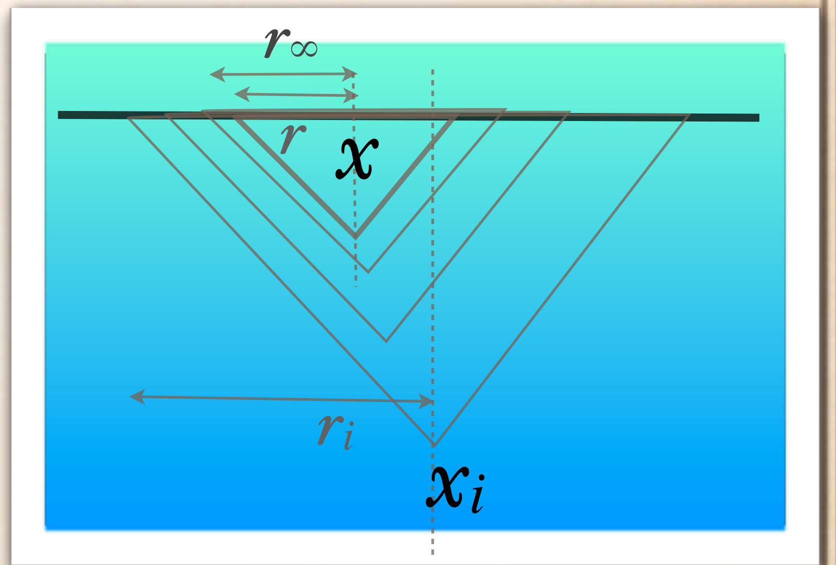


PROOF OF K-W (3)

- **Lemma 1**. Let $(x_i, r_i)_{i\in I}$ be a monotone net in **B**(X). If $(x_i)_{i\in I}$ has a d-limit x and r=inf r_i, then (x, r) = sup $(x_i, r_i)_{i\in I}$
- **Lemma 2**. If **B**(X) is a dcpo, the converse holds.
- **Proof**: We first show r = inf r_i.
- Let r_∞ = inf r_i. Since $r_i \geq r$ for every i, $r_\infty \geq r$. **Translate** by r_∞: the net $(x_i, r_i - r_\infty)_{i\in I}$ has a **sup** (x', r'). By the same argument, inf $(r_i - r_\infty) \geq r'$. So r'=0. $(x', r') \geq (x_i, r_i - r_\infty) \Rightarrow (x', r_\infty) \geq (x_i, r_i)$, so $(x', r_\infty) \geq (x, r)$: $r_\infty \leq r$.



PROOF OF K-W (3)

- **Proof**: (continued)
  - We claim that **translation** $(y, s) \longmapsto (y, s+a)$ is Scott-continuous, for every $a \geq 0$.
    - Since order isomorphism with $\mathbf{B}_a$ = {formal balls of radius $\geq a$}
    - and sups are computed in $\mathbf{B}_a$ as in $\mathbf{B}(X)$ ... because $\mathbf{B}_a$ Scott-closed, by previous slide.

PROOF OF K-W (3)

- **Proof**: (continued)
  - If $(x, r) = \sup\,(x_i, r_i)_{i \in I}$ then $\sup\,(d\,(x_i, y) - r_i) = d(x, y) - r$.
  - $s \leq d(x, y) - r$ is easy.
    $d\,(x_i, y) - r_i \leq s$ implies $(x_i, r_i{+}s) \leq (y, 0)$
    Take sups.
    By continuity of **translation** by s, $(x, r{+}s) \leq (y, 0)$,
    that is, $d(x, y) \leq r{+}s$. So $s \geq d(x, y) - r$.

PROOF OF K-W (3)

- **Proof**: (continued)

Call that s

  - If $(x, r) = \sup (x_i, r_i)_{i \in I}$ then $\sup (d(x_i, y) - r_i) = d(x, y) - r$.
  - $s \leq d(x, y) - r$ is easy.
    $d(x_i, y) - r_i \leq s$ implies $(x_i, r_i+s) \leq (y, 0)$
    Take sups.
    By continuity of **translation** by s, $(x, r+s) \leq (y, 0)$,
    that is, $d(x, y) \leq r+s$. So $s \geq d(x, y) - r$.

PROOF OF K-W (3)

- **Lemma 2**. If $\mathbf{B}(X)$ is a dcpo, the converse holds, namely, if $(x, r) = \sup\,(x_i, r_i)_{i \in I}$ then x=d-lim $(x_i)_{i \in I}$ and $r = \inf r_i$.
- **Proof**: (finished)
  - If $(x, r) = \sup\,(x_i, r_i)_{i \in I}$ then $\sup\,(d\,(x_i, y) - r_i) = d(x, y) - r$.
  - Hence $\sup\,(d\,(x_i, y) + r - r_i) = d(x, y)$ (**translation**).
  - Since $r = \inf r_i$, $\limsup d\,(x_i, y) = d(x, y)$, i.e., x=d-lim $(x_i)_{i \in I}$.

THE K-W THEOREM

- We have proved:
- **Theorem** (Kostanek, Waszkiewicz 2010): X, d is Yoneda-complete iff $\mathbf{B}(X)$ is a **dcpo**.
- Moreover, $\sup (x_i, r_i)_{i \in I} = (d\text{-}\lim (x_i)_{i \in I}, \inf r_i)$.
- And translation $(y, s) \longmapsto (y, s+a)$ is Scott-continuous, for every $a \geq 0$.

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
Yoneda-complete	$\mathbb{R}^+ \cup \{+\infty\}, d_{\mathbb{R}}$	dcpo
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

OP AND SYM

- $d^{\text{op}}(x, y) = d(y, x)$
- $d^{\text{sym}}(x, y) = \max\,(d(y, x), d^{\text{op}}(x, y))$
- d^{sym} is a **metric**
- **Lemma**: for Cauchy nets,
  d^{op}-limit = d^{sym}-limit = limit in the d^{sym}-open ball topology.
- Asymmetry stems from definition of Cauchy net

SMYTH-COMPLETENESS

- X, d is **Smyth-complete** iff every Cauchy net has a d^{op}-limit (equivalently, a d^{sym}-limit)
- In general much stronger than Yoneda-completeness
- **Example 1**: =usual notion of completeness for metrics
- **Example 2**: an ordering ($x \leq y$ iff $d(x, y) = 0$) is Smyth-complete iff it is **equality** (!)
- **Example 3**: $\mathbb{R}, d_{\mathbb{R}}$ is not Smyth-complete, neither are $\mathbb{R} \cup \{+\infty\}$ is, or $\mathbb{R}^+ \cup \{+\infty\}$ (!) but [a, b] is.

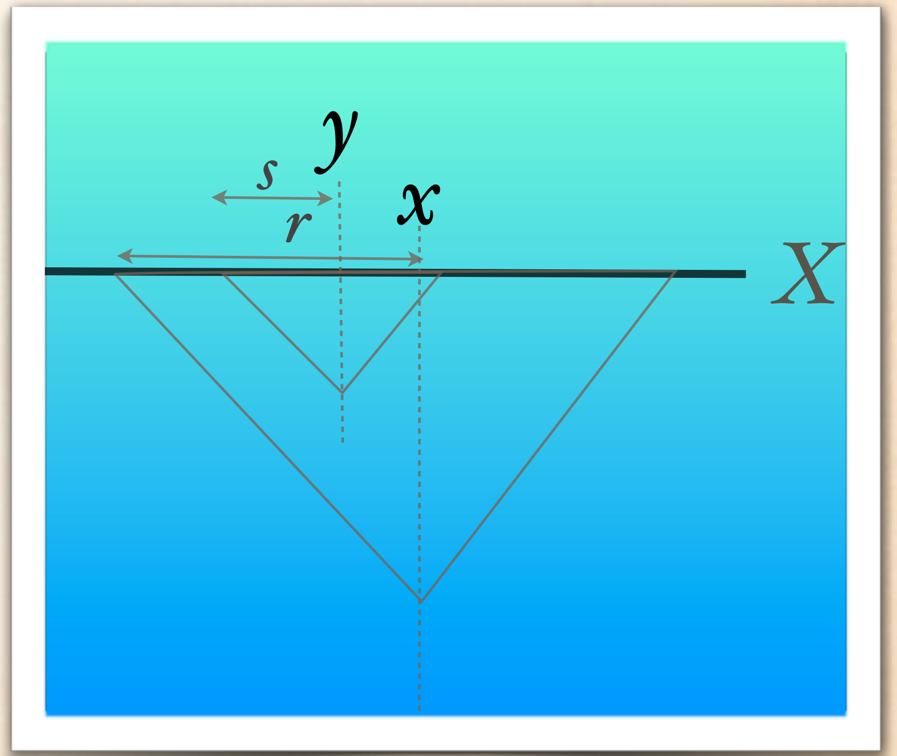
THE R-V THEOREM

- **Theorem** (Romaguera, Valero 2010):
  X, d is Smyth-complete iff $\mathbf{B}(X)$ is a **continuous dcpo**
  and $(x, r) \ll (y, s)$ iff $d(x, y) < r - s$

- Remember that
  $$(x, r) \leq (y, s) \text{ iff } d(x, y) \leq r - s$$

- Proof: as for Kostanek-Waszkiewicz
  (with extra epsilons and deltas)

- In general, let
  $$(x, r) < (y, s) \text{ iff } d(x, y) < r - s$$



A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
Smyth-complete	$[a, b]$	cont. dcpo with $\ll \; = \; \prec$
Yoneda-complete	$\mathbb{R}^+ \cup \{+\infty\}, d_{\mathbb{R}}$	dcpo
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
d^{sym}-compact	[a, b]	
Smyth-complete		cont. dcpo with $\ll\ = \prec$
Yoneda-complete	$\mathbb{R}^+ \cup \{+\infty\}, d_{\mathbb{R}}$	dcpo
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
d^{sym}-compact	$[a, b]$	
Smyth-complete		cont. dcpo with $\ll = \prec$
?		continuous dcpo
Yoneda-complete	$\mathbb{R}^+ \cup \{+\infty\}, d_{\mathbb{R}}$	dcpo
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
d^{sym}-compact	$[a, b]$	
Smyth-complete		cont. dcpo with $\ll\; = \;\prec$
d-continuous Y.-c.		continuous dcpo
Yoneda-complete	$\mathbb{R}^+ \cup \{+\infty\}, d_{\mathbb{R}}$	dcpo
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

D-CONTINUITY

- Let us call X, d ***d*-continuous** Yoneda-complete iff $\mathbf{B}(X)$ is a continuous dcpo — no constraint on $\ll$ [KW2010].
- **Proposition**. The Sorgenfrey line $\mathbb{R}_\ell$ is d_ℓ-continuous but not Smyth-complete.
- $\mathbb{R}_\ell$ is just $\mathbb{R}$, $d_\ell(x, y) = x - y$ if $x \geq y$, $+\infty$ (not 0!) otherwise.
- Open ball topology has subbase of half-open intervals [a, b)

THE SORGENFREY LINE

- A famous counterexample in topology:
- Hausdorff, regular, zero-dimensional
- first-countable
- has a countable dense subset ($\mathbb{Q}$)
- paracompact, hence normal
- Choquet-complete, hence Baire
- **not** locally compact (not even consonant)
- **not** second-countable
- **not** metrizable (although Hausdorff quasi-metric)
- $\mathbb{R}_\ell^2$ **not** normal
- **not** Smyth-complete (d_ℓ^{sym} discrete)

$\mathbb{R}_\ell$ IS D-CONTINUOUS

- **Theorem**. $\mathbb{R}_\ell$ is d_ℓ-continuous Yoneda-complete.

- The map $\mathbf{B}(\mathbb{R}_\ell) \longrightarrow \mathbb{R} \times \mathbb{R}$

  $$(x, r) \longmapsto (-x, x-r)$$

  is an order isomorphism
  onto the subset $\{(a, b) \mid a+b \leq 0\}$.

- The latter is a continuous dcpo.

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  $$(x, r) \longmapsto (-x, x-r)$$

  is an order isomorphism
  onto the subset $\{(a, b) \mid a+b \leq 0\}$.

- The latter is a continuous dcpo.

Note:
$(x, r) \ll (y, s)$ iff $x>y$ and $d_\ell(x, y)<r-s$
$(x, r) < (y, s)$ iff $d_\ell(x, y)<r-s$
Hence $\mathbb{R}_\ell$ is indeed
not Smyth-complete
(Romaguera-Valero: $\ll \neq <$).

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
d^{sym}-compact	$[a, b]$	
Smyth-complete		cont. dcpo with $\ll = \prec$
d-continuous Y.-c.	$\mathbb{R}_\ell$ $\mathbb{R}^+ \cup \{+\infty\}, d_\mathbb{R}$	continuous dcpo
Yoneda-complete		dcpo
not complete	$\mathbb{R}, d_\mathbb{R}$	mere poset?

THE MEANING OF <

- **B**(X) is itself quasi-metric:

$$d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)$$

... with specialization $(x, r) \leq (y, s)$ iff $d(x, y) \leq r - s$

- **Fact**. (y, s) is in the interior of $\uparrow(x, r)$ [in open ball topol.]
  iff $(x, r) < (y, s)$ [reminder: iff $d(x, y) < r - s$]

C-SPACES

- **Fact**. With its open ball topology, **B**(X) is a **c-space**: every open neighborhood $\mathcal{U}$ of a point (y, s) contains a point $(x, r) \prec (y, s)$ [$\prec$ = «in the interior of upward closure of»^{op}]
- Notion due to {Ershov73, Erné91}
  Proof. $\mathcal{U}$ contains an open ball B((y, s), R).
  Take $x=y$, and any r such that $s < r < R+s$.
- **Fact.** Continuous dcpos are sober c-spaces.
  A sober c-space is a continuous dcpo, has the Scott topology.
  Furthermore, $\ll$ = $\prec$.

R-V REVISITED

- **Theorem** (Romaguera, Valero 2010):
  X, d is Smyth-complete iff $\mathbf{B}(X)$ is a **continuous dcpo**
  and $\ll \; = \; \prec$.

- **Corollary**.
  X, d is Smyth-complete iff $\mathbf{B}(X)$ is **sober**
  in its open ball topology.

D-FINITENESS

- A point x is d-finite iff
  for every net $(y_i)_{i \in I}$ with d-limit y, $d(x, y) = \liminf d(x, y_i)$
  [d-limit: for every z, $d(y, z) = \limsup d(y_i, z)$]
- **Example 1**: in a metric space, every point is d-finite
- **Example 2**: in an ordering, d-finite = finite
- **Example 3**: in $\mathbb{R}^+ \cup \{+\infty\}$,$\mathrm{d}_{\mathbb{R}}$, d-finite = all except $+\infty$.
- **Example 4**: in $\mathbb{R}_\ell$, no point is d_ℓ-finite.

D-ALGEBRAICITY

- X, d is ***d*-algebraic** Yoneda-complete iff every point is the *d*-limit of a Cauchy net of *d*-finite points. [BvBR1998]
- **Example 1**: every complete metric space is *d*-algebraic Y.-c.
- **Example 2**: in an ordering, *d*-algebraic Y.-c.= algebraic dcpo
- **Example 3**: $\mathbb{R}^+ \cup \{+\infty\}, d_\mathbb{R}$ is *d*-algebraic Y.-c.
- **Example 4**: $\mathbb{R}_\ell$ is not *d*-algebraic.

ALI-AKBARI ET AL.

- **Theorem** [Ali-Akbari, Honary, Pourmahdian, Rezaii 2009].
  X, d is Smyth-complete iff
  X, d is Yoneda-complete and **every** point is d-finite.
- Hence Smyth-complete strictly stronger than
  Yoneda-complete d-algebraic.

FORMAL BALLS

- No point (x, r) in $\mathbf{B}(X)$ is finite, ever.
- **Fact**. A point x is d-finite iff
  the open ball centered at (x, r) with radius R is **Scott-open**
  [for some/for all r, R]
- Compare this with: in a dcpo, x is finite iff $\uparrow x$ is Scott-open.
- **Corollary**. X, d is d-algebraic Yoneda-complete iff there are enough open balls to generate the Scott topology on $\mathbf{B}(X)$.

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
d^{sym}-compact	$[a, b]$	
Smyth-complete		cont. dcpo with $\ll\ = \ \prec$
d-algebraic Y.-c.		continuous dcpo with enough open balls
d-continuous Y.-c.	$\mathbb{R}_\ell$ $\mathbb{R}^+ \cup \{+\infty\}, d_\mathbb{R}$	continuous dcpo
Yoneda-complete		dcpo
not complete	$\mathbb{R}, d_\mathbb{R}$	mere poset?

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
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d-continuous Y.-c.	$\mathbb{R}_\ell$	continuous dcpo
Yoneda-complete		dcpo
Choquet-complete		
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

A SPECTRUM OF COMPLETENESS

X, d		$\mathbf{B}(X)$
d^{sym}-compact	[a, b]	
Smyth-complete		cont. dcpo with ≪ = <
d-algebraic Y.-c.	$\mathbb{R}^+$	continuous dcpo with enough open balls
d-continuo[us]		continuous dcpo
Yoneda-com[plete]		dcpo
Choquet-complete		
not complete	$\mathbb{R}, d_{\mathbb{R}}$	mere poset?

All these notions equivalent for metric spaces

OUTLINE

- Notions of completeness
- Low-hanging fruit: fixed point theorems
- The formal ball completion
- Low-handing fruit: miscellanea

OUTLINE

- Notions of completeness
- Low-hanging fruit: fixed point theorems
- The formal ball completion
- Low-handing fruit: miscellanea

YONEDA-CONTINUITY

- A uniformly cont. map $f : X \longrightarrow Y$ is **Yoneda-continuous** iff f preserves d-limits of Cauchy nets [BvBR1998]

- Lipschitz ⇒ unif. continuity ⇒ continuity in metric spaces

- ... **not** elsewhere! In orderings:
  Lipschitz = unif. cont. = monotonic
  Yoneda-continuous = Scott-continuous

YONEDA=SCOTT

- Let $\mathbf{B}_c(f) : \mathbf{B}(X) \longrightarrow \mathbf{B}(Y)$

  $$(x, r) \longmapsto (f(x), c.r)$$

- f is c-Lipschitz iff $\mathbf{B}_c(f)$ is monotonic
- f is c-Lipschitz Yoneda-continuous iff $\mathbf{B}_c(f)$ Scott-continuous

RUTTEN = KLEENE

- Remember? If g Scott-continuous on a dcpo and $x \leq g(x)$, then g has a least fixed point above x [Kleene].

- **Theorem** [Rutten1996]. Let X, d be Y.-c., with $d(\_, \_) < +\infty$. Let $f : X \longrightarrow Y$ be c-Lipschitz Yoneda-continuous, c<1. Then f has a unique fixed point.

- Fix x_0 in X. For r_0 large enough, $d(x_0, f(x_0)) \leq (1\text{-}c)\, r_0$.
  That means $(x_0, r_0) \leq (f(x_0), c.r_0) = \mathbf{B}_c(f)\,(x_0, r_0)$.
  Let (y, s) be the least fixed point of $\mathbf{B}_c(f)$ above (x_0, r_0).
  In particular, $y{=}f(y)$.

CARISTI-WASZKIEWICZ =BOURBAKI-WITT

- **Theorem** [Bourbaki 1949, Witt 1951]. Let Y be a dcpo, $g : Y \longrightarrow Y$ be inflationary [$y \leq g(y)$ for every y]. Then g has a fixed point above any y in Y.

- Apply Kostanek-Waszkiewicz again, and obtain:

- **Theorem** [Waszkiewicz2010].
Let X, d be Yoneda-complete, and $f : X \longrightarrow X$ be **any** map.
Assume f has a d-lsc. **potential** $\varphi : X \longrightarrow \mathbb{R}^+$
$$[\varphi(f(x)) + d(x, f(x)) \leq \varphi(x) \quad \text{for every } x]$$
Then f has a fixed point.

CARISTI-WASZKIEWICZ

- φ is ***d*-lsc.** iff $\varphi(d\text{-limit } x_i) \leq \liminf \varphi(x_i)$ for every Cauchy net

- **Theorem** [Waszkiewicz 2010].
  Let X, d be Yoneda-complete, and $f : X \longrightarrow X$ be **any** map.
  Assume f has a d-lsc. potential $\varphi : X \longrightarrow \mathbb{R}^+$.
  Then f has a fixed point.

- Consider $g(x, r) = (f(x), r - \varphi(x) + \varphi(f(x)))$

  g is inflationary on subdcpo $\{(x, r) \in \mathbf{B}(X) \mid r \geq \varphi(x)\}$.

- Generalizes [Caristi 1976] on complete metric spaces.

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CAUCHY COMPLETION

- In metric spaces, a simple notion of completion:
  - Take the space of Cauchy nets [OK, sequences, really]
  - Define a «metric» d' on it
  - Quotient by the «at d'-distance zero» relation
- We can replace the quotient by a **canonical** representative: next slide.

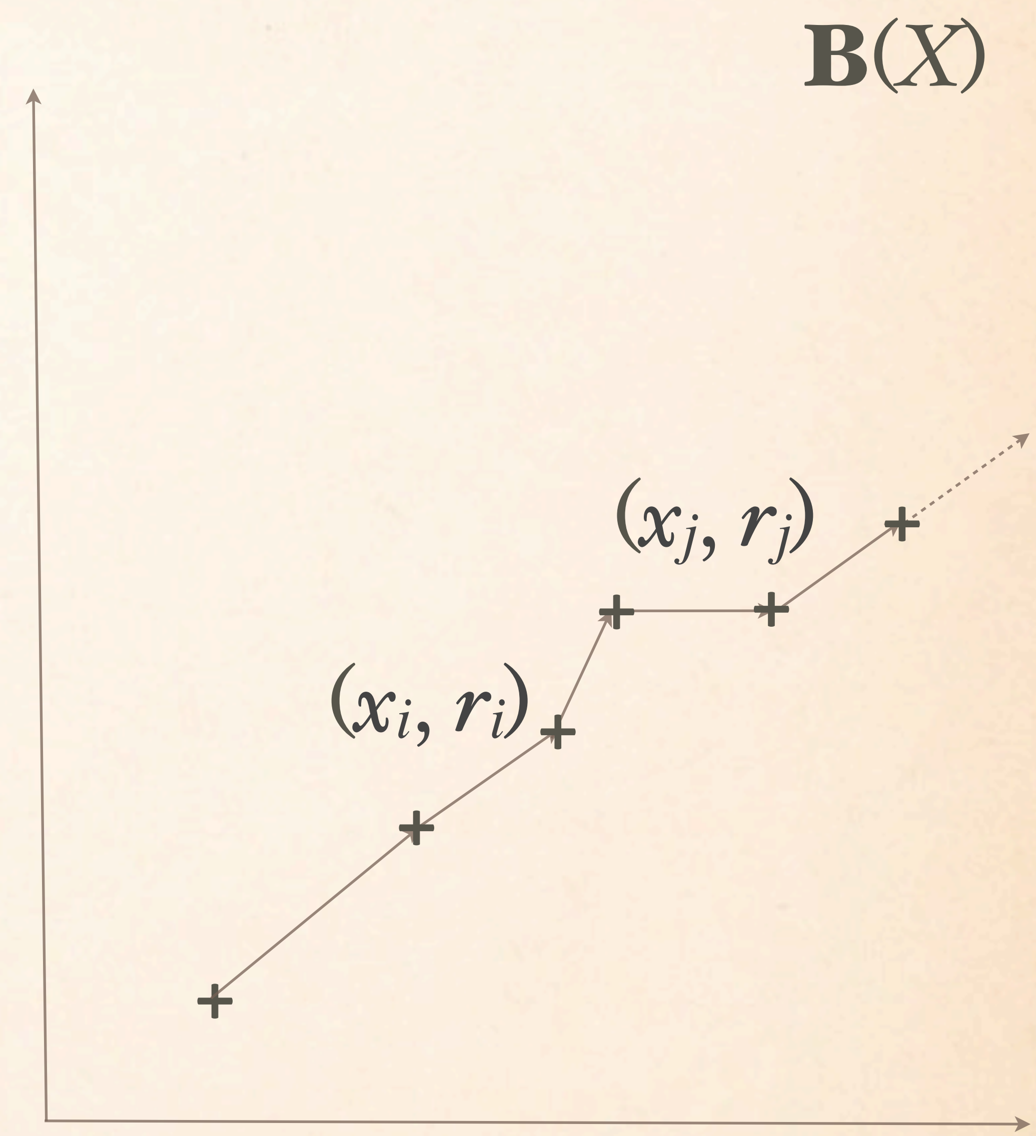
CAUCHY COMPLETION REVISITED

- Recall abstract basis
  $(x, r) < (y, s)$ iff $d(x, y) < r - s$

- Let $\mathbf{S}(X)$ be the space of all rounded ideals D in $\mathbf{B}(X)$ **with aperture zero**.

- aperture = $\inf \{r \mid (x, r) \in D\}$
  ... so D is a Cauchy-weighted net

- $\mathbf{S}(X)$ is a subset of the rounded ideal completion $\mathbf{RI}(\mathbf{B}(X))$



= sobrification of $\mathbf{B}(X)$/open ball topology by [Lawson 1997]: good start!

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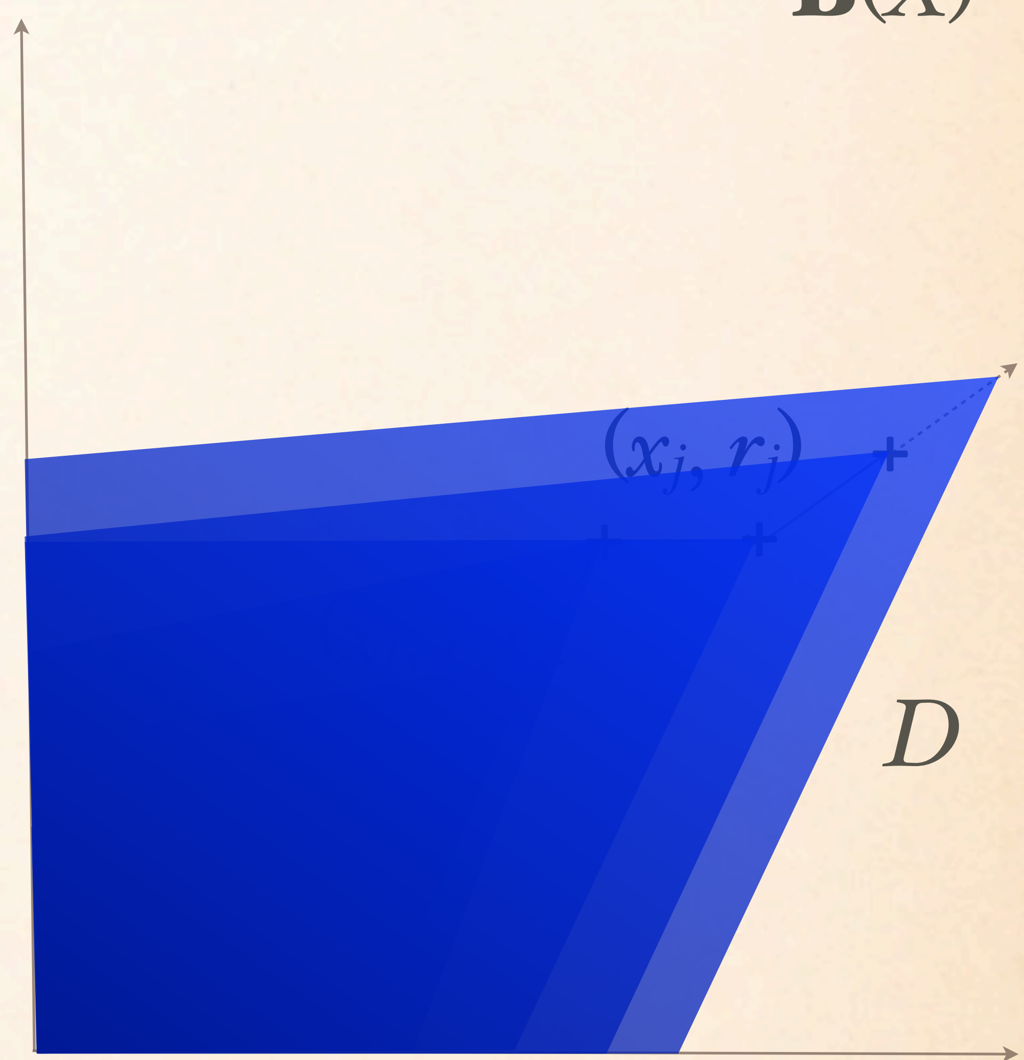

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CAUCHY COMPLETION REVISITED

- $\mathbf{S}(X)$ = {rounded ideals D in $\mathbf{B}(X)$ with aperture zero}

- Define $d^+_{\mathcal{H}}$ on $\mathbf{S}(X)$ by the ½-Hausdorff metric formula

$$d^+_{\mathcal{H}}(D, D') = \sup_{B \in D} \inf_{B' \in D'} d^+(B, B')$$

[recall $d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)$]

- **Theorem**. $\mathbf{B}(\mathbf{S}(X))$ is isomorphic to $\mathbf{RI}(\mathbf{B}(X))$

  through $(D, s) \longmapsto D{+}s = \{(x, r{+}s) \mid (x, r) \in D\}$

- **Corollary**. $\mathbf{S}(X)$ is **Yoneda-complete** $d^+_{\mathcal{H}}$-continuous.

  ... in fact $d^+_{\mathcal{H}}$-algebraic with $d^+_{\mathcal{H}}$-finite elements $\Downarrow(x, 0)$.

FREE YONEDA-C. SPACES

- Let $\mathbf{YCQMet}_c$: Yoneda-complete quasi-metric spaces
  + c-Lipschitz Yoneda-continuous maps

- Let $\mathbf{QMet}_c$: quasi-metric spaces + c-Lipschitz maps

- **Theorem**. $\mathbf{QMet}_c : \mathbf{S} \dashv \text{Forget} : \mathbf{YCQMet}_c$.

$\Downarrow(x, 0)$ $\quad \mathbf{S}(X)$

isometric embedding $\uparrow$ $\quad \uparrow$

x $\quad X$

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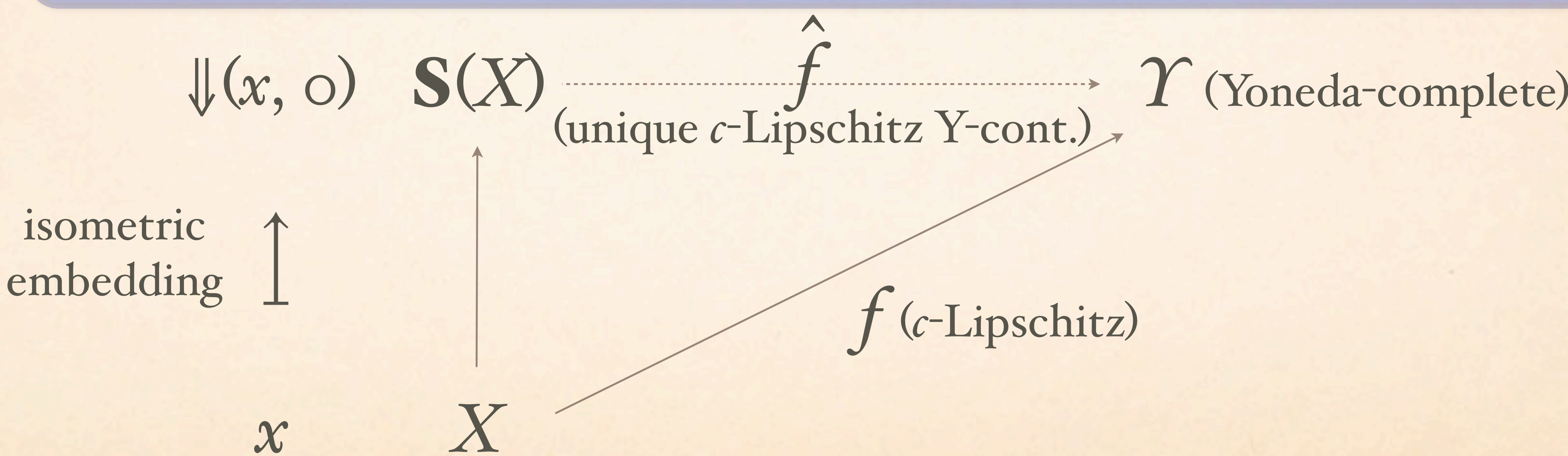

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- Let $\mathbf{YCQMet}_u$: Yoneda-complete quasi-metric spaces + **uniformly** Yoneda continuous maps

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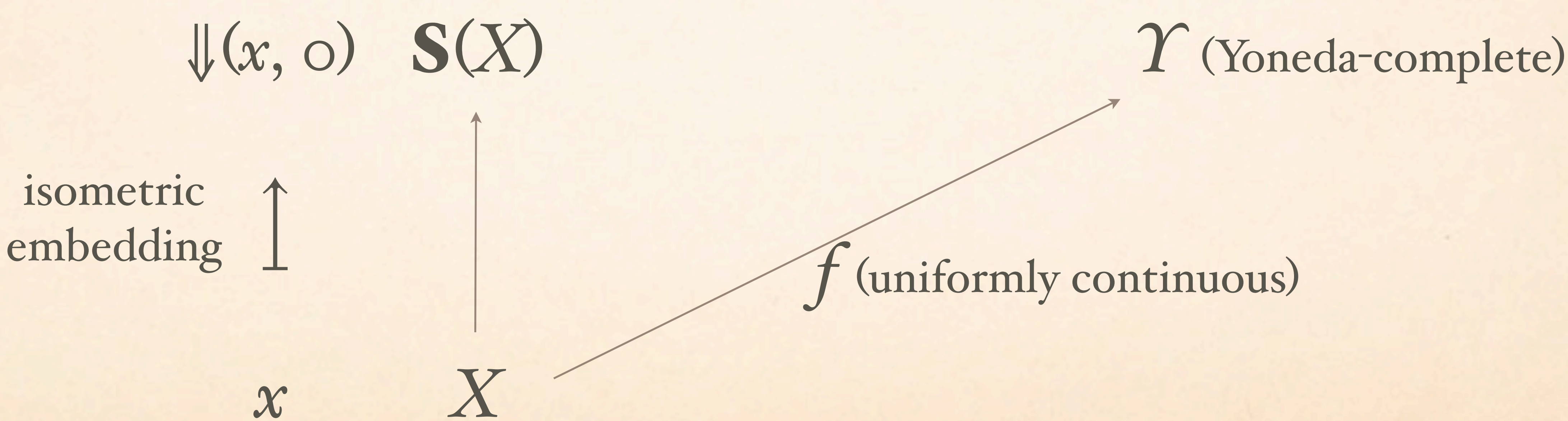

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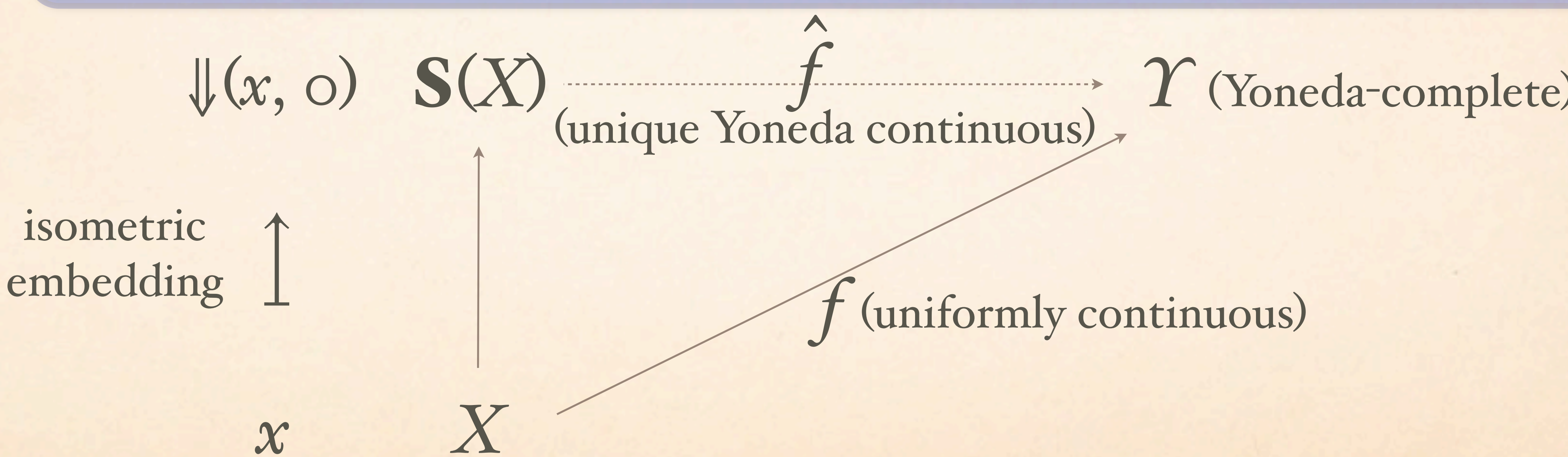

FREE YONEDA-C. SPACES

- Let $\mathbf{YCQMet}_u$: Yoneda-complete quasi-metric spaces
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- Let $\mathbf{QMet}_u$: quasi-metric spaces + **uniformly cont**. maps

- **Theorem**. $\mathbf{QMet}_u : \mathbf{S} \dashv \text{Forget} : \mathbf{YCQMet}_u$.

$\Downarrow(x, 0)$ $\mathbf{S}(X)$ $\xrightarrow{\hat{f}}$ Y (Yoneda-complete)

(unique Yoneda continuous)

isometric embedding $\uparrow$

f (uniformly

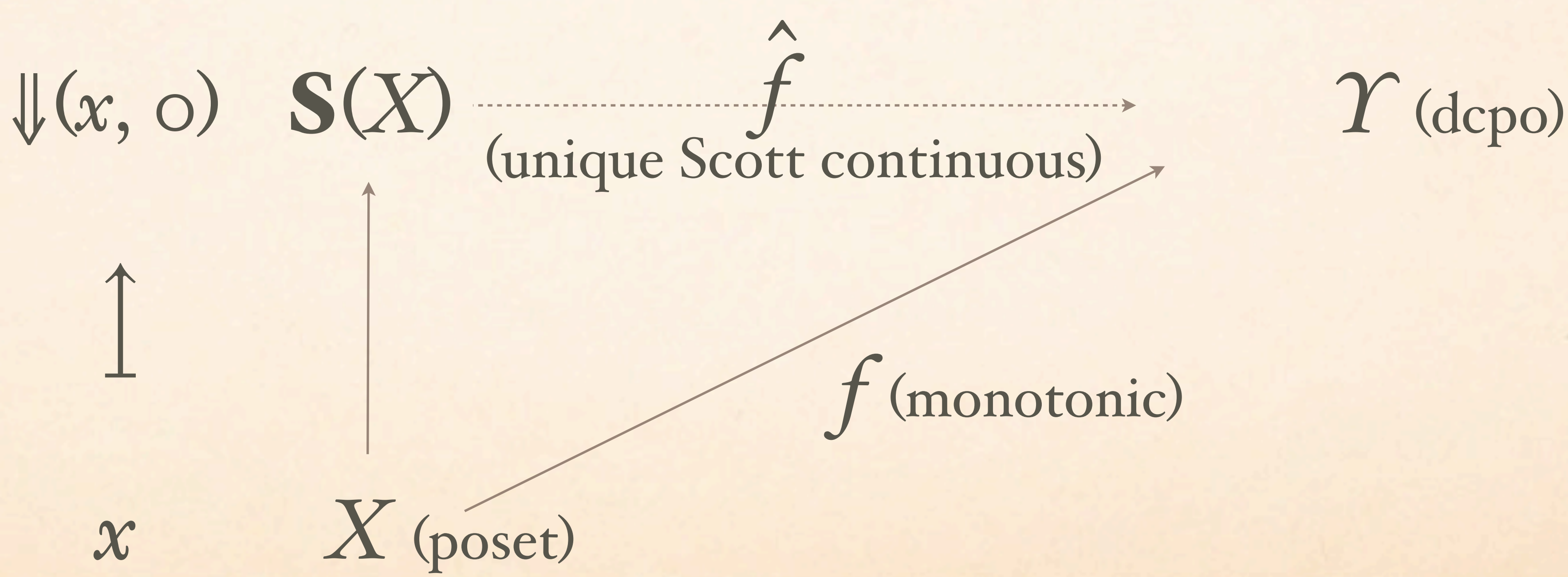
x X

In particular $\mathbf{S}(X)$
$\cong$ Cauchy completion
on metric spaces

FREE YONEDA-C. SPACES

- On orderings, uniformly continuous = monotonic
  Yoneda continuous = Scott-continuous
  Yoneda-complete = dcpo

- **Theorem**. For posets X, $\mathbf{S}(X)$ ...



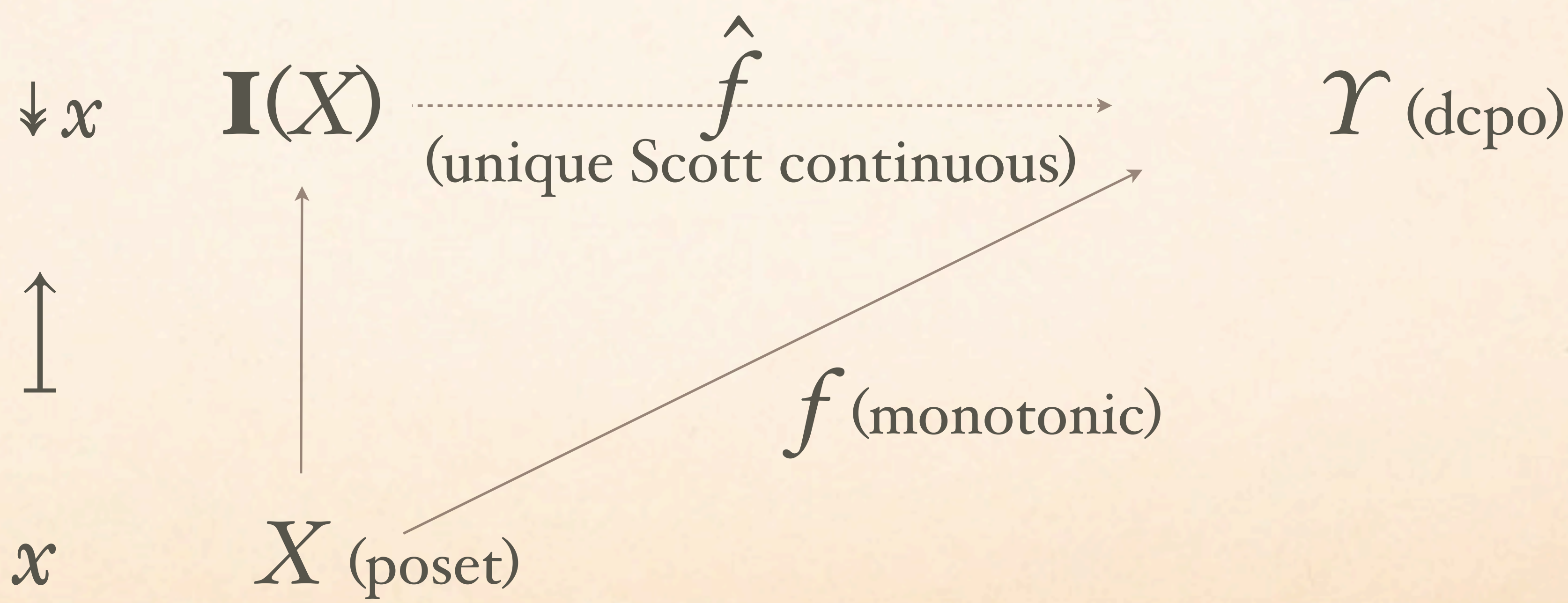
FREE YONEDA-C. SPACES

- On orderings, uniformly continuous = monotonic

  Yoneda continuous = Scott-continuous

  Yoneda-complete = dcpo

- **Theorem**. For posets X, $\mathbf{S}(X) \cong$ ideal completion $\mathbf{I}(X)$



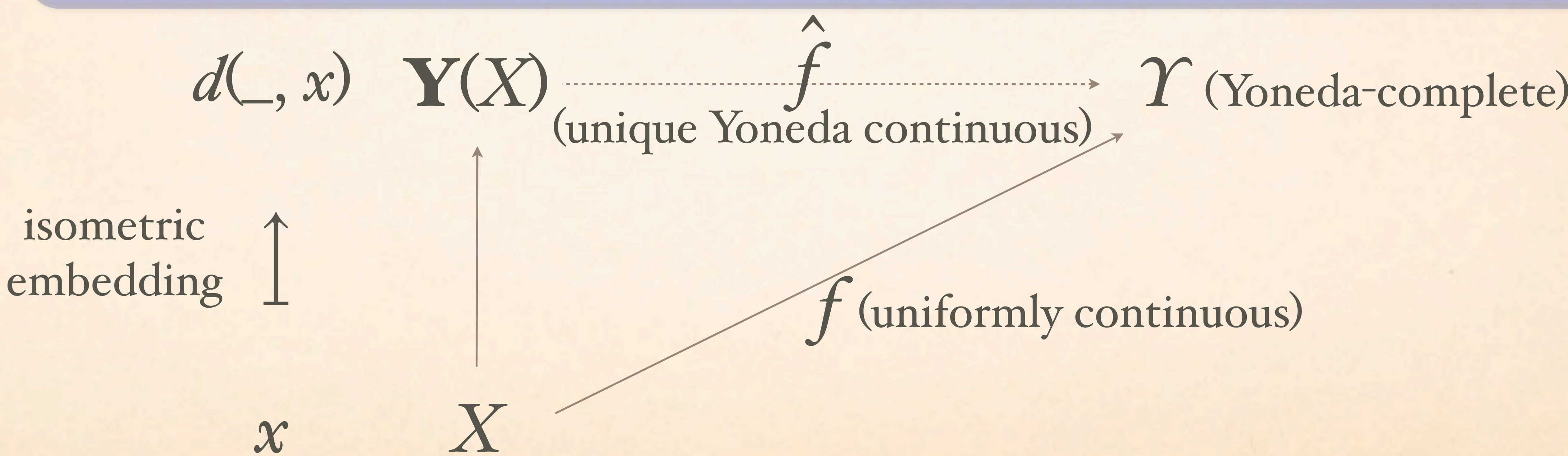
YONEDA COMPLETION

- The Yoneda map $\mathrm{Yon} : X \longrightarrow [X \to \mathbb{R}^+ \cup \{+\infty\}]_{1\text{-Lipschitz}}$

$$x \longmapsto d(\_, x)$$

  Let $\mathbf{Y}(X)$ = d-closure of image of Yon.

- **Theorem** [BvBR1998]. $\mathbf{QMet}_u : \mathbf{Y} \dashv \text{Forget} : \mathbf{YCQMet}_u$.



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$$x \longmapsto d(\_, x)$$

Let $\mathbf{Y}(X)$ = d-closure of image of Yon.

- **Theorem** [… $\dashv$ Forget : $\mathbf{YCQMet}_u$.

Hence $\mathbf{Y}(X) \cong \mathbf{S}(X)$

$d(\_, x)$ $\mathbf{Y}(X)$ $\overset{f}{\dashrightarrow}$ Υ (Yoneda-complete)

(unique Yoneda continuous)

isometric embedding ↑

f (uniformly continuous)

x X

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IDEMPOTENCE

- Cauchy completion is idempotent:

  $$\text{Cauchy}(\text{Cauchy}(X)) \cong \text{Cauchy}(X)$$

- Formal ball completion **S** is **not**: $\mathbf{S}(\mathbf{S}(X)) \ncong \mathbf{S}(X)$

  ... unless X is metric for example

  Counterexample: orderings (**I** is not idempotent)

- What are the fixed points of **S**?

  ... at least the complete metric spaces,
  but there are more.

FIXED POINTS OF S

- **Theorem** [Flagg Sünderhauf 1996; Künzi Schellekens 2002]
  $\mathbf{S}(X) \cong X$ if and only if X is **Smyth-complete**.

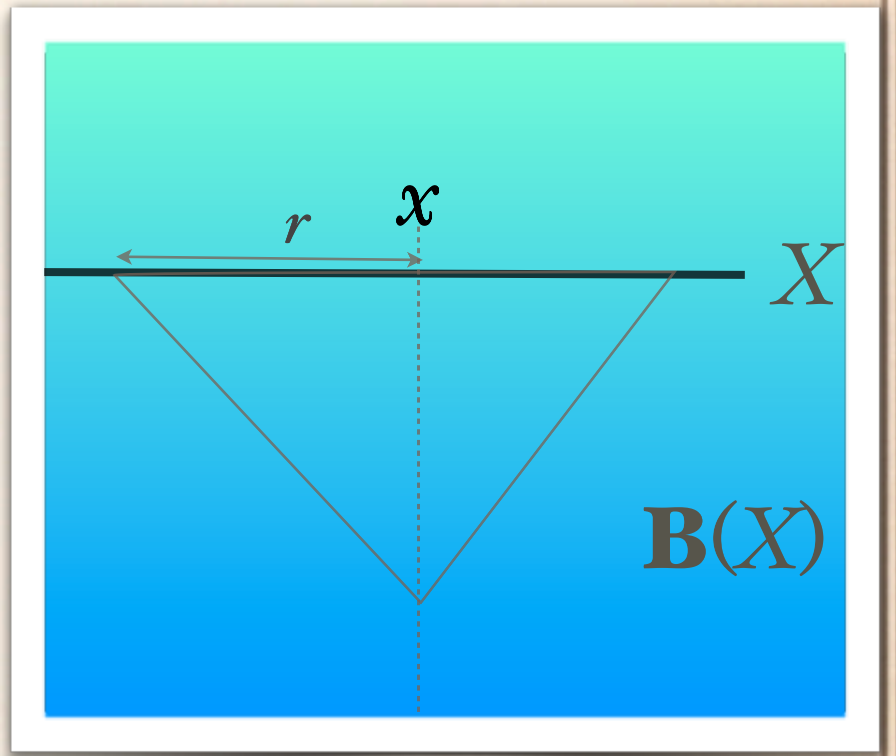
- If X Smyth-complete, then $\mathbf{B}(X)$ sober [remember?]
  So $\mathbf{B}(\mathbf{S}(X)) \cong \mathbf{RI}(\mathbf{B}(X)) \cong$ sobrification of $\mathbf{B}(X) \cong \mathbf{B}(X)$,
  because sobrification is idempotent.
  In particular, looking at elements of radius 0, $\mathbf{S}(X) \cong X$.

- If $\mathbf{S}(X) \cong X$, then
  $\mathbf{B}(X) \cong \mathbf{B}(\mathbf{S}(X)) \cong \mathbf{RI}(\mathbf{B}(X)) \cong$ sobrification of $\mathbf{B}(X)$
  is sober, so X is Smyth-complete.

THE D-SCOTT TOPOLOGY

- Let the ***d*-Scott topology** on X be induced by Scott topology on $\mathbf{B}(X)$ through embedding $\eta : X \longrightarrow \mathbf{B}(X)$

  $x \longmapsto (x, 0)$



- **Example 1**: = open ball topology
  if X metric, or if X Smyth-complete
- **Example 2**: = Scott topology if X, d ordering
- **Ex.** 3: base of open balls with d-finite centers if Y.-c. d-algebr.
- **Ex.** 4:=generalized Scott topology [BvBR1998] if Y.-c. d-cont.

SOBRIETY

- Among sober spaces, we find Hausdorff spaces, continuous dcpos, and:
- **Theorem**. Every Yoneda-complete *d*-continuous space X is sober in its *d*-Scott topology.
- **Proof**. $\mathbf{B}(X)$ is continuous hence sober in its Scott topology.

$$X \xrightarrow{\ \eta\ } \mathbf{B}(X) \underset{0}{\overset{\text{radius}}{\rightrightarrows}} \mathbb{R}^{+op}$$

shows η as an equalizer map ($(x, r) \in \mathrm{Im}\,\eta$ iff $r=0$) in **Top**.
Subspaces of sober spaces obtained as equalizers are sober.

SOBRIETY

- Among sober spaces, we find Hausdorff spaces,
  continuous dcpos, and:
- **Theorem**. Every Yoneda-complete *d*-continuous space *X* is sober in its *d*-Scott topology.
- **Corollary**. Every Smyth-complete space is sober in its open ball topology.
- Because in that case *d*-Scott = open ball (every point *d*-finite [Ali-Akbari et al.2009]).

RETRACTS

- A 1-Lipschitz Y.-continuous retraction:

$$Y \underset{s}{\overset{r}{\rightleftarrows}} X \qquad (r \circ s = \mathrm{id})$$

- induces a Scott-continuous retraction:

$$\mathbf{B}(Y) \underset{\mathbf{B}_1(s)}{\overset{\mathbf{B}_1(r)}{\rightleftarrows}} \mathbf{B}(X) \qquad (\mathbf{B}_1(r) \circ \mathbf{B}_1(s) = \mathrm{id})$$

RETRACTS OF D-CONT. SPACES

- If Y is Y.-c. d-continuous, then **B**(Y) is a continuous dcpo
- If **B**(X) is a continuous dcpo, then X is Y.-c. d-continuous. Hence:
- **Proposition**. Every 1-Lipschitz Y.-continuous retract of a Yoneda-complete d-continuous space is Yoneda-complete d-continuous.

$Y \underset{s}{\overset{r}{\rightleftarrows}} X$

$\mathbf{B}(Y) \underset{\mathbf{B}_1(s)}{\overset{\mathbf{B}_1(r)}{\rightleftarrows}} \mathbf{B}(X)$

CONTINUITY/ALGEBRAICITY

- **Theorem**. The *d*-**continuous** Y.-comp. spaces are exactly the 1-Lipschitz Y.-continuous retracts of *d*-**algebraic** Yoneda-complete spaces.

$D = (x_i, r_i)_{i \in I}$ $\mathbf{S}(X)$ $\Downarrow (x, 0)$

$\downarrow$ r $\downarrow\uparrow$ s $\uparrow$

$d\text{-}\lim x_i$ X x

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$D=(x_i, r_i)_{i \in I}$ **S**(X) $\Downarrow(x, 0)$

r s

$d\text{-}\lim x_i$ X x

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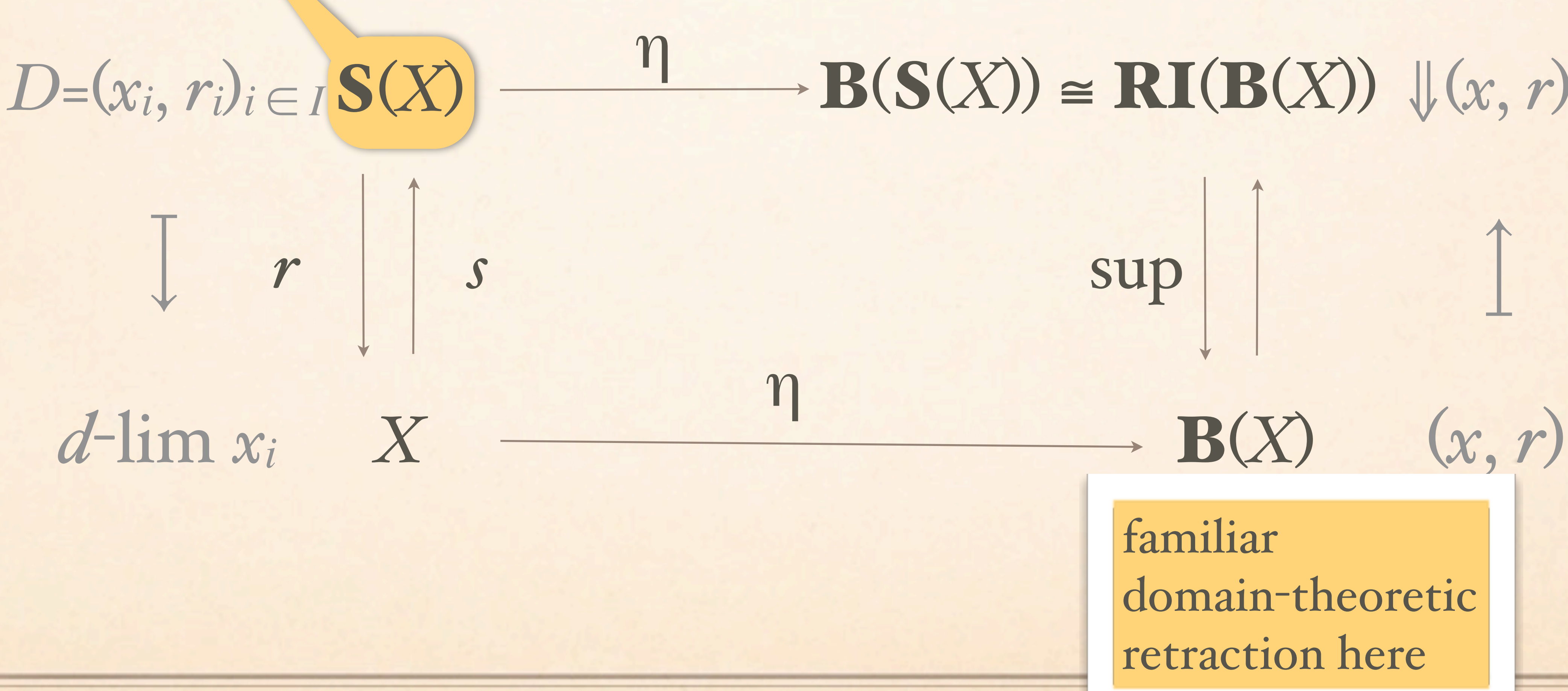

CONCLUSION

- Formal balls provide
  a **unifying** view of quasi-metric theory
  through **domain theory**

- Plenty of opportunities of generalization of theorems
  on continuous dcpos
  to Yoneda-complete d-continuous spaces

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