Jean Goubault-Larrecq

Topological functors and Cartesian-closed categories

Séminaire invité à l'université de Puebla (BUAP), Mexique, 24 mai 2023

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Outline

- **Top** is not Cartesian-closed
- … but some full subcategories are,
  e.g. compactly generated spaces [Brown 61, 63; Steenrod 67; Kelley 55]
- A general construction
  due to [Escardó, Lawson, Simpson 04]
- … generalizes to **topological functors** [JGL 14]
- Applications: streams (very briefly).

Cartesian-closed categories

Terminal objects

- A **terminal object** in a category **C** is an object **1** such that

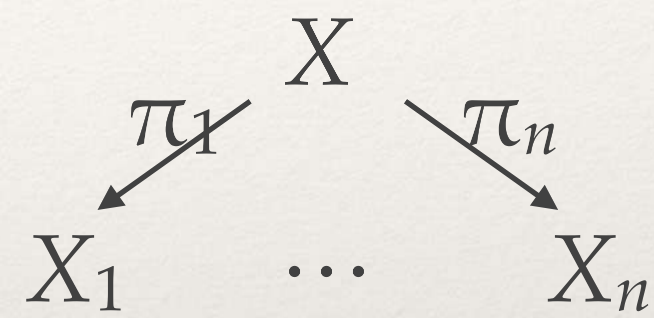
  for every object X, there is a unique morphism

$$X \xrightarrow{!} \mathbf{1}$$

- In **Set**, **Ord**, **Top**: any one-element set/poset/space.

Finite products

- A **product** of objects $X_1, \ldots, X_n$ is any object
  $$X \stackrel{\text{def}}{=} X_1 \times \ldots \times X_n$$
  with **projections** $\pi_i : X \to X_i$



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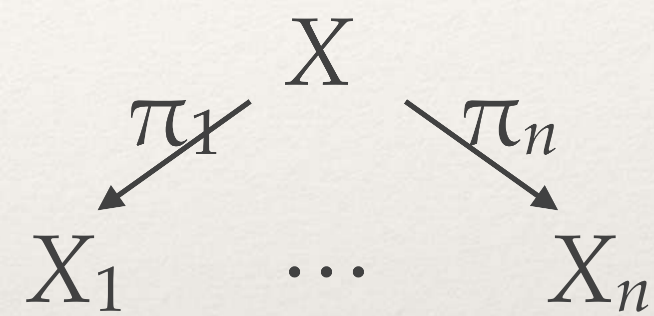

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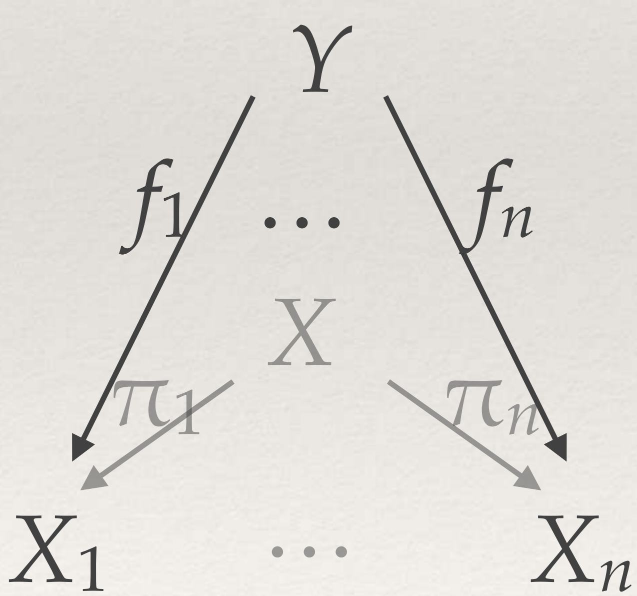

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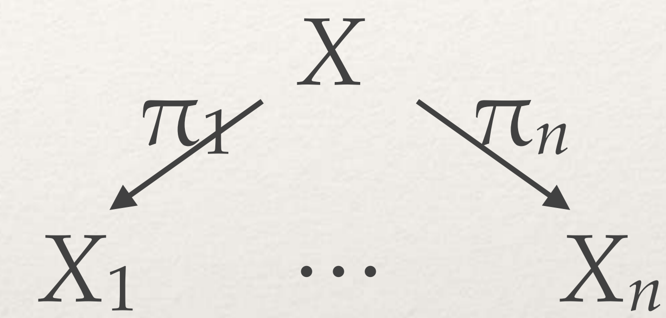

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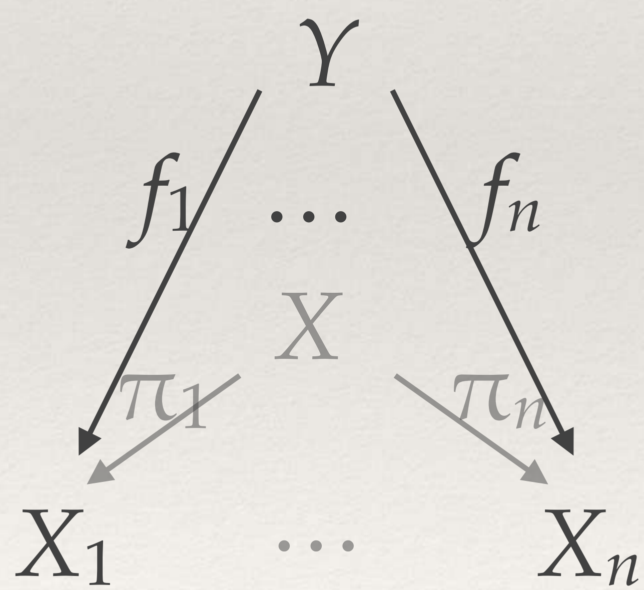
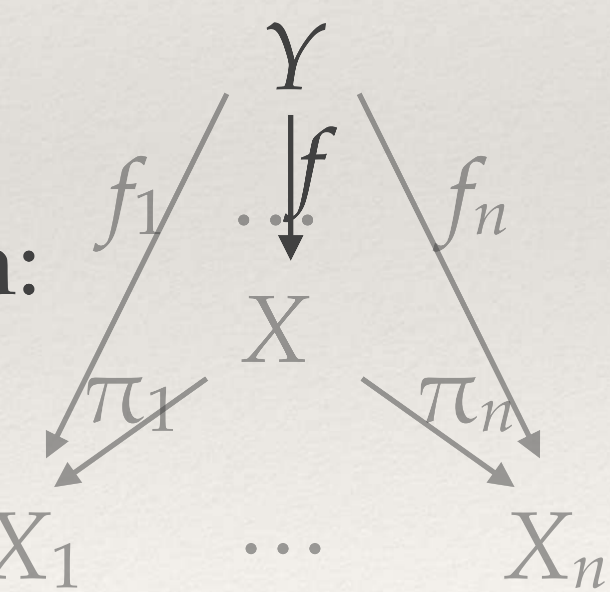
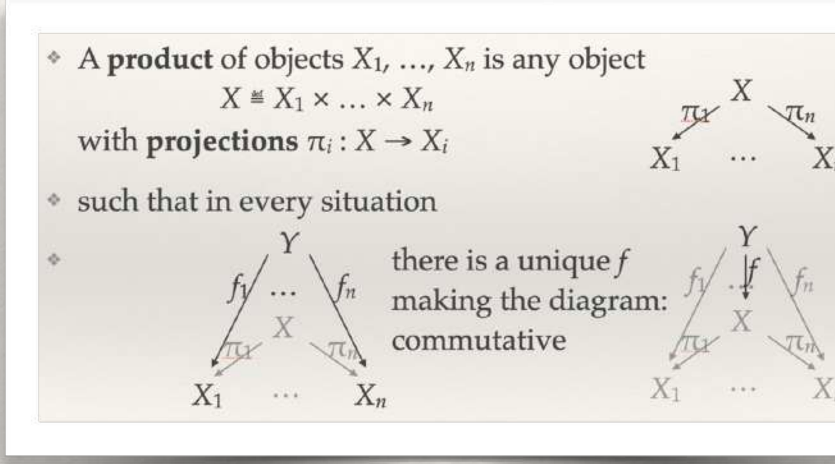


- there is a unique f making the diagram: commutative



Finite products

- Finite product in **Set** (**Ord**, **Top**)
  = Cartesian product (with product ordering, product topology)



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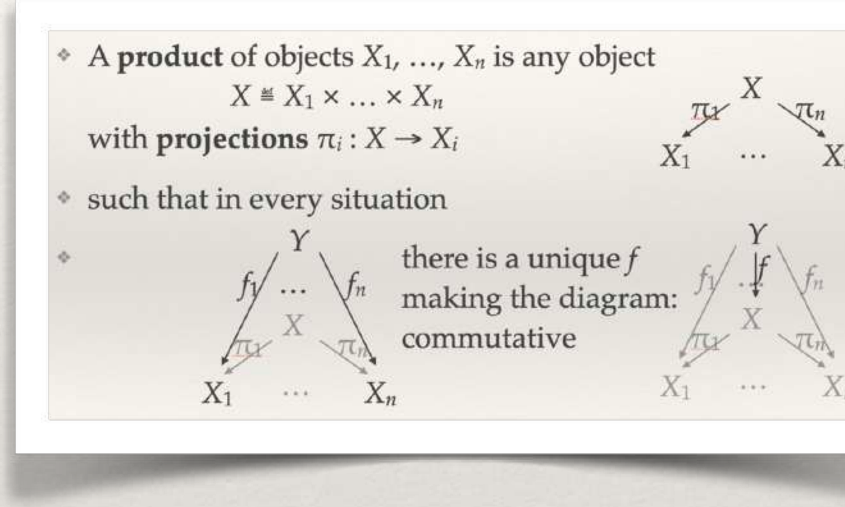

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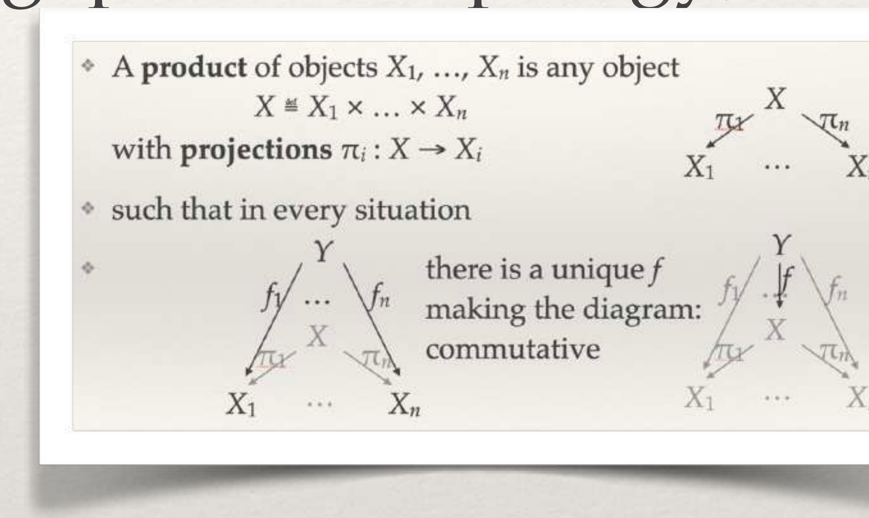

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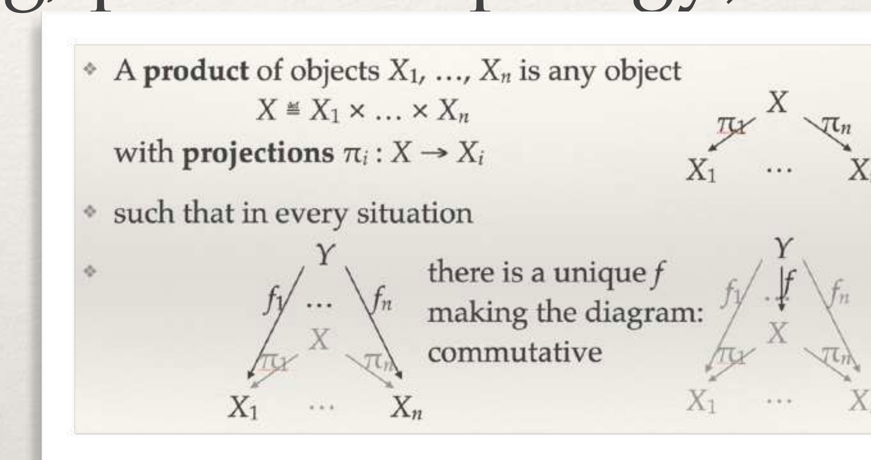

- Products are special cases of **limits**
- Product of n=0 object = **terminal** object
- Equivalent definition: $f = \langle f_1, \ldots, f_n \rangle$ **axiomatized** by:

$$\pi_i \circ \langle f_1, \ldots, f_n \rangle = f_i$$

$$\langle f_1, \ldots, f_n \rangle \circ g = \langle f_1 \circ g, \ldots, f_n \circ g \rangle$$

$$\langle \pi_1, \ldots, \pi_n \rangle = \mathrm{id}$$

Cartesian-closed categories

- A category **C** with finite products
  and with **internal homs**, a.k.a. **exponentials**
  (« function space objects »)

- E.g., **Set** will be Cartesian-closed, because
  there is an **object** « set of all maps from X to Y »
  with the expected properties (next slides)

- Applications:
  (1) **convenient** categories for algebraic topology
  (2) models of the **lambda-calculus** in logic and computer science

Exponentials

- An **exponential** Y^X is an object with:
- an **application** morphism $\mathrm{App} : Y^X \times X \to Y$
- for every morphism $f : Z \times X \to Y$,
  a **currification** morphism $\Lambda(f) : Z \to Y^X$
- such that:
  (β) $\mathrm{App} \circ (\Lambda(f) \times \mathrm{id}) = f$ (« $\Lambda(f)(z)$ applied to x is $f(z,x)$ »)
  (η) $\Lambda(\mathrm{App}) = \mathrm{id}$
  (σ) $\Lambda(f) \circ g = \Lambda(f \circ (g \times \mathrm{id}))$

Exponentiable objects

- An object X is **exponentiable** iff the exponential Y^X exists for every object Y

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- In **Set**, every object is exponentiable
  $Y^X = \{\text{all maps from } X \text{ to } Y\}$
  $\mathrm{App} : (f, x) \mapsto f(x) \qquad \Lambda(f) : z \mapsto (x \mapsto f(z,x))$

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- In **Ord**, every object is exponentiable, too
  Y^X = {all monotonic maps from X to Y}, pointwise ordering
- In **Top**, the exponentiable objects are
  the **core-compact** spaces [Day, Kelly 70]
  every locally compact space is core-compact,
  every core-compact and sober (e.g., Hausdorff) space is locally compact

Cartesian-closed categories

- A category with finite products is **Cartesian-closed** iff every object is exponentiable

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Cartesian-closed categories

- A category with finite products is **Cartesian-closed** iff every object is exponentiable

- **Set** is Cartesian-closed

- **Ord** is Cartesian-closed

- **Top** is not: any Hausdorff, non-locally compact space fails to be exponentiable (e.g., **Q**, or the Sorgenfrey line)

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- In **Top**, the exponentiable objects are the **core-compact** spaces [Day, Kelly 70]

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Cartesian-closed categories in topology?

- Several solutions to the conundrum:
- Find **larger** categories than **Top** that would be Cartesian-closed
  - e.g., **Conv** (see F. Mynard's talk)
- Find **smaller** categories than **Top** that would be Cartesian-closed
  - e.g., compactly-generated spaces
    - or more generally $\mathscr{C}$-generated spaces (this talk)
    - or dcpos (directed-complete partial orders, with Scott topology)

The Escardó-Lawson-Simpson construction

The Escardó-Lawson-Simpson construction: outline

- Fix a suitable class $\mathscr{C}$ of topological spaces
- Build a Cartesian-closed category $\mathbf{Map}_{\mathscr{C}}$

  … intuitively larger than **Top**
- Show that $\mathbf{Map}_{\mathscr{C}}$ is equivalent to

  a full subcategory $\mathbf{Top}_{\mathscr{C}}$ of **Top**
- so $\mathbf{Top}_{\mathscr{C}}$ will be Cartesian-closed, too.

The category $\mathbf{Map}_{\mathscr{C}}$

- Fix a class $\mathscr{C}$ of spaces

  in the case of compactly-generated spaces, $\mathscr{C}$={all compact Hausdorff spaces}

- Call $\mathscr{C}$**-probe** any continuous map $k : C \to X$

  where $C \in \mathscr{C}$

- A function $f \colon X \to Y$ is $\mathscr{C}$**-continuous** iff

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⚠️ X and Y live in **Top**, but f is a function, in **Set**

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- **Definition. $\mathbf{Map}_{\mathscr{C}}$** has:

  — as objects, all topological spaces

  — a morphisms, all $\mathscr{C}$-continuous maps

« larger » than **Top**: every continuous map is $\mathscr{C}$ -continuous

Strongly productive classes

- The class $\mathscr{C}$ is **strongly productive** iff:

  — all the objects of $\mathscr{C}$ are **exponentiable**

  (core-compact, in **Top**)

  — $\mathscr{C}$ is closed under binary products

  certainly true if $\mathscr{C}$={all compact Hausdorff spaces}, right?

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- **Theorem** [ELS 04]. If $\mathscr{C}$ is a strongly productive class of topological spaces, then $\mathbf{Map}_{\mathscr{C}}$ is Cartesian-closed.

$\mathscr{C}$-generated spaces

- For every topological space X,
  define $\mathscr{C}X$ as X, **retopologized** with
  the finest topology that makes
  all the $\mathscr{C}$-probes $k : C \to X$ continuous
  … this is a finer topology than the original topology on X
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- Let $\mathbf{Top}_{\mathscr{C}}$ be the full subcategory of $\mathbf{Top}$
  consisting of $\mathscr{C}$-generated spaces
- There is a functor $\mathscr{C} : \mathbf{Map}_{\mathscr{C}} \to \mathbf{Top}_{\mathscr{C}}$
  we have just defined it on objects; on morphisms, $\mathscr{C}(f) = f$

Coreflective CCCs of topological spaces

- There is also an inclusion functor $I : \mathbf{Top}_{\mathscr{C}} \to \mathbf{Map}_{\mathscr{C}}$
- **Lemma.** $\mathscr{C}$ and I together define an equivalence of categories between $\mathbf{Top}_{\mathscr{C}}$ and $\mathbf{Map}_{\mathscr{C}}$.

i.e., there are isos $X \to \mathscr{C}IX$ (in $\mathbf{Top}_{\mathscr{C}}$) for every $\mathscr{C}$-generated space
and $X \to I\mathscr{C}X$ (in $\mathbf{Map}_{\mathscr{C}}$) for every space X

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- **Corollary** [ELS 04]. If $\mathscr{C}$ is a strongly productive class of topological spaces, then $\mathbf{Top}_{\mathscr{C}}$ is a Cartesian-closed, coreflective full subcategory of $\mathbf{Top}$.

… the right adjoint to the inclusion functor is defined just like $\mathscr{C}$

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« strongly » not needed — allow me to skip this.

Examples

- If $\mathscr{C}$={all compact Hausdorff spaces}, $\mathbf{Top}_{\mathscr{C}}$ is very close to compactly-generated spaces

  … we reobtain the usual compactly-generated spaces
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- If $\mathscr{C}$={all core-compact spaces},
  $\mathbf{Top}_{\mathscr{C}}$ is the largest CCC obtainable by this construction

  … = full subcategory of **Top** consisting of quotients of core-compact spaces
  (in general, $\mathbf{Top}_{\mathscr{C}}$ is the full subcategory of **Top**
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- If $\mathscr{C}$={one-point compactification of $\mathbb{N}$},
  $\mathbf{Top}_{\mathscr{C}}$ = {sequential spaces}

Topological functors

Topological functors

- See *The Joy of Cats* [Adámek, Herrlich, Strecker 90]
- **Topological functors** attempt to capture topology abstractly by properties of the **forgetful functor Top → Set**

  … we will use this as a running example
- What we will see:
  The Escardó-Lawson-Simpson construction
  generalizes to a large class of topological functors

Topological functors

- See *The Joy of Cats* [Adámek, Herrlich, Strecker 90]
- A **topological functor** $|\_| : \mathbf{C} \to \mathbf{D}$ is a functor that is:
  (1) faithful
  (2) amnestic
  (3) every $|\_|$-source has a $|\_|$-initial lift

i.e., injective on hom sets
In the example of
$|\_| : \mathbf{Top} \to \mathbf{Set}$,
check that if $|f| = |g|$
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- Lifts are **unique** when they exist
- For $|\_| : \mathbf{Top} \to \mathbf{Set}$,
  g **has a lift** iff it is continuous

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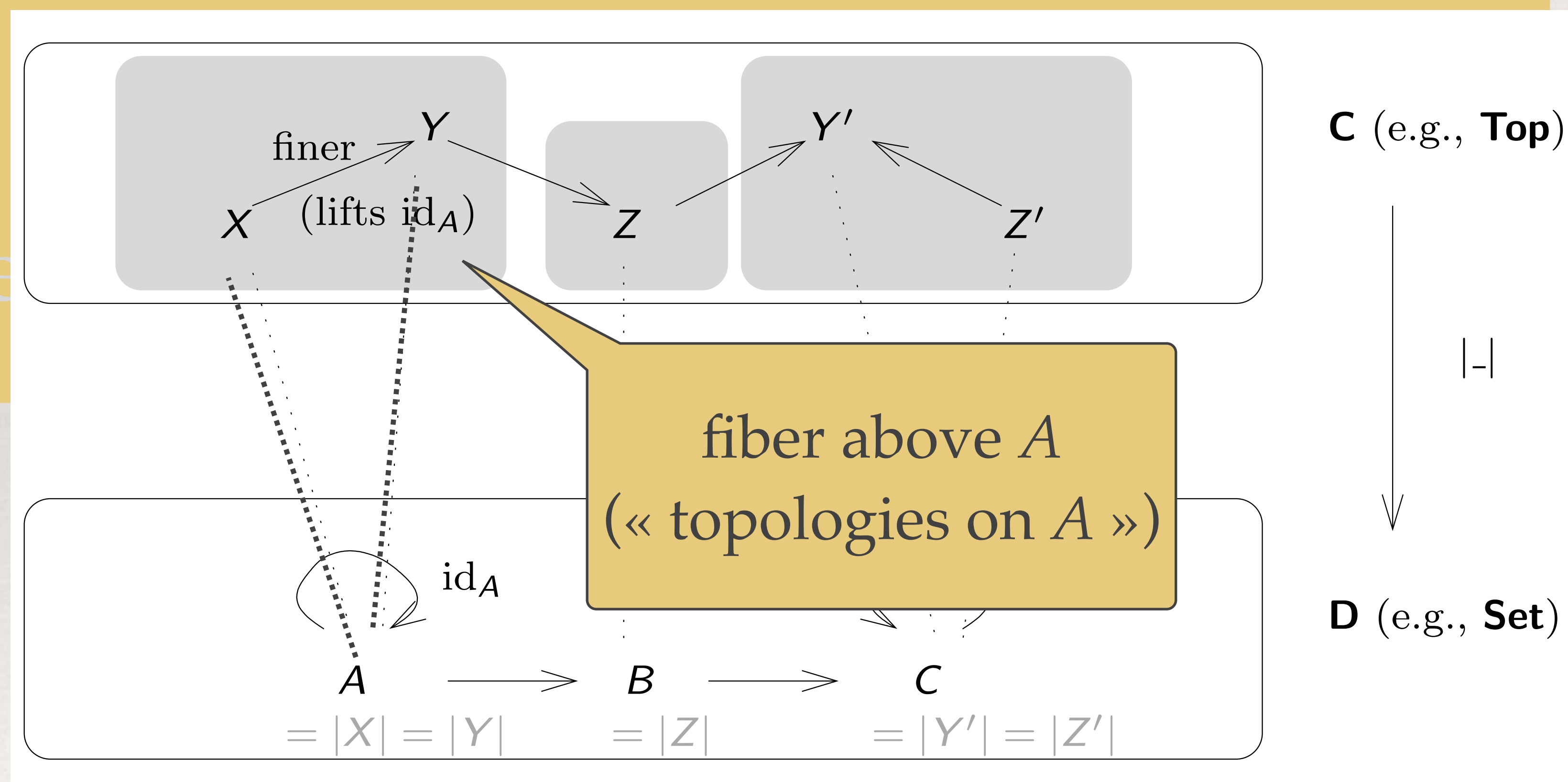
- A **topological functor** $|\_| : \mathbf{C} \to \mathbf{D}$ is a functor that is:
  (1) faithful
  (2) amnestic
  (3) every $|\_|$-source

This is more interesting.
Let us look at **fibers.**



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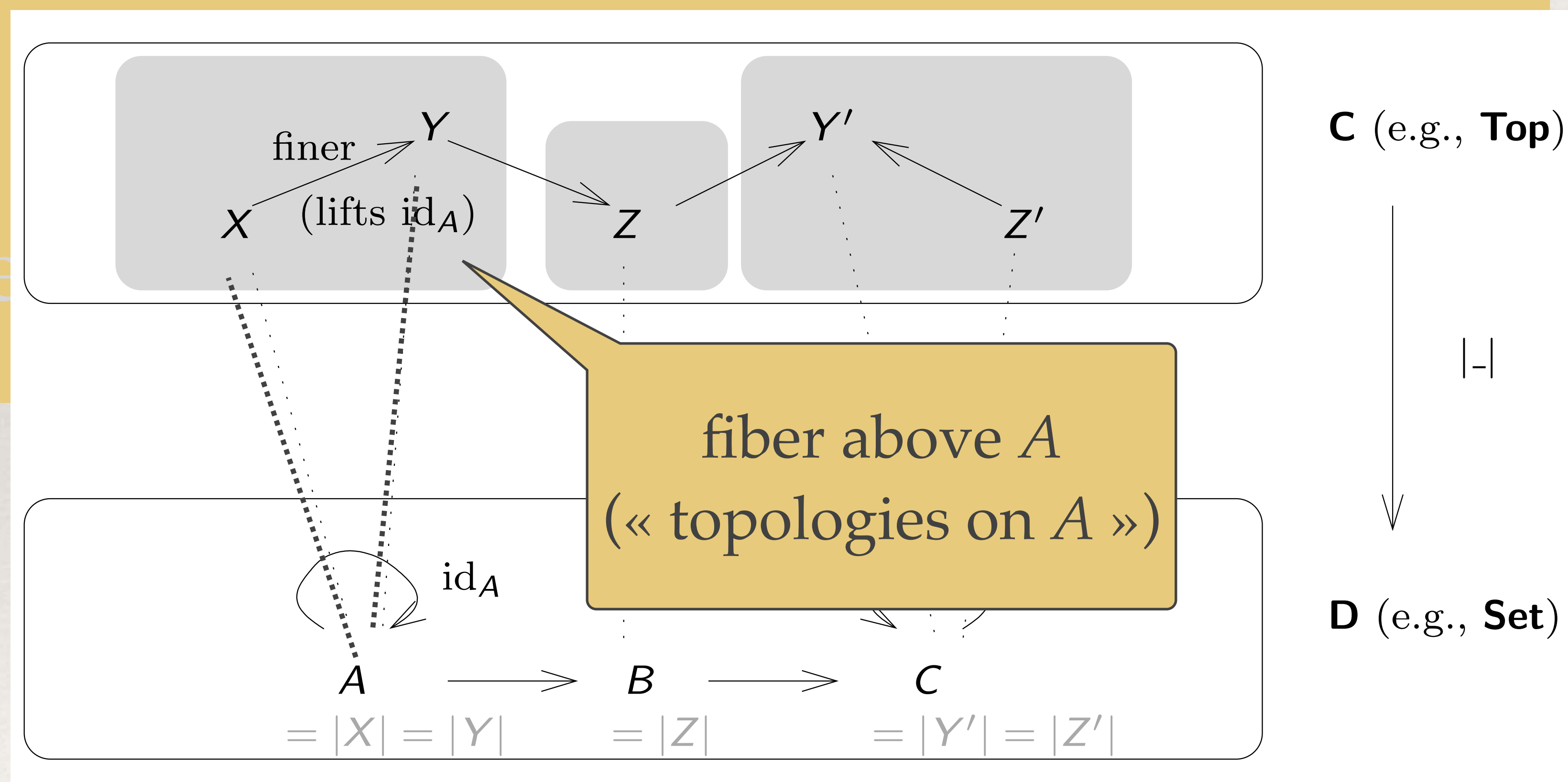

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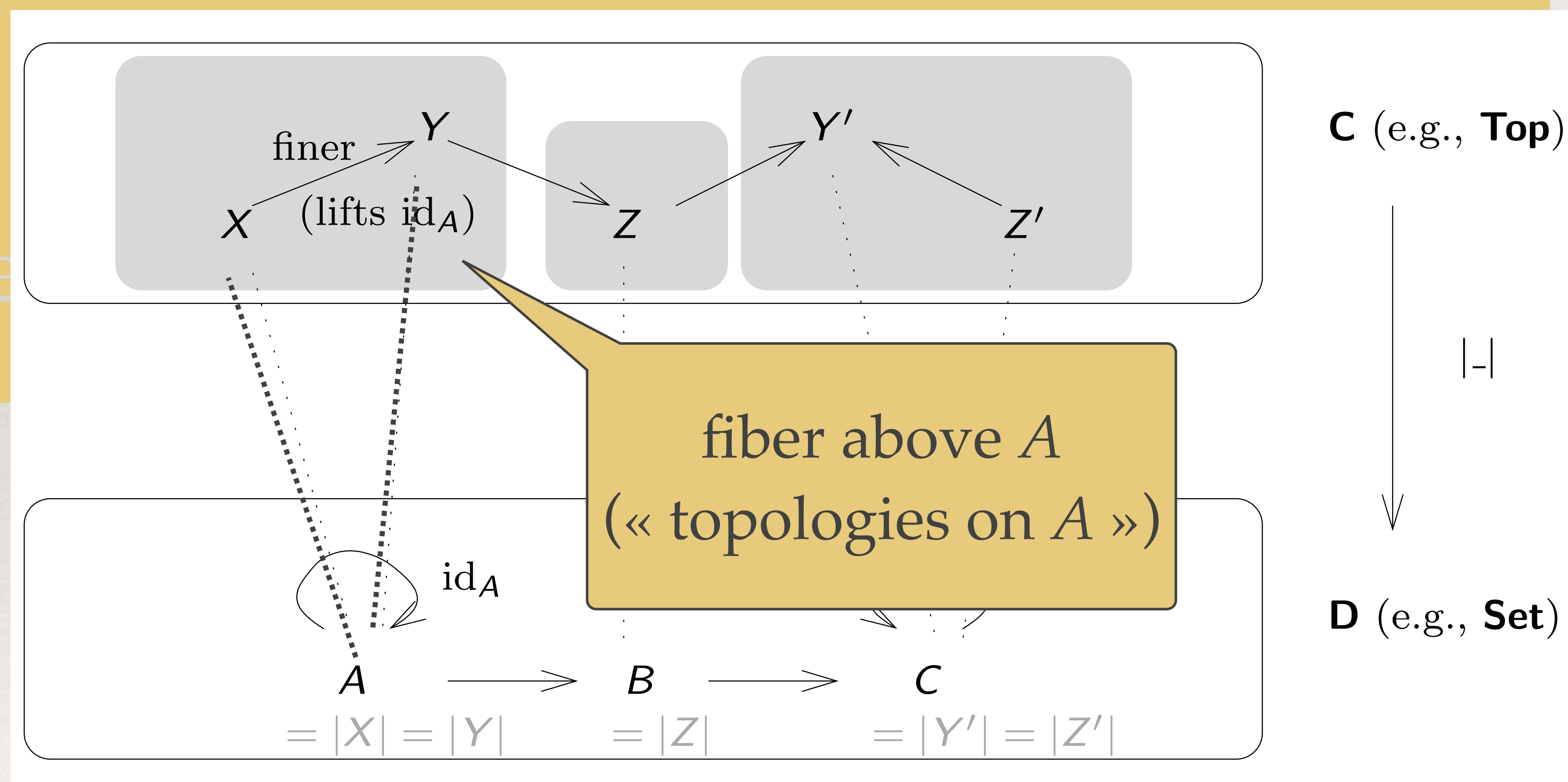

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Let us look at **fibers.**

- In the fiber above A, $X \leq Y$ (« is finer than ») iff id_A has a lift to $X \to Y$
- **Amnesticity** means that $\leq$ is **antisymmetric** (hence a partial ordering)

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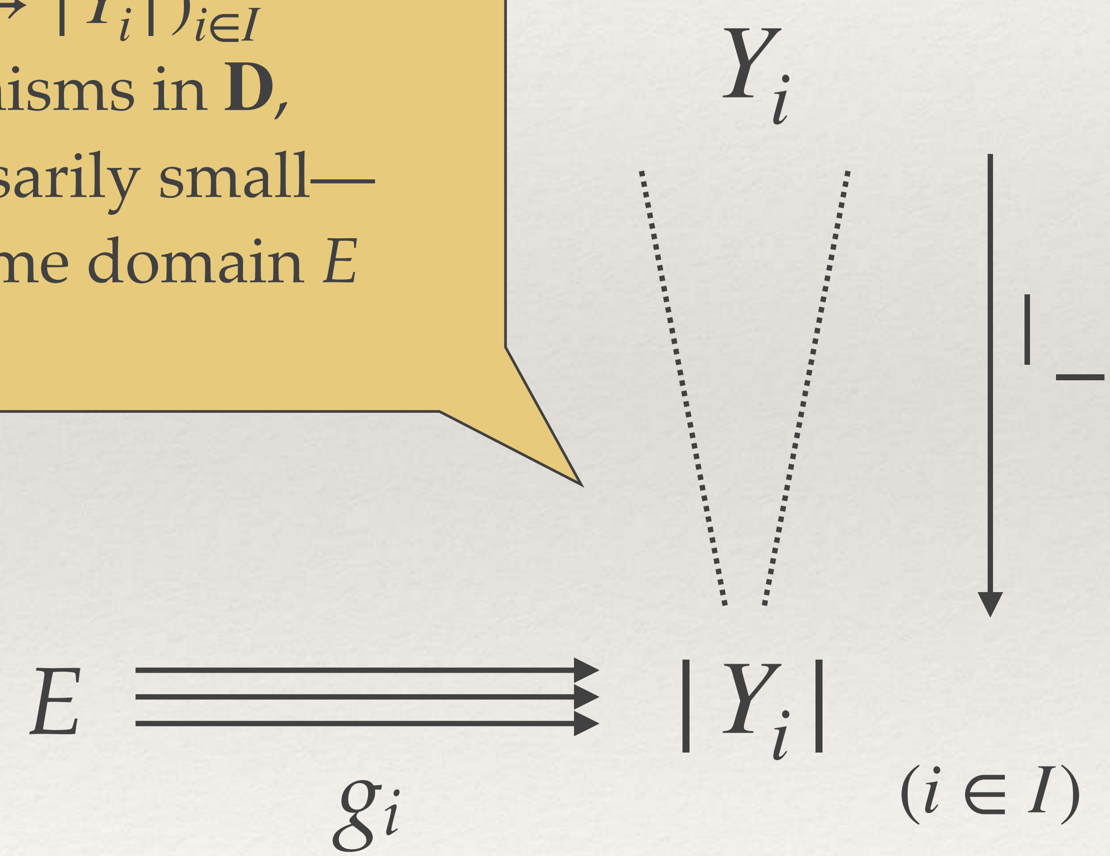
Now this deserves an explanation.
Intuitively: on any set E there is a **coarsest topology** that makes any given collection of maps $g_i : E \to Y_i$ continuous

Every |_|-source …

A |_|-**source** is a collection
$(g_i : E \to |Y_i|)_{i \in I}$
of morphisms in **D**,
—not necessarily small—
with the same domain E



- In the canonical example $|\_| : \mathbf{Top} \to \mathbf{Set}$,
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Note: the objects Y_i (in **C**) are part of the data.

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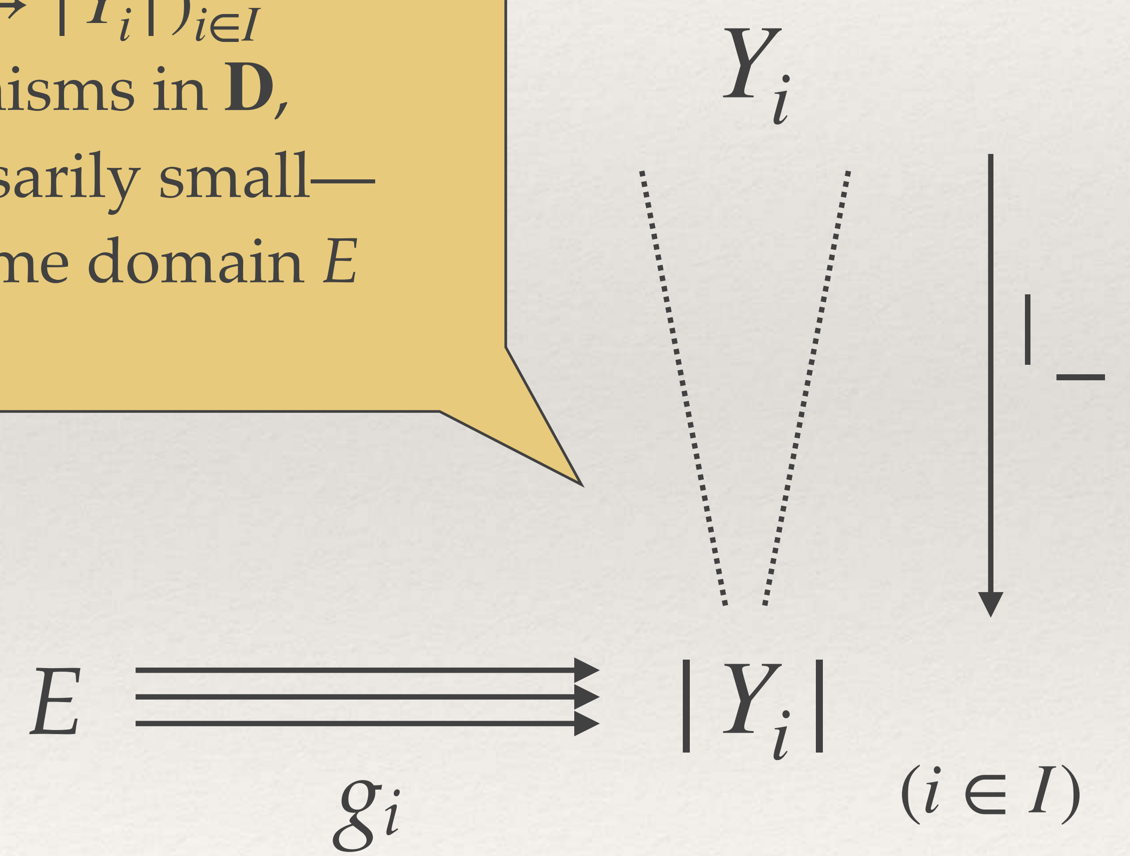
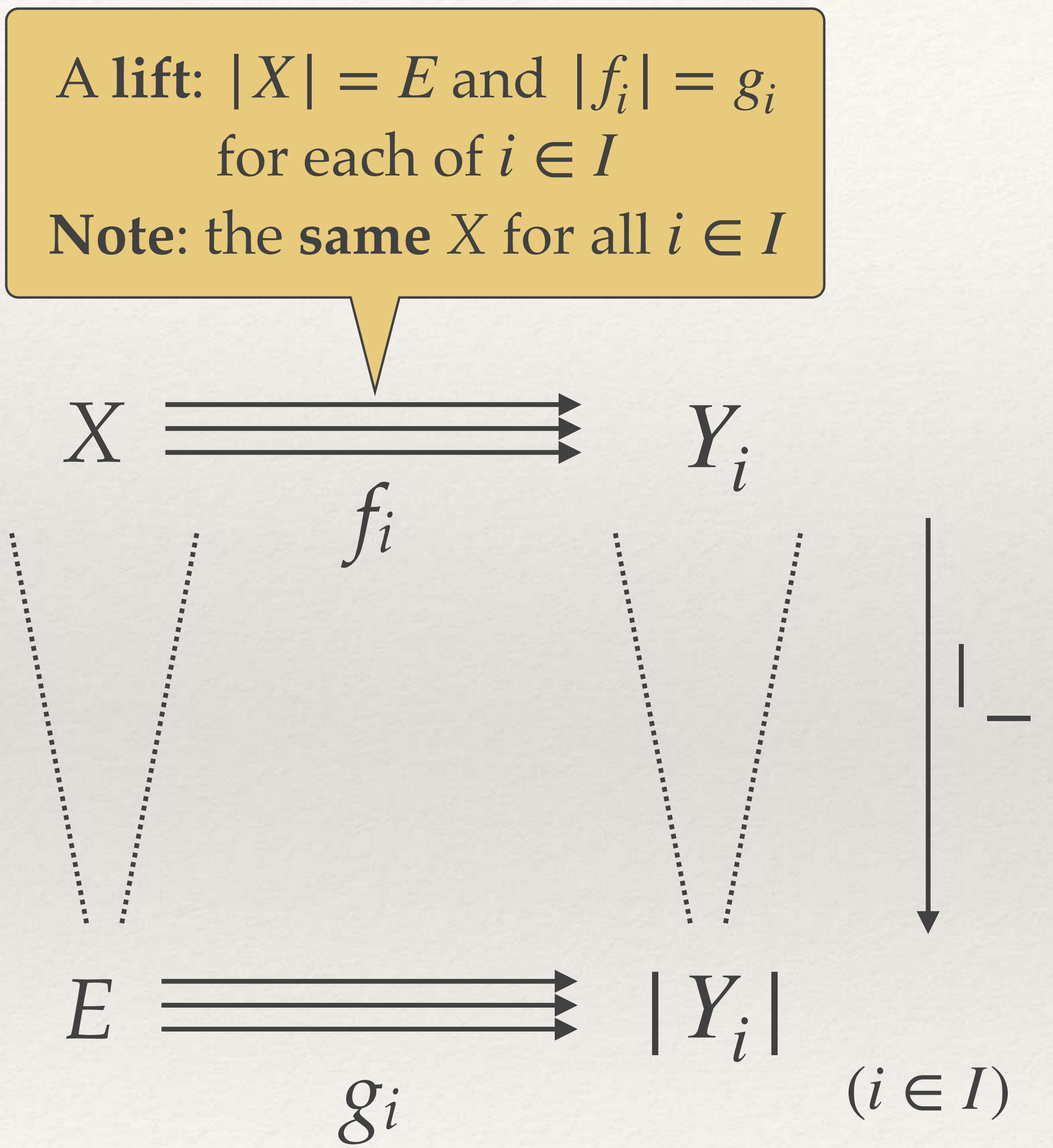


Every |_|-source has a … lift



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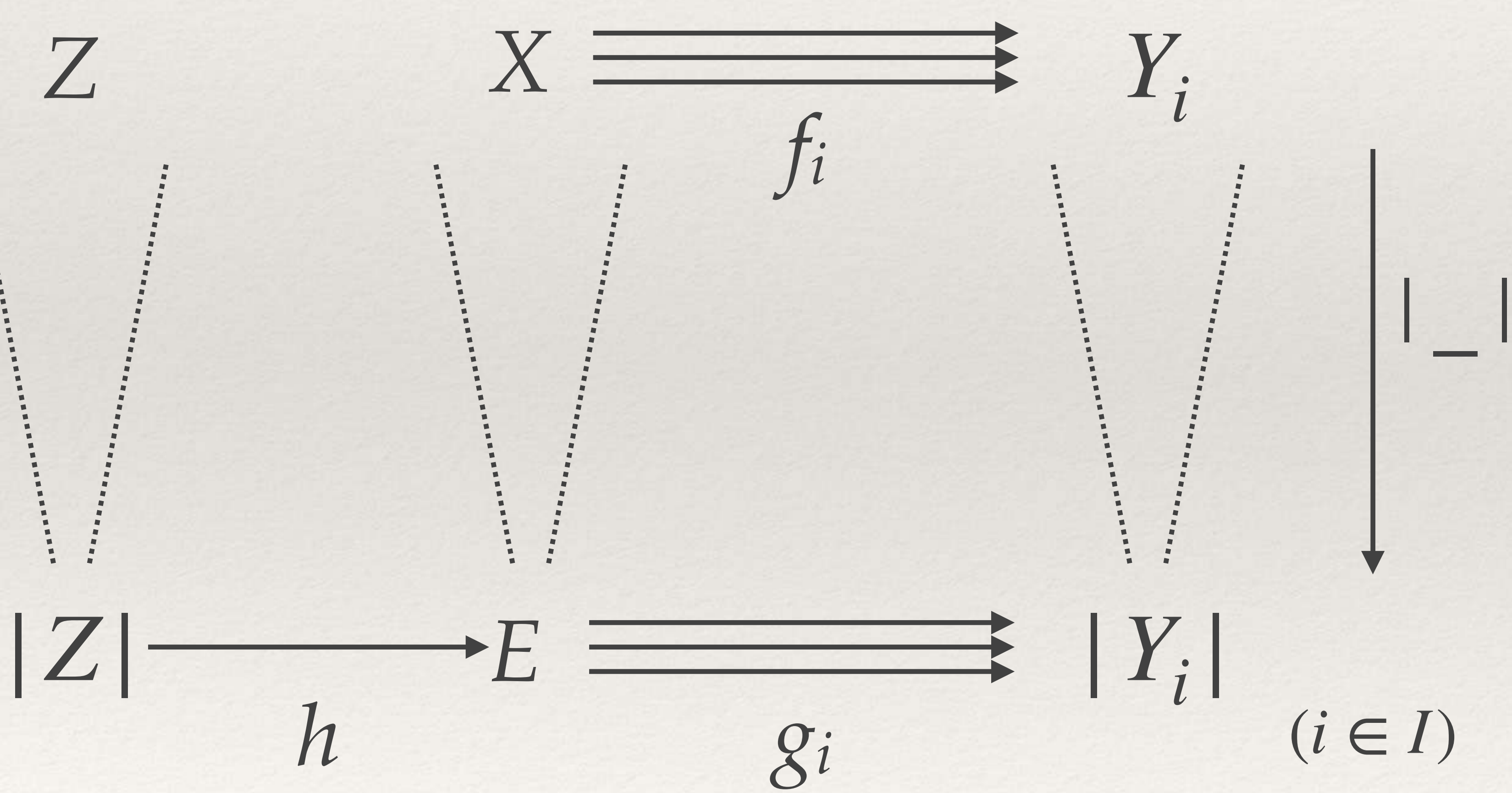
- Each g_i is a (set) function from E to a top. space Y_i
- A **lift** is a topology on E (turning E into a topological space X) that makes each g_i continuous : $X \to Y_i$

Every $|\_|$-source has a $|\_|$-initial lift

$|\_|$ **-initial**: for every $h : |Z| \to E$,
h has a lift : $Z \to X$ iff every $g_i \circ h$ has a lift



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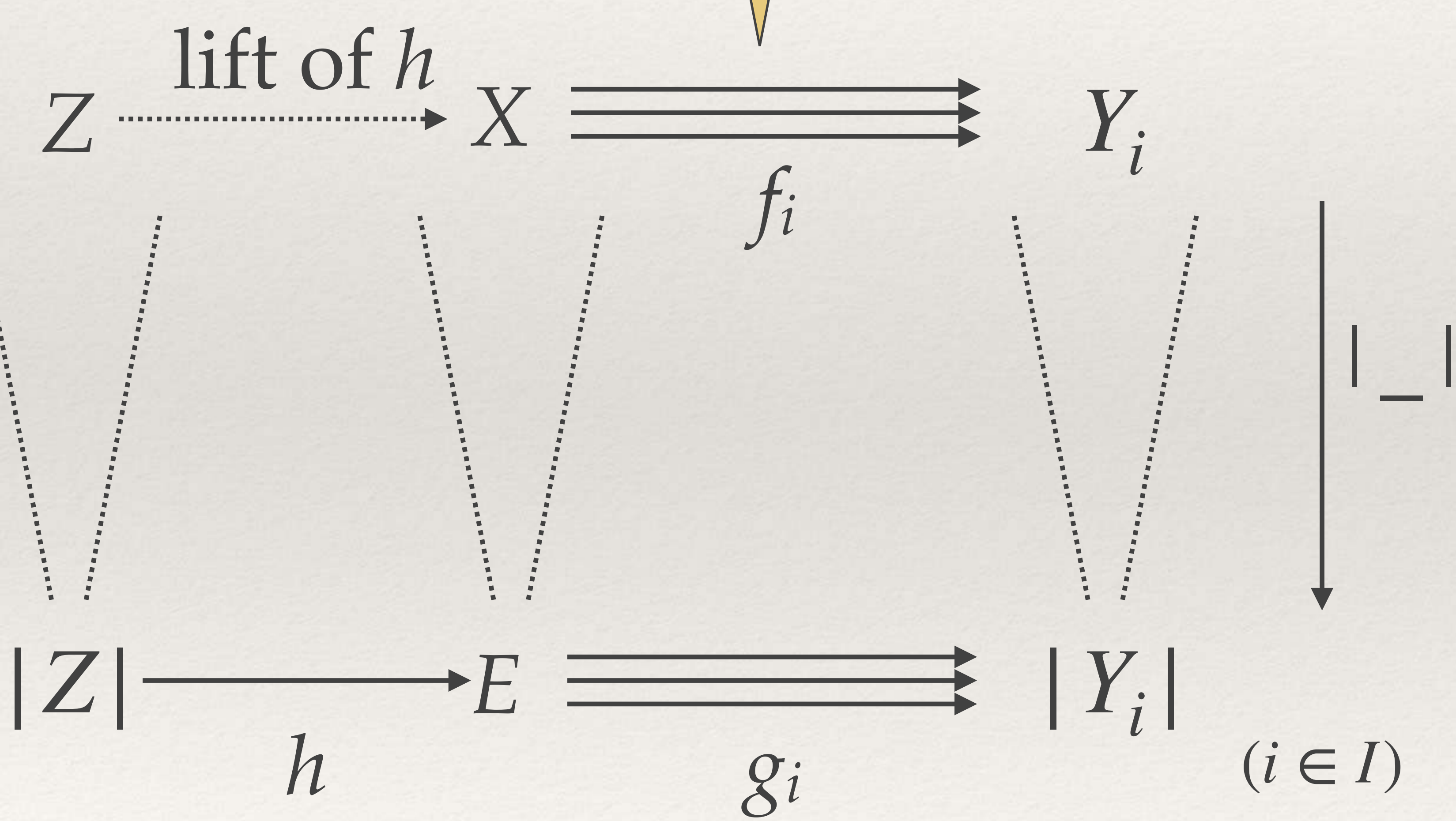

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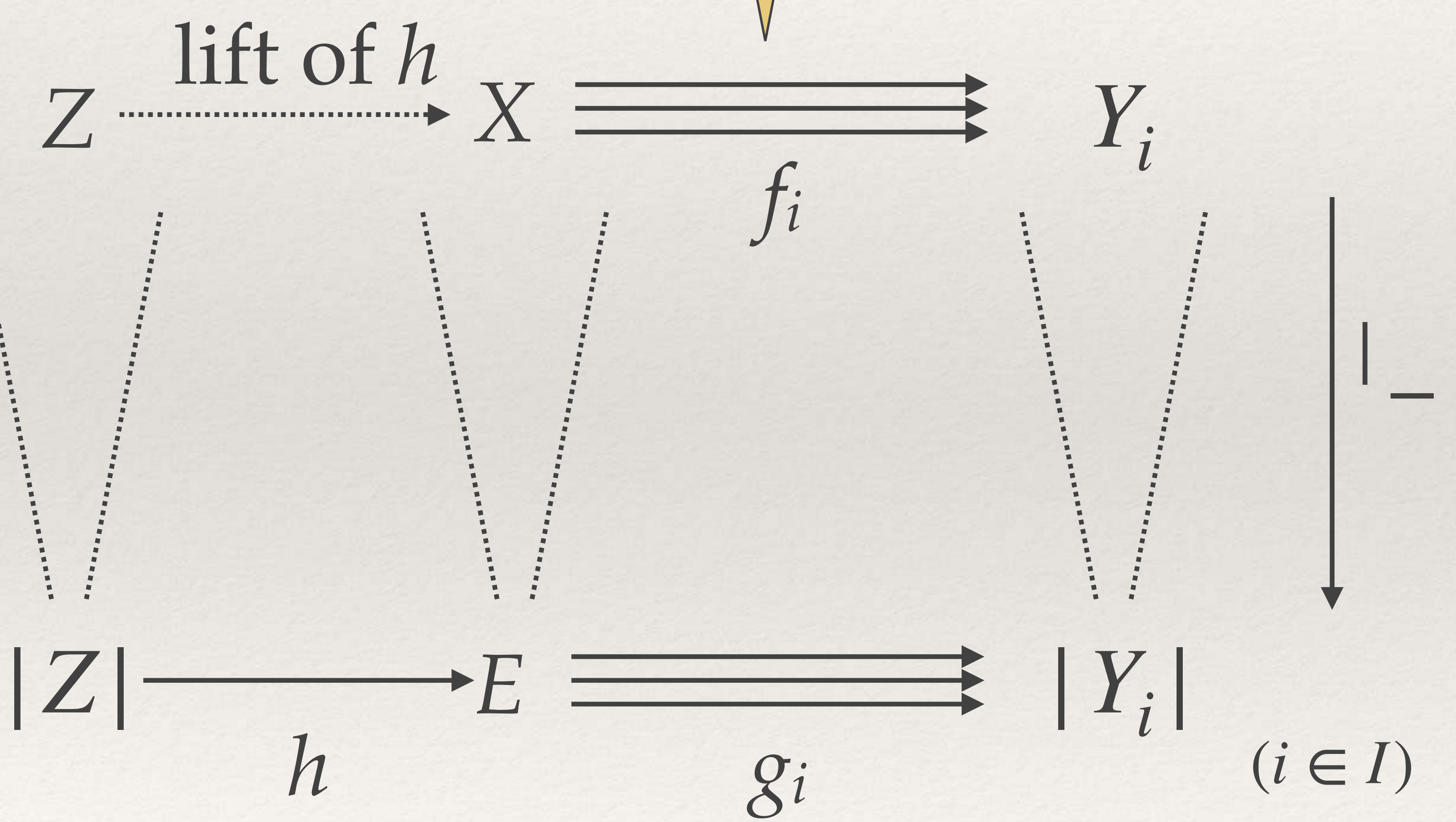

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- and X is **coarsest**: a map $h : Z \to X$ is continuous iff every $g_i \circ h$ is continuous

Properties of topological functors

Fibers are complete lattices

- Given a topological functor $|\_| : \mathbf{C} \to \mathbf{D}$, the fiber of any object E of $\mathbf{D}$ has all **suprema** (with respect to the finer than relation $\leq$)

  … proof: consider $|\_|$-sources consisting of identity morphisms.

- Hence all **infima**

  … the infimum of a collection F is the supremum of its family of lower bounds

- Therefore fibers are **complete lattices** (possibly large)

Self-duality

- A functor $|\_| : \mathbf{C} \to \mathbf{D}$ is topological iff
  $|\_| : \mathbf{C}^{op} \to \mathbf{D}^{op}$ is topological. In other words:
- A functor $|\_| : \mathbf{C} \to \mathbf{D}$ is topological iff:
  (1) faithful
  (2) amnestic
  (3) every $|\_|$-**sink** has a $|\_|$-**final** lift
- Proof: similar to the order-theoretical result that a poset that has all suprema also has all infima
- In the **Top** $\to$ **Set** case: on any set E there is a **finest topology** that makes any given collection of maps $g_i : X_i \to E$ continuous

Discrete, indiscrete objects

- Since the fiber of any object E of $\mathbf{D}$ is a complete lattice, it has:

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- **Corollary.** $|\_|$ preserves all existing limits and colimits

Limits, colimits

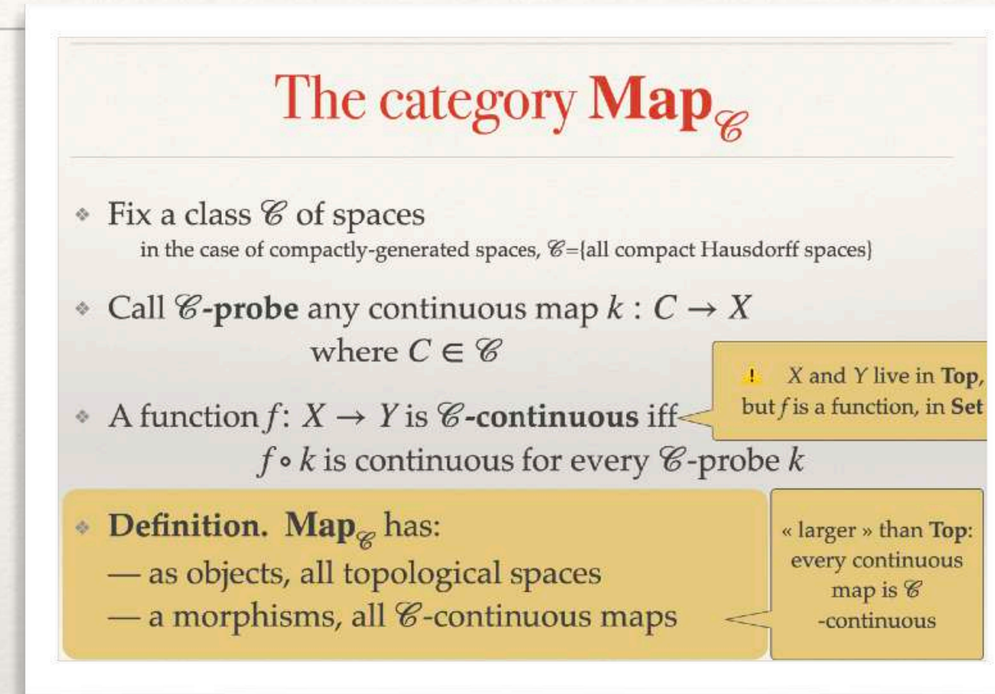
- $|\_|$ preserves all existing limits and colimits, but we also have:
- **Proposition.** $|\_|$ lifts all existing limits and colimits.
- Limits in **C**: take the corresponding limits in **D**, then take the coarsest lift
  Colimits in **C**: take the corresponding colimits in **D**, then take the finest lift

The category $\mathbf{Map}_{\mathscr{C}}$, categorically

Generalizing the Escardó-Lawson-Simpson construction

- We fix a topological functor $|\_| : \mathbf{C} \to \mathbf{D}$
- We will require that $\mathbf{D} = \mathbf{Set}$:
  $|\_|$ is a **topological construct**
- We will also require that $|\_|$
  has **discrete terminal objects**, i.e. $\mathbf{1}_0 = \mathbf{1}_1$
  (« only one topology on a one-element set »)
- **Objective**: generalize [Escardó, Lawson, Simpson 04] and obtain Cartesian-closed full subcategories of $\mathbf{C}$

The (new) category $\mathbf{Map}_{\mathscr{C}}$



- Fix a class $\mathscr{C}$ of objects of $\mathbf{C}$
- Call $\mathscr{C}$-probe any morphism
  $k : C \to X$ where $C \in \mathscr{C}$
- A function $f : |X| \to |Y|$ is
  a $\mathscr{C}$**-map** : $X \to Y$ iff $f \circ |k|$ lifts to $C \to Y$
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- **Definition.** $\mathbf{Map}_{\mathscr{C}}$ has:
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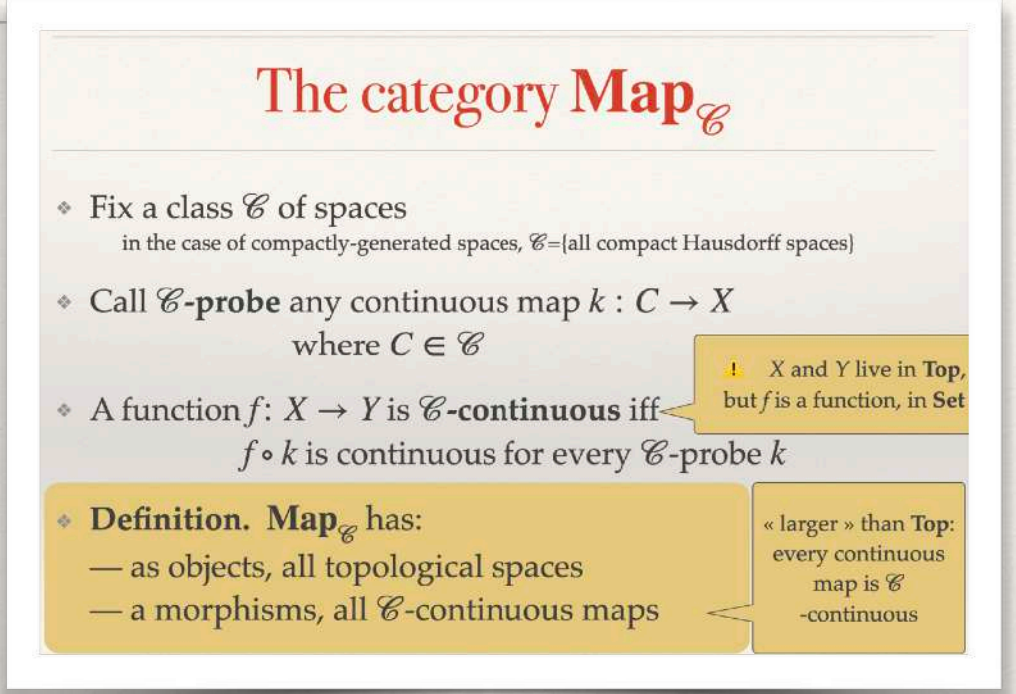

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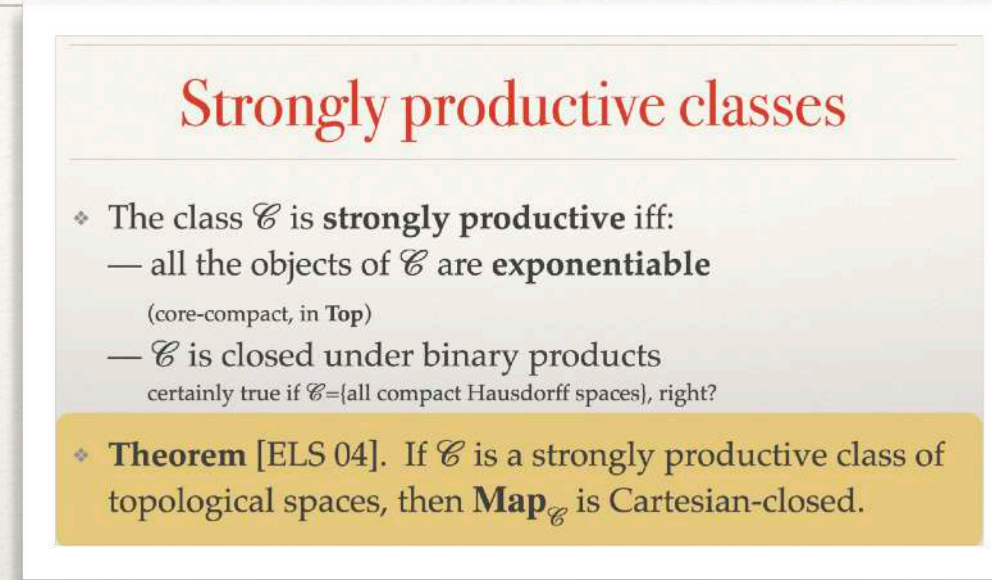
« larger » than $\mathbf{C}$: for every morphism f in $\mathbf{C}$, $|f|$ is a $\mathscr{C}$-map

$\mathbf{Map}_{\mathscr{C}}$ is Cartesian-closed



- The class $\mathscr{C}$ is **strongly productive** iff:
  — all the objects of $\mathscr{C}$ are **exponentiable** in **C**
  — $\mathscr{C}$ is closed under binary products
- **Theorem** [JGL 14]. If | _ | is topological,
  has discrete terminal objects,
  and if $\mathscr{C}$ is strongly productive,
  then $\mathbf{Map}_{\mathscr{C}}$ is Cartesian-closed.
- We give a brief sketch of the construction in the next slide

$\mathbf{Map}_{\mathscr{C}}$ is Cartesian-closed

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- *Proof sketch.* Let $\mathscr{C}(X, Y) \overset{\text{def}}{=} \{\text{all } \mathscr{C}\text{-maps} : X \to Y\}$
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- Hence we define $[Y^X]_{\mathscr{C}}$ as the $|\_|$-initial lift of
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- The rest is a pretty tedious check.
  See https://projects.lsv.ens-cachan.fr/topology/?page_id=6613 ☐

$\mathscr{C}$-generated objects

- For every object X of $\mathbf{C}$,
  let $\mathscr{C}X$ be the finest
  in the fiber of $|X|$
  such that
  for every $\mathscr{C}$-probe $k : C \to X$,
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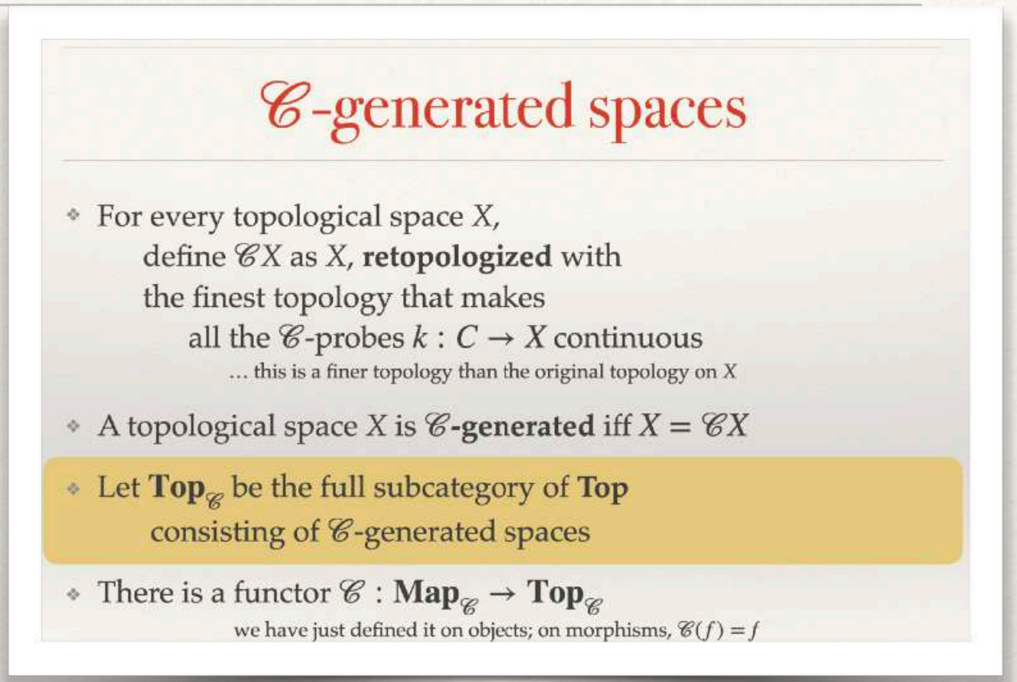

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- Technically, this is obtained
  as a $|\_|$-final lift of
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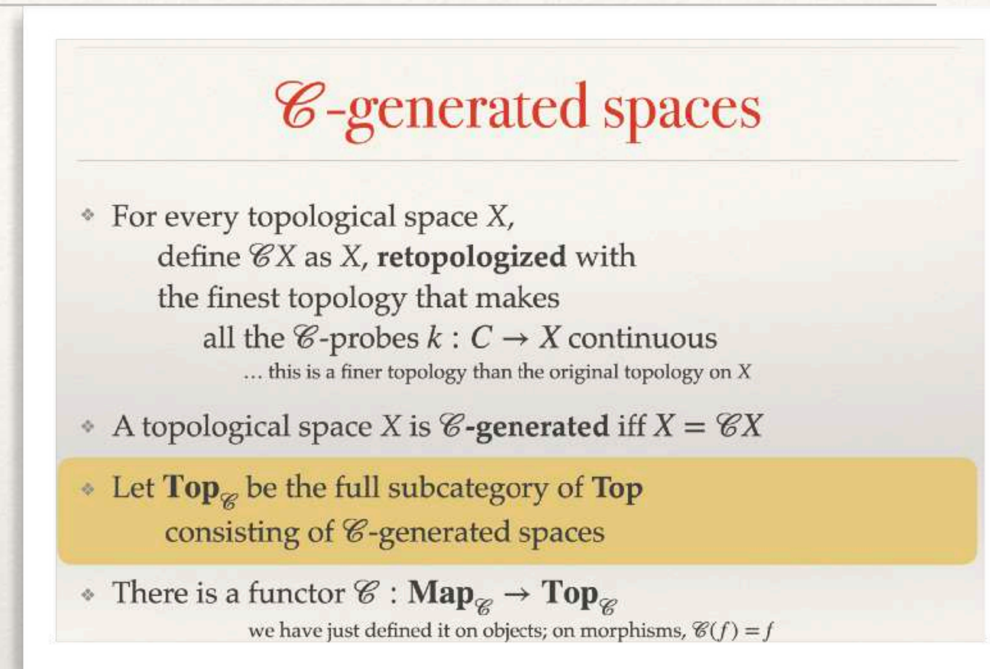
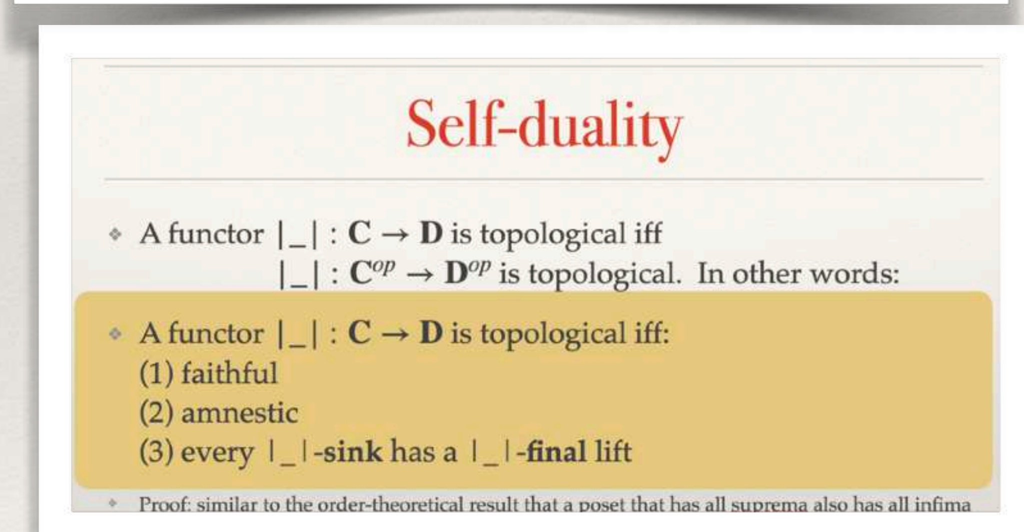




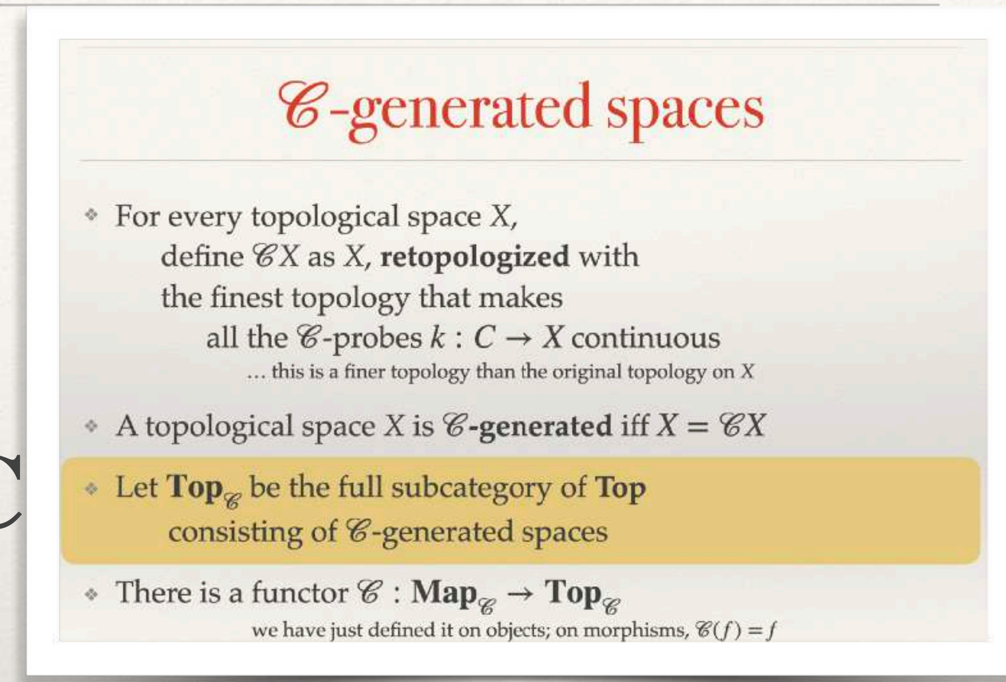
The category $\mathbf{C}_{\mathscr{C}}$

- An object X of $\mathbf{C}$ is $\mathscr{C}$**-generated** iff $X = \mathscr{C}X$

- Let $\mathbf{C}_{\mathscr{C}}$ be the full subcategory of $\mathbf{C}$ consisting of $\mathscr{C}$-generated objects

- There is a functor $\mathscr{C} : \mathbf{Map}_{\mathscr{C}} \to \mathbf{C}_{\mathscr{C}}$

  … on morphisms, $\mathscr{C}(f)$ is the unique lift of $f : |X| \to |Y|$ to $\mathscr{C}X \to \mathscr{C}Y$ (exercise)



$\mathbf{C}_{\mathscr{C}}$ is coreflective in $\mathbf{C}$

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  … and an inclusion functor

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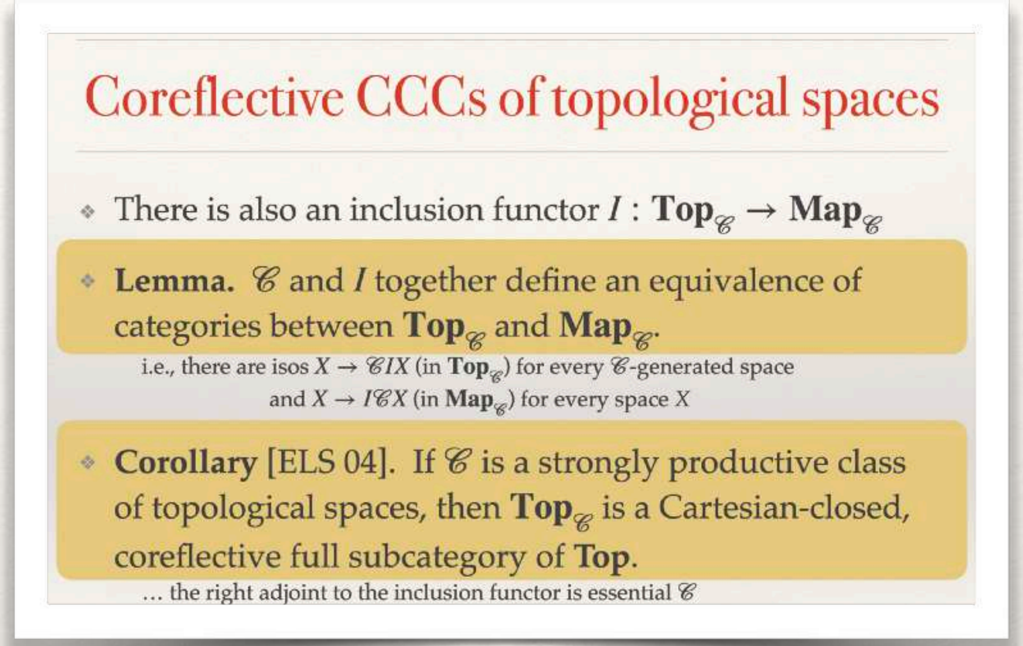

Coreflective CCCs of topological spaces

- There is also an inclusion functor $I : \mathbf{Top}_{\mathscr{C}} \to \mathbf{Map}_{\mathscr{C}}$
- **Lemma.** $\mathscr{C}$ and I together define an equivalence of categories between $\mathbf{Top}_{\mathscr{C}}$ and $\mathbf{Map}_{\mathscr{C}}$.

  i.e., there are isos $X \to \mathscr{C}IX$ (in $\mathbf{Top}_{\mathscr{C}}$) for every $\mathscr{C}$-generated space and $X \to I\mathscr{C}X$ (in $\mathbf{Map}_{\mathscr{C}}$) for every space X
- **Corollary** [ELS 04]. If $\mathscr{C}$ is a strongly productive class of topological spaces, then $\mathbf{Top}_{\mathscr{C}}$ is a Cartesian-closed, coreflective full subcategory of $\mathbf{Top}$.

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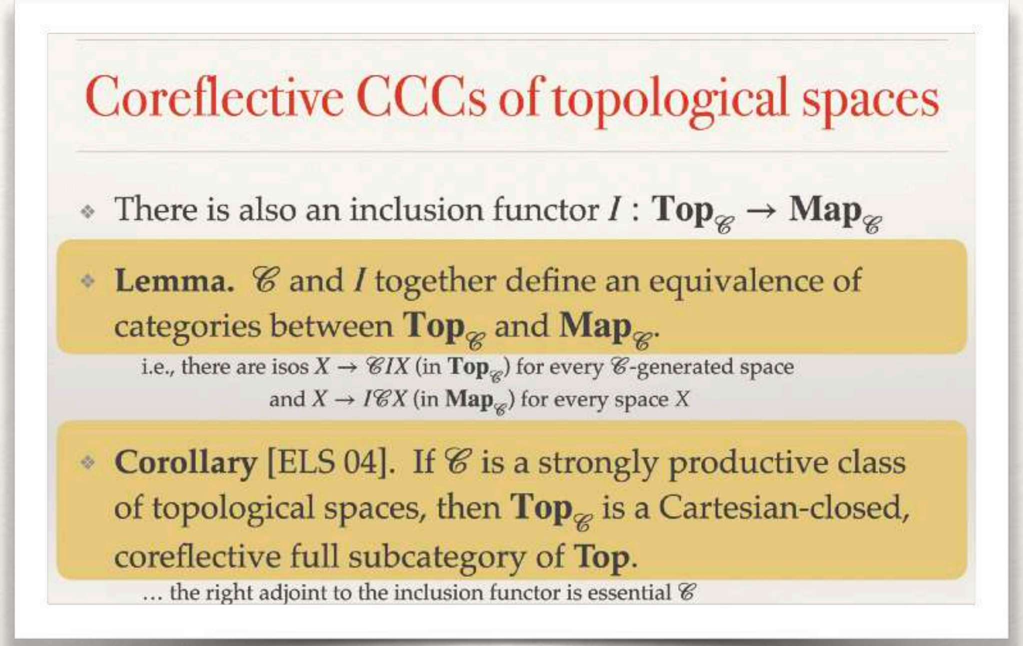

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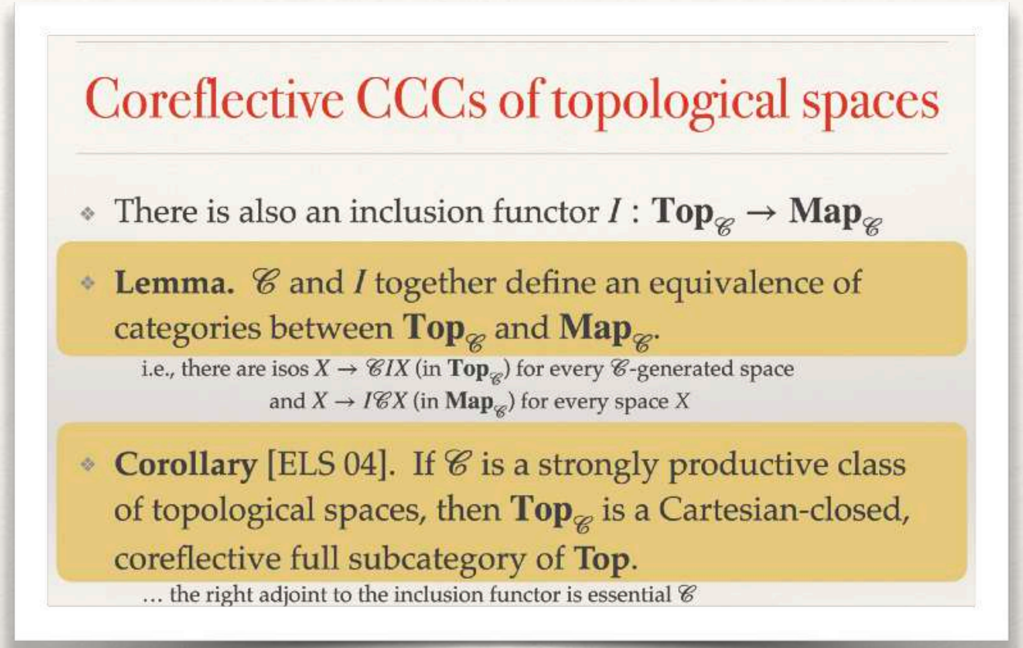

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- **Lemma.** $\mathscr{C}$ and I together define an equivalence of categories between $\mathbf{C}_{\mathscr{C}}$ and $\mathbf{Map}_{\mathscr{C}}$.

- **Corollary** [JGL 14]. If $|\_|$ is topological, has discrete terminal objects, and if $\mathscr{C}$ is strongly productive, then $\mathbf{C}_{\mathscr{C}}$ is a Cartesian-closed, coreflective full subcategory of $\mathbf{C}$.

  … the right adjoint to the inclusion functor is defined just like $\mathscr{C}$

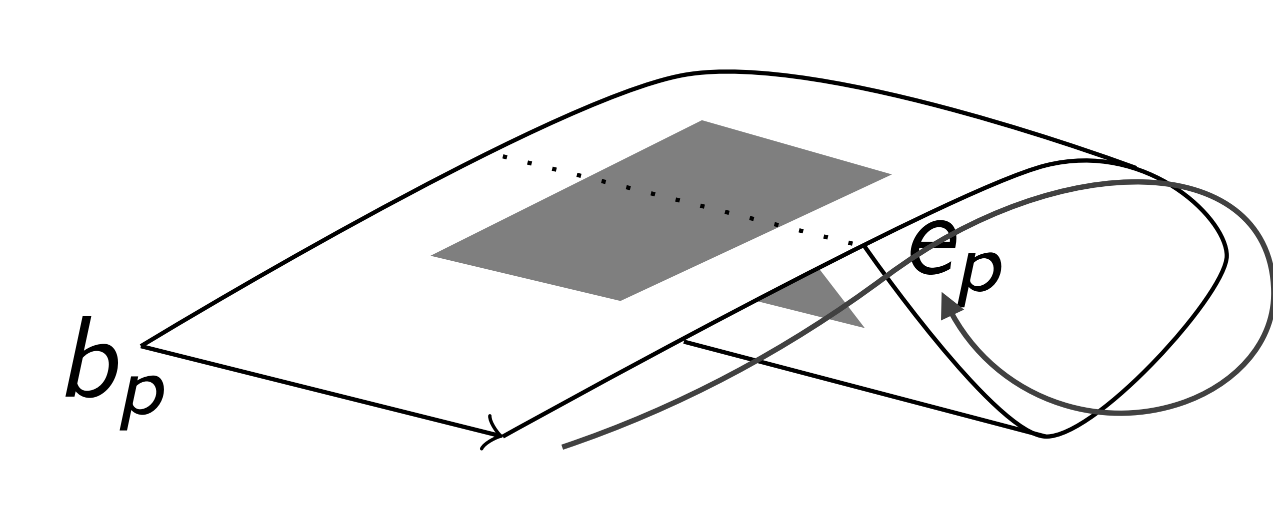
$\mathscr{C}$-generation by colimits

- Also, we have:
- **Proposition** [JGL 14]. Under the previous assumptions, $\mathbf{C}_{\mathscr{C}}$ is complete, cocomplete,
  and the objects of $\mathbf{C}_{\mathscr{C}}$ (the $\mathscr{C}$-generated objects of $\mathbf{C}$) are exactly the colimits of objects of $\mathscr{C}$, taken in $\mathbf{C}$.
- Generalizes a similar result in [Escardó, Lawson, Simpson 04].

An application to streams

Directed spaces

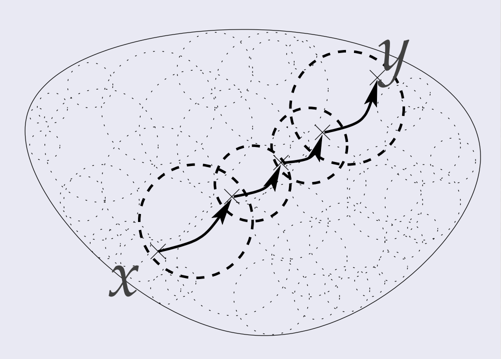
- How do we model spaces with **local directions of time** (which may be cyclic, globally) as used in **directed** algebraic topology?
- Now two standard proposals: **d-spaces** [Grandis 09], **streams** [Krishnan 08], related by an adjunction [Haucourt 09]



Krishnan's streams

- A **stream** [Krishnan 08] is a **cosheaf** of preorders on a topological space X, namely:
  — for each open set U, a preordering $\sqsubseteq_U$ on U
  — if $U \subseteq V$ then $x \sqsubseteq_U y$ implies $x \sqsubseteq_V y$
  — if $U = \bigcup_{i \in I} U_i$ and $x \sqsubseteq_U y$ then there is a chain

$$x = x_0 \sqsubseteq_{U_{i_1}} x_1 \sqsubseteq_{U_{i_2}} \cdots \sqsubseteq_{U_{i_n}} x_n = y$$



- A **stream morphism** is a continuous map that is **locally monotonic**: $x \sqsubseteq_{f^{-1}(V)} y \implies f(x) \sqsubseteq_V f(y)$

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- Hence our construction applies…

Exponentiable streams

- **Theorem** [JGL 14]. The exponentiable streams are exactly those whose underlying topological space is **core-compact**.
- We recall that the core-compact spaces are those that are exponentiable in **Top**.
- Hence…

Cartesian-closed categories of streams

- **Theorem** [JGL 14]. Let $\mathscr{C}$ be any class of streams that is strongly productive (= closed under binary products, consisting of core-compact streams).
Then the full subcategory $\mathbf{Str}_{\mathscr{C}}$ of $\mathscr{C}$-generated streams (=colimits of streams in $\mathscr{C}$) is Cartesian-closed.
- E.g., if $\mathscr{C}$={compact Hausdorff streams},
(and further restricting to Hausdorff $\mathscr{C}$-generated streams),
we get Krishnan's CCC of **compactly-flowing streams**

Conclusion

Conclusion, and open questions

- **Topological functors**: effectively generalize topology… enough to lift (apparently) purely topological arguments such as [Lawson, Escardó, Simpson 04]
- **Q1.** I only obtained that $\mathbf{Map}_{\mathscr{C}}$ is Cartesian-closed for $|\_| : \mathbf{C} \to \mathbf{D}$ where $\mathbf{D} = \mathbf{Set}$: what about **other** Cartesian-closed categories $\mathbf{D}$?
  (Seems like laziness on my part.)
- **Q2.** Is the restriction to having **discrete terminal objects** necessary?
- **Q3.** Other **applications**, beyond streams?
  (The case of d-spaces should be easy, but look outside directed spaces, too.)

Main references

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- Martín Escardó, Jimmie Lawson, and Alex Simpson. *Comparing Cartesian Closed Categories of (Core) Compactly Generated Spaces*. Topology and Its Applications, 143(1–3):105–146, 2004.