

Jean Goubault-Larrecq

Topological functors and Cartesian-closed categories



Outline

- ❖ **Top** is not Cartesian-closed
- ❖ ... but some full subcategories are,
e.g. compactly generated spaces [Brown 61, 63; Steenrod 67; Kelley 55]
- ❖ A general construction
due to [Escardó, Lawson, Simpson 04]
- ❖ ... generalizes to **topological functors** [JGL 14]
- ❖ Applications: streams (very briefly).

Cartesian-closed categories

Terminal objects

- ❖ A **terminal object** in a category **C** is an object **1** such that for every object X , there is a unique morphism

$$X \xrightarrow{!} \mathbf{1}$$

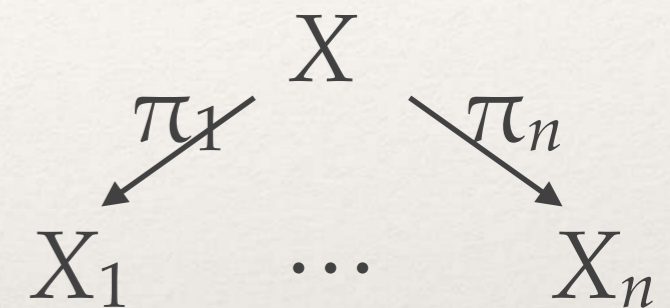
- ❖ In **Set**, **Ord**, **Top**: any one-element set / poset / space.

Finite products

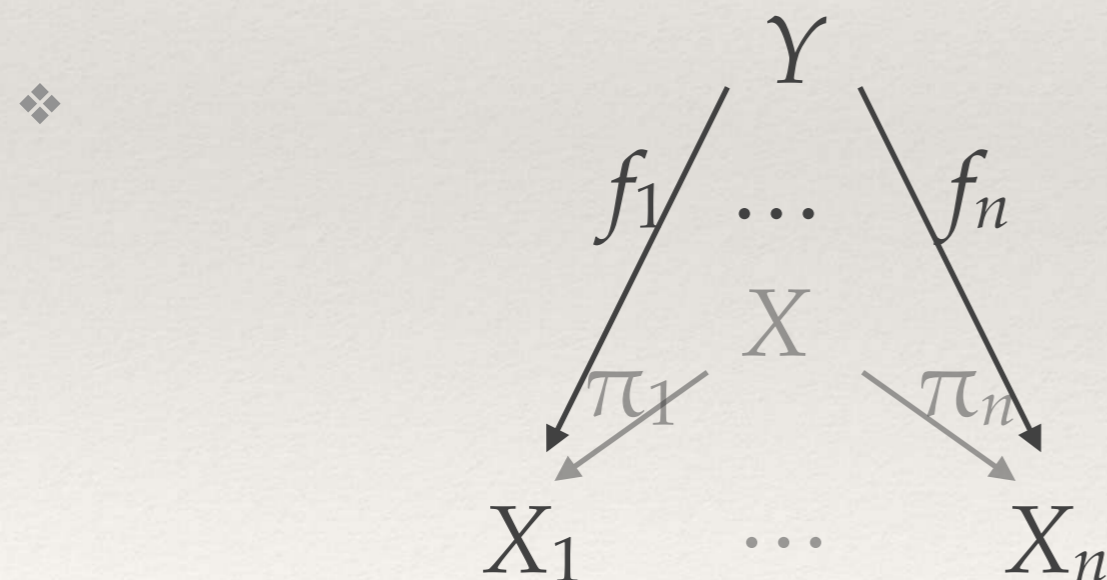
- ❖ A **product** of objects X_1, \dots, X_n is any object

$$X \stackrel{\text{def}}{=} X_1 \times \dots \times X_n$$

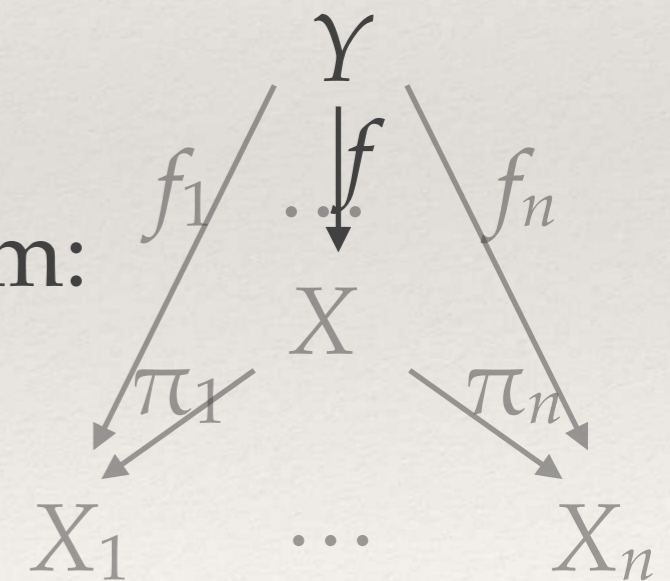
with **projections** $\pi_i : X \rightarrow X_i$



- ❖ such that in every situation



there is a unique f making the diagram:
commutative



Finite products

- ❖ Finite product in **Set (Ord, Top)**
= Cartesian product (with product ordering, product topology)
- ❖ Products are special cases of **limits**
- ❖ Product of $n=0$ object = **terminal object**
- ❖ Equivalent definition: $f = \langle f_1, \dots, f_n \rangle$ **axiomatized by:**

$$\pi_i \circ \langle f_1, \dots, f_n \rangle = f_i$$

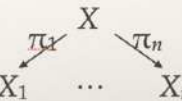
$$\langle f_1, \dots, f_n \rangle \circ g = \langle f_1 \circ g, \dots, f_n \circ g \rangle$$

$$\langle \pi_1, \dots, \pi_n \rangle = \text{id}$$

❖ A **product** of objects X_1, \dots, X_n is any object

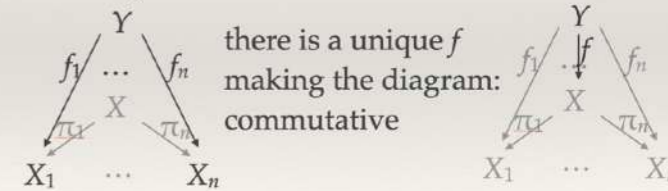
$$X \cong X_1 \times \dots \times X_n$$

with **projections** $\pi_i : X \rightarrow X_i$



❖ such that in every situation

❖ there is a unique f making the diagram: commutative



Cartesian-closed categories

- ❖ A category \mathbf{C} with finite products and with **internal homs**, a.k.a. **exponentials**
(« function space objects »)
- ❖ E.g., **Set** will be Cartesian-closed, because there is an **object** « set of all maps from X to Y » with the expected properties (next slides)
- ❖ Applications:
 - (1) **convenient** categories for algebraic topology
 - (2) models of the **lambda-calculus** in logic and computer science

Exponentials

- ❖ An **exponential** Y^X is an object with:
- ❖ an **application** morphism $\text{App} : Y^X \times X \rightarrow Y$
- ❖ for every morphism $f : Z \times X \rightarrow Y$,
a **currification** morphism $\Lambda(f) : Z \rightarrow Y^X$
- ❖ such that:
 - $(\beta) \text{App} \circ (\Lambda(f) \times \text{id}) = f$ (« $\Lambda(f)(z)$ applied to x is $f(z,x)$ »)
 - $(\eta) \Lambda(\text{App}) = \text{id}$
 - $(\sigma) \Lambda(f) \circ g = \Lambda(f \circ (g \times \text{id}))$

Exponentiable objects

❖ An object X is **exponentiable** iff the exponential Y^X exists for every object Y

❖ In **Set**, every object is exponentiable
 $Y^X = \{\text{all maps from } X \text{ to } Y\}$

$$\text{App} : (f, x) \mapsto f(x) \quad \Lambda(f) : z \mapsto (x \mapsto f(z, x))$$

❖ In **Ord**, every object is exponentiable, too

$Y^X = \{\text{all monotonic maps from } X \text{ to } Y\}$, pointwise ordering

❖ In **Top**, the exponentiable objects are the **core-compact** spaces [Day, Kelly 70]

every locally compact space is core-compact,

every core-compact and sober (e.g., Hausdorff) space is locally compact

❖ An **exponential** Y^X is an object with:

❖ an **application** morphism $\text{App} : Y^X \times X \rightarrow Y$

❖ for every morphism $f : Z \times X \rightarrow Y$, a **currification** morphism $\Lambda(f) : Z \rightarrow Y^X$

❖ such that:

(β) $\text{App} \circ (\Lambda(f) \times \text{id}) = f$ ($\ll \Lambda(f)(z)$ applied to x is $f(z, x)$ \gg)

(η) $\Lambda(\text{App}) = \text{id}$

(σ) $\Lambda(f) \circ g = \Lambda(f \circ (g \times \text{id}))$

Cartesian-closed categories

❖ A category with finite products is **Cartesian-closed** iff every object is exponentiable

❖ **Set** is Cartesian-closed

❖ **Ord** is Cartesian-closed

❖ **Top** is not: any Hausdorff, non-locally compact space fails to be exponentiable (e.g., \mathbb{Q} , or the Sorgenfrey line)

- ❖ An **exponential** Y^X is an object with:
- ❖ an **application** morphism $\text{App} : Y^X \times X \rightarrow Y$
- ❖ for every morphism $f : Z \times X \rightarrow Y$, a **currification** morphism $\Lambda(f) : Z \rightarrow Y^X$
- ❖ such that:
 - (β) $\text{App} \circ (\Lambda(f) \times \text{id}) = f$ ($\ll \Lambda(f)(z)$ applied to x is $f(z,x) \gg$)
 - (η) $\Lambda(\text{App}) = \text{id}$
 - (σ) $\Lambda(f) \circ g = \Lambda(f \circ (g \times \text{id}))$

- ❖ In **Top**, the exponentiable objects are the **core-compact** spaces [Day, Kelly 70]
 - every locally compact space is core-compact,
 - every core-compact and sober (e.g., Hausdorff) space is locally compact

Cartesian-closed categories in topology?

- ❖ Several solutions to the conundrum:
- ❖ Find **larger** categories than **Top** that would be Cartesian-closed
e.g., **Conv** (see F. Mynard's talk)
- ❖ Find **smaller** categories than **Top** that would be Cartesian-closed
e.g., compactly-generated spaces
or more generally \mathcal{C} -generated spaces (this talk)
or dcpos (directed-complete partial orders, with Scott topology)

The Escardó-Lawson-Simpson construction

The Escardó-Lawson-Simpson construction: outline

- ❖ Fix a suitable class \mathcal{C} of topological spaces
- ❖ Build a Cartesian-closed category $\mathbf{Map}_{\mathcal{C}}$
... intuitively larger than \mathbf{Top}
- ❖ Show that $\mathbf{Map}_{\mathcal{C}}$ is equivalent to
a full subcategory $\mathbf{Top}_{\mathcal{C}}$ of \mathbf{Top}
- ❖ so $\mathbf{Top}_{\mathcal{C}}$ will be Cartesian-closed, too.

The category $\text{Map}_{\mathcal{C}}$

- ❖ Fix a class \mathcal{C} of spaces
in the case of compactly-generated spaces, $\mathcal{C} = \{\text{all compact Hausdorff spaces}\}$
- ❖ Call \mathcal{C} -probe any continuous map $k : C \rightarrow X$
where $C \in \mathcal{C}$
- ❖ A function $f : X \rightarrow Y$ is \mathcal{C} -continuous iff
 $f \circ k$ is continuous for every \mathcal{C} -probe k

! X and Y live in **Top**,
but f is a function, in **Set**

- ❖ **Definition.** $\text{Map}_{\mathcal{C}}$ has:
 - as objects, all topological spaces
 - as morphisms, all \mathcal{C} -continuous maps

« larger » than **Top**:
every continuous
map is \mathcal{C} -
continuous

Strongly productive classes

- ❖ The class \mathcal{C} is **strongly productive** iff:
 - all the objects of \mathcal{C} are **exponentiable**
(core-compact, in **Top**)
 - \mathcal{C} is closed under binary products
certainly true if $\mathcal{C} = \{\text{all compact Hausdorff spaces}\}$, right?

Note that this makes sense in any category with finite products

- ❖ **Theorem [ELS 04].** If \mathcal{C} is a strongly productive class of topological spaces, then $\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed.

\mathcal{C} -generated spaces

- ❖ For every topological space X ,
define $\mathcal{C}X$ as X , **retopologized** with
the finest topology that makes
all the \mathcal{C} -probes $k : C \rightarrow X$ continuous
... this is a finer topology than the original topology on X
- ❖ A topological space X is **\mathcal{C} -generated** iff $X = \mathcal{C}X$
- ❖ Let **$\mathbf{Top}_{\mathcal{C}}$** be the full subcategory of **\mathbf{Top}**
consisting of \mathcal{C} -generated spaces
- ❖ There is a functor **$\mathcal{C} : \mathbf{Map}_{\mathcal{C}} \rightarrow \mathbf{Top}_{\mathcal{C}}$**
we have just defined it on objects; on morphisms, $\mathcal{C}(f) = f$

Coreflective CCCs of topological spaces

❖ There is also an inclusion functor $I : \mathbf{Top}_{\mathcal{C}} \rightarrow \mathbf{Map}_{\mathcal{C}}$

❖ **Lemma.** \mathcal{C} and I together define an equivalence of categories between $\mathbf{Top}_{\mathcal{C}}$ and $\mathbf{Map}_{\mathcal{C}}$.

i.e., there are isos $X \rightarrow \mathcal{C}IX$ (in $\mathbf{Top}_{\mathcal{C}}$) for every \mathcal{C} -generated space
and $X \rightarrow I\mathcal{C}X$ (in $\mathbf{Map}_{\mathcal{C}}$) for every space X

❖ **Corollary** [ELS 04]. If \mathcal{C} is a strongly productive class of topological spaces, then $\mathbf{Top}_{\mathcal{C}}$ is a Cartesian-closed, coreflective full subcategory of \mathbf{Top} .

... the right adjoint to the inclusion functor is defined just like \mathcal{C}

« strongly » not needed
— allow me to skip this.

Examples

- ❖ If $\mathcal{C} = \{\text{all compact Hausdorff spaces}\}$,
 $\mathbf{Top}_{\mathcal{C}}$ is very close to compactly-generated spaces
... we reobtain the usual compactly-generated spaces
as the full subcategory of $\mathbf{Top}_{\mathcal{C}}$ consisting of (weak) Hausdorff spaces
- ❖ If $\mathcal{C} = \{\text{all core-compact spaces}\}$,
 $\mathbf{Top}_{\mathcal{C}}$ is the largest CCC obtainable by this construction
... = full subcategory of \mathbf{Top} consisting of quotients of core-compact spaces
(in general, $\mathbf{Top}_{\mathcal{C}}$ is the full subcategory of \mathbf{Top}
spanned by colimits in \mathbf{Top} of diagrams in \mathcal{C})
- ❖ If $\mathcal{C} = \{\text{one-point compactification of } \mathbb{N}\}$,
 $\mathbf{Top}_{\mathcal{C}} = \{\text{sequential spaces}\}$

Topological functors

Topological functors

- ❖ See *The Joy of Cats* [Adámek, Herrlich, Strecker 90]
- ❖ **Topological functors** attempt to capture topology abstractly by properties of the **forgetful functor**
Top \rightarrow **Set**
... we will use this as a running example
- ❖ What we will see:
The Escardó-Lawson-Simpson construction
generalizes to a large class of topological functors

Topological functors

❖ See *The Joy of Cats* [Adámek, Herrlich, Strecker 90]

❖ A **topological functor** $|_|\ : \mathbf{C} \rightarrow \mathbf{D}$ is a functor that is:

(1) faithful

(2) amnestic

(3) every $|_|\$ -source has a $|_|\$ -initial lift

i.e., injective on hom sets

In the example of

$|_|\ : \mathbf{Top} \rightarrow \mathbf{Set}$,

check that if $|f| = |g|$

then $f = g$

❖ We say that $f : X \rightarrow Y$ (in \mathbf{C})

lifts $g : |X| \rightarrow |Y|$

iff $|f| = g$

❖ Lifts are **unique** when they exist

❖ For $|_|\ : \mathbf{Top} \rightarrow \mathbf{Set}$,

g **has a lift** iff it is continuous

Topological functors

❖ See *The Joy of Cats* [Adámek, Herrlich, Strecker 90]

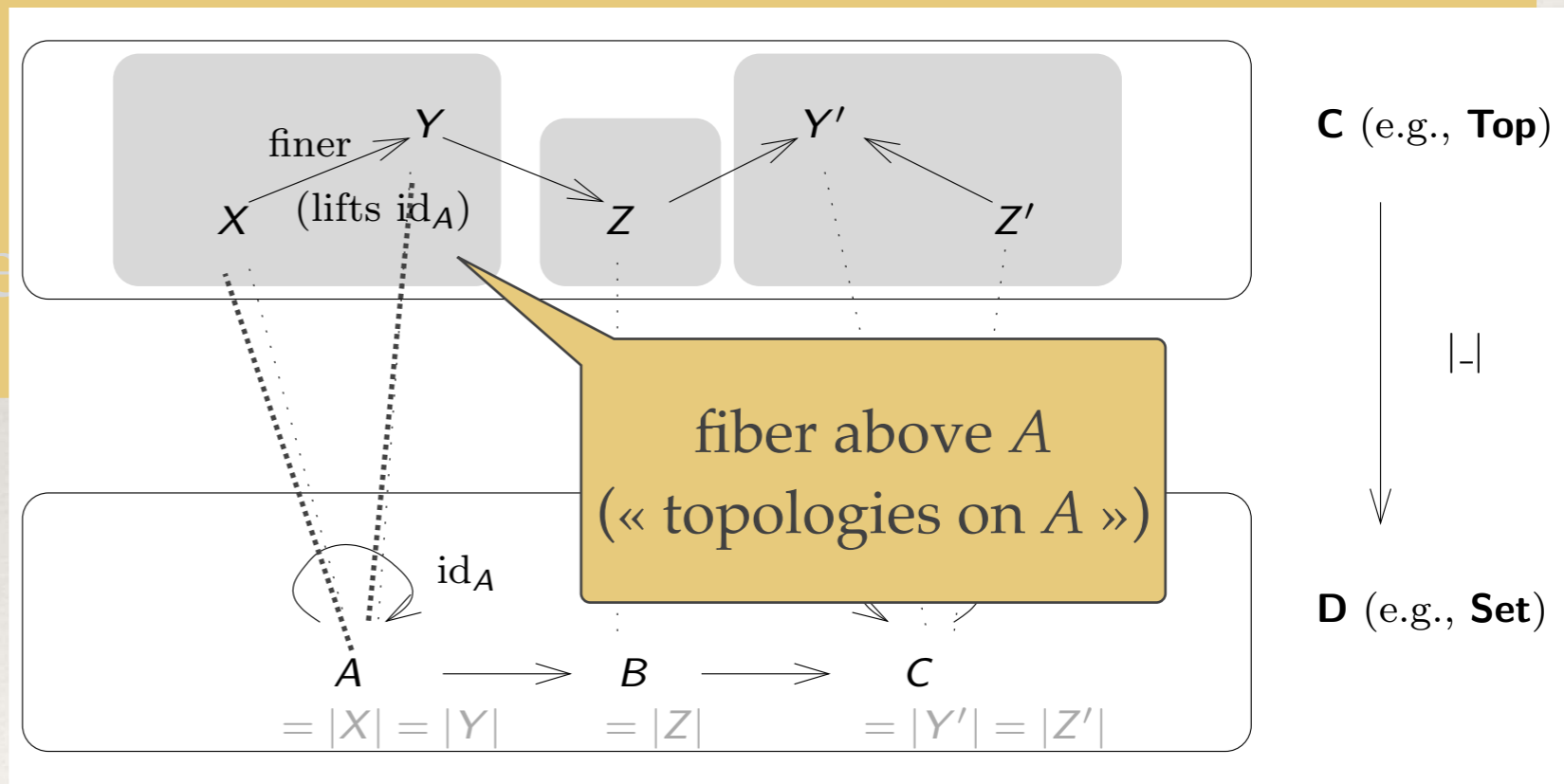
❖ A **topological functor** $|-| : \mathbf{C} \rightarrow \mathbf{D}$ is a functor that is:

(1) faithful

(2) amnestic

(3) every $|-|$ -source

This is more interesting.
Let us look at **fibers**.



- ❖ In the fiber above A , $X \leq Y$ (« is finer than ») iff id_A has a lift to $X \rightarrow Y$
- ❖ **Amnesticity** means that \leq is **antisymmetric** (hence a partial ordering)

Topological functors

❖ See *The Joy of Cats* [Adámek, Herrlich, Strecker 90]

❖ A **topological functor** $|_ | : \mathbf{C} \rightarrow \mathbf{D}$ is a functor that is:

- (1) faithful
- (2) amnestic
- (3) every $|_ |$ -source has a $|_ |$ -initial lift

Now this deserves an explanation.

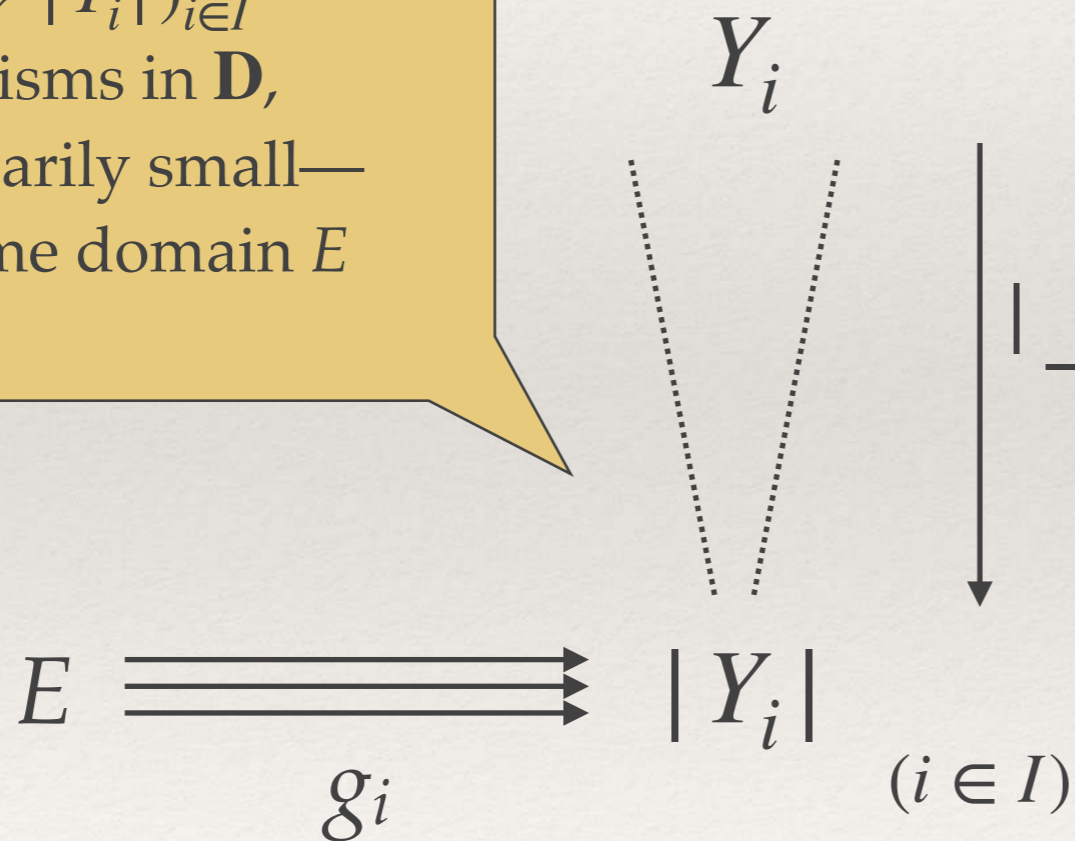
Intuitively: on any set E there is a **coarsest topology** that makes any given collection of maps $g_i : E \rightarrow Y_i$ continuous

Every $|-|$ -source ...

A $|-|$ -**source** is a collection
 $(g_i : E \rightarrow |Y_i|)_{i \in I}$
of morphisms in \mathbf{D} ,
—not necessarily small—
with the same domain E

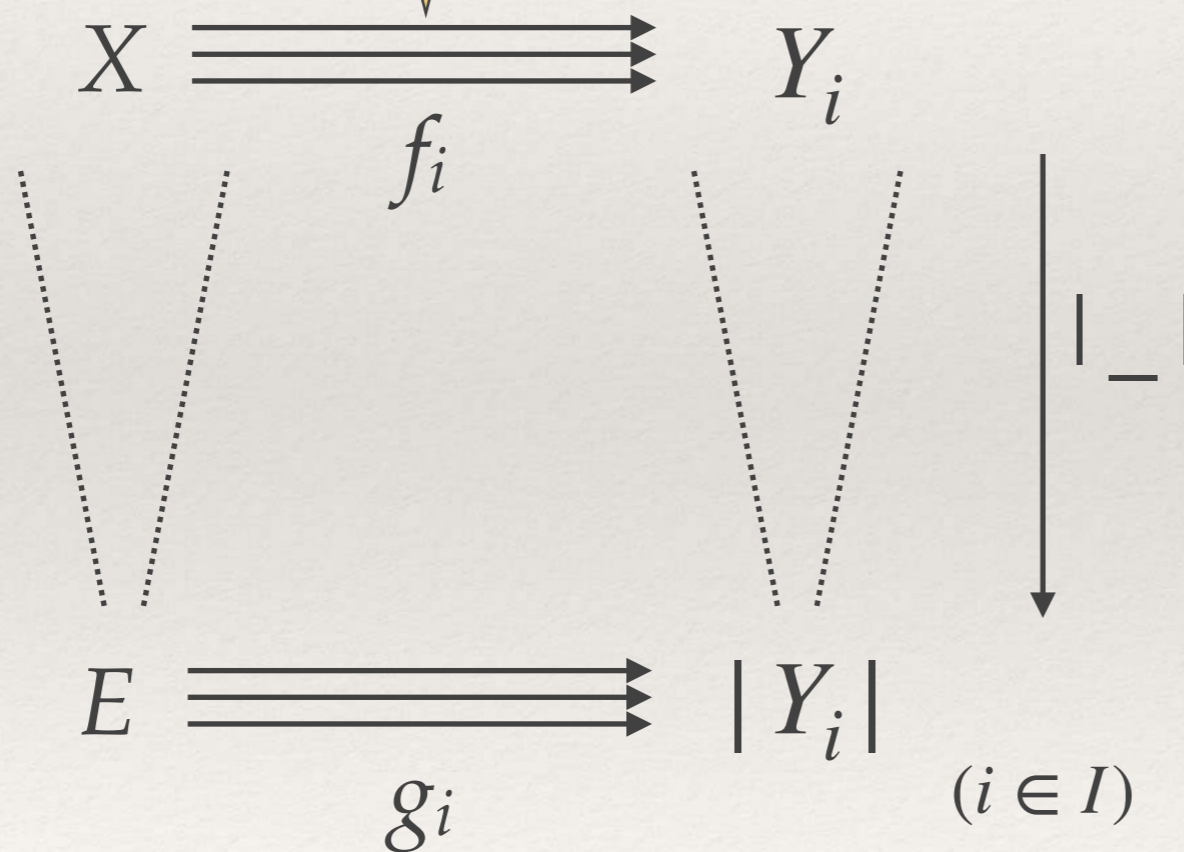
Note: the objects Y_i (in \mathbf{C})
are part of the data.

- ❖ In the canonical example
 $|-| : \mathbf{Top} \rightarrow \mathbf{Set}$,
- ❖ Each g_i is a (set) function
from E to a top. space Y_i



Every $|_|\text{-source}$ has a ... lift

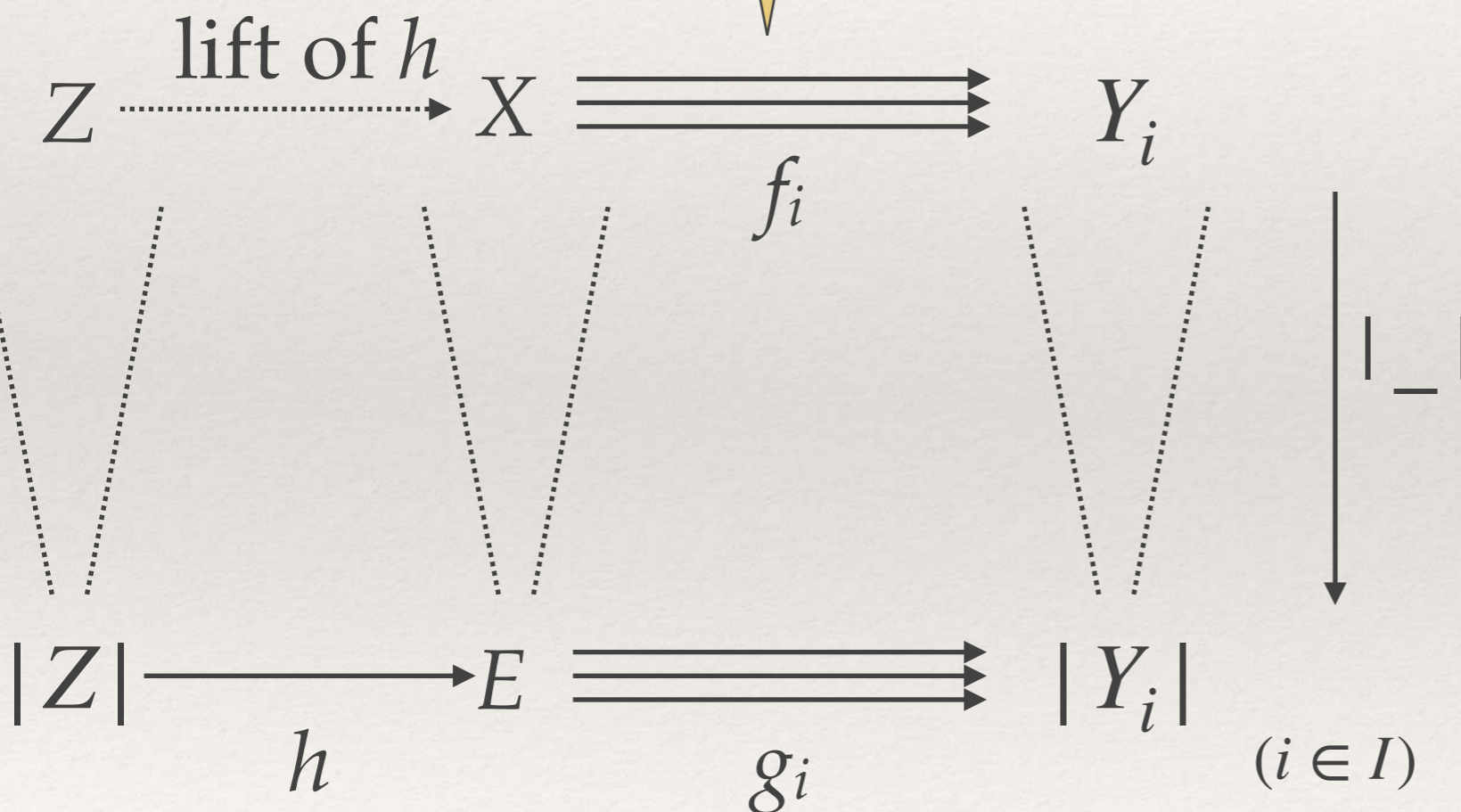
A lift: $|X| = E$ and $|f_i| = g_i$
for each of $i \in I$
Note: the same X for all $i \in I$



- ❖ In the canonical example $|_| : \mathbf{Top} \rightarrow \mathbf{Set}$,
- ❖ Each g_i is a (set) function from E to a top. space Y_i
- ❖ A lift is a topology on E (turning E into a topological space X) that makes each g_i continuous : $X \rightarrow Y_i$

Every $|_|\text{-source}$ has a $|_|\text{-initial lift}$

$|_|\text{-initial}$: for every $h : |Z| \rightarrow E$,
 h has a lift $Z \rightarrow X$ iff every $g_i \circ h$ has a lift



- ❖ In the canonical example $|_|\text{-} : \mathbf{Top} \rightarrow \mathbf{Set}$,
- ❖ Each g_i is a (set) function from E to a top. space Y_i
- ❖ A **lift** is a topology on E (turning E into a topological space X) that makes each g_i continuous $: X \rightarrow Y_i$

- ❖ and X is **coarsest**: a map $h : Z \rightarrow X$ is continuous iff every $g_i \circ h$ is continuous

Properties of topological functors

Fibers are complete lattices

- ❖ Given a topological functor $|_ | : \mathbf{C} \rightarrow \mathbf{D}$,
the fiber of any object E of \mathbf{D} has all **suprema**
(with respect to the finer than relation \leq)
... proof: consider $|_ |$ -sources consisting of identity morphisms.
- ❖ Hence all **infima**
... the infimum of a collection F is the supremum of its family of lower bounds
- ❖ Therefore fibers are **complete lattices** (possibly large)

Self-duality

- ❖ A functor $|_|_ : \mathbf{C} \rightarrow \mathbf{D}$ is topological iff $|_|_ : \mathbf{C}^{op} \rightarrow \mathbf{D}^{op}$ is topological. In other words:
 - ❖ A functor $|_|_ : \mathbf{C} \rightarrow \mathbf{D}$ is topological iff:
 - (1) faithful
 - (2) amnestic
 - (3) every $|_|_$ -sink has a $|_|_$ -final lift
 - ❖ Proof: similar to the order-theoretical result that a poset that has all suprema also has all infima
 - ❖ In the $\mathbf{Top} \rightarrow \mathbf{Set}$ case: on any set E there is a **finest topology** that makes any given collection of maps $g_i : X_i \rightarrow E$ continuous

Discrete, indiscrete objects

- ❖ Since the fiber of any object E of \mathbf{D} is a complete lattice, it has:
- ❖ a **smallest** (finest) element: the **discrete** object E_0
- ❖ a **largest** (coarsest) element: the **indiscrete** object E_1
- ❖ $_0 : E \mapsto E_0$ and $_1 : E \mapsto E_1$ define functors $: \mathbf{D} \rightarrow \mathbf{C}$
- ❖ ... and there is a string of adjunctions $_0 \dashv |_| \dashv _1$
- ❖ **Corollary.** $|_|$ preserves all existing limits and colimits

Limits, colimits

- ❖ $|_!$ preserves all existing limits and colimits, but we also have:

❖ **Proposition.** $|_!$ lifts all existing limits and colimits.

- ❖ Limits in \mathbf{C} : take the corresponding limits in \mathbf{D} , then take the coarsest lift
- ❖ Colimits in \mathbf{C} : take the corresponding colimits in \mathbf{D} , then take the finest lift

The category $\mathbf{Map}_{\mathcal{C}}$, categorically

Generalizing the Escardó-Lawson-Simpson construction

- ❖ We fix a topological functor $|_|\ : \mathbf{C} \rightarrow \mathbf{D}$
- ❖ We will require that $\mathbf{D} = \mathbf{Set}$:
 - $|_|\$ is a **topological construct**
- ❖ We will also require that $|_|\$
 - has **discrete terminal objects**, i.e. $\mathbf{1}_0 = \mathbf{1}_1$
(« only one topology on a one-element set »)
- ❖ **Objective:** generalize [Escardó, Lawson, Simpson 04] and obtain Cartesian-closed full subcategories of \mathbf{C}

The (new) category $\mathbf{Map}_{\mathcal{C}}$

- ❖ Fix a class \mathcal{C} of objects of \mathbf{C}
- ❖ Call \mathcal{C} -probe any morphism $k : C \rightarrow X$ where $C \in \mathcal{C}$
- ❖ A function $f : |X| \rightarrow |Y|$ is a \mathcal{C} -map : $X \rightarrow Y$ iff $f \circ |k|$ lifts to $C \rightarrow Y$ for every \mathcal{C} -probe $k : C \rightarrow X$

The category $\mathbf{Map}_{\mathcal{C}}$

- ❖ Fix a class \mathcal{C} of spaces
in the case of compactly-generated spaces, $\mathcal{C} = \{\text{all compact Hausdorff spaces}\}$
- ❖ Call \mathcal{C} -probe any continuous map $k : C \rightarrow X$
where $C \in \mathcal{C}$
- ❖ A function $f : X \rightarrow Y$ is \mathcal{C} -continuous iff $f \circ k$ is continuous for every \mathcal{C} -probe k

! X and Y live in \mathbf{Top} , but f is a function, in \mathbf{Set}

❖ **Definition.** $\mathbf{Map}_{\mathcal{C}}$ has:

- as objects, all topological spaces
- as morphisms, all \mathcal{C} -continuous maps

« larger » than \mathbf{Top} : every continuous map is \mathcal{C} -continuous

- ❖ **Definition.** $\mathbf{Map}_{\mathcal{C}}$ has:
 - as objects, all the objects of \mathbf{C}
 - as morphisms, all the \mathcal{C} -maps

« larger » than \mathbf{C} : for every morphism f in \mathbf{C} , $|f|$ is a \mathcal{C} -map

$\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed

- ❖ The class \mathcal{C} is **strongly productive** iff:
 - all the objects of \mathcal{C} are **exponentiable** in \mathbf{C}
 - \mathcal{C} is closed under binary products

Strongly productive classes

- ❖ The class \mathcal{C} is **strongly productive** iff:
 - all the objects of \mathcal{C} are **exponentiable**
(core-compact, in \mathbf{Top})
 - \mathcal{C} is closed under binary products
certainly true if $\mathcal{C} = \{\text{all compact Hausdorff spaces}\}$, right?
- ❖ **Theorem** [ELS 04]. If \mathcal{C} is a strongly productive class of topological spaces, then $\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed.

- ❖ **Theorem** [JGL 14]. If $|_ |$ is topological, $_$ has discrete terminal objects, and if \mathcal{C} is strongly productive, then $\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed.

- ❖ We give a brief sketch of the construction in the next slide

Map _{\mathcal{C}} is Cartesian-closed

\mathcal{C} is strongly productive iff:
 — all objects in \mathcal{C} are **exponentiable**
 — \mathcal{C} is closed under binary products

❖ *Proof sketch.* Let $\mathcal{C}(X, Y) \stackrel{\text{def}}{=} \{\text{all } \mathcal{C}\text{-maps } : X \rightarrow Y\}$

We define $[Y^X]_{\mathcal{C}}$ so that $|[Y^X]_{\mathcal{C}}| = \mathcal{C}(X, Y)$

« the exponential is the set of \mathcal{C} -maps $: X \rightarrow Y$, with some structure »

❖ Let $_ \bullet k : \mathcal{C}(X, Y) \rightarrow |Y^C|$
 send every \mathcal{C} -map $f : X \rightarrow Y$
 to the unique lift of $f \circ |k|$

Y^C makes sense because every $C \in \mathcal{C}$ is **exponentiable**

A function $f : |X| \rightarrow |Y|$ is
 a \mathcal{C} -map $: X \rightarrow Y$ iff $f \circ |k|$ lifts to $C \rightarrow Y$
 for every \mathcal{C} -probe $k : C \rightarrow X$

❖ We wish $_ \bullet k$ to « be continuous » from $[Y^X]_{\mathcal{C}}$ to Y^C
 for every \mathcal{C} -probe $k : C \rightarrow X$ (and every $C \in \mathcal{C}$)

and, up to iso,
 Y^C can be chosen so
 $|Y^C| = \text{Hom}_{\mathcal{C}}(C, Y)$
 (admitted; needs
discrete terminal objects)

❖ Hence we define $[Y^X]_{\mathcal{C}}$ as the $|_|$ -initial lift of
 the $|_|$ -source $(_ \bullet k : \mathcal{C}(X, Y) \rightarrow |Y^C|)_{C \in \mathcal{C}, k : C \rightarrow X \text{ } \mathcal{C}\text{-probe}}$

❖ The rest is a pretty tedious check.

See https://projects.lsv.ens-cachan.fr/topology/?page_id=6613 □

\mathcal{C} -generated objects

- ❖ For every object X of \mathbf{C} , let $\mathcal{C}X$ be the finest topology on $|X|$ in the fiber of $|X|$ such that

for every \mathcal{C} -probe $k : C \rightarrow X$, $|k|$ lifts to $C \rightarrow \mathcal{C}X$

- ❖ Technically, this is obtained as a $|_ |$ -final lift of

the $|_ |$ -sink $(|k| : |C| \rightarrow |X|)_{C \in \mathcal{C}, k : C \rightarrow X \text{ } \mathcal{C}\text{-probe}}$

\mathcal{C} -generated spaces

- ❖ For every topological space X , define $\mathcal{C}X$ as X , **retopologized** with the finest topology that makes all the \mathcal{C} -probes $k : C \rightarrow X$ continuous
... this is a finer topology than the original topology on X
- ❖ A topological space X is **\mathcal{C} -generated** iff $X = \mathcal{C}X$
- ❖ Let $\mathbf{Top}_{\mathcal{C}}$ be the full subcategory of \mathbf{Top} consisting of \mathcal{C} -generated spaces
- ❖ There is a functor $\mathcal{C} : \mathbf{Map}_{\mathcal{C}} \rightarrow \mathbf{Top}_{\mathcal{C}}$
we have just defined it on objects; on morphisms, $\mathcal{C}(f) = f$

Self-duality

- ❖ A functor $|_ | : \mathbf{C} \rightarrow \mathbf{D}$ is topological iff $|_ | : \mathbf{C}^{op} \rightarrow \mathbf{D}^{op}$ is topological. In other words:
- ❖ A functor $|_ | : \mathbf{C} \rightarrow \mathbf{D}$ is topological iff:
 - (1) faithful
 - (2) amnesic
 - (3) every $|_ |$ -sink has a $|_ |$ -final lift
- ❖ Proof: similar to the order-theoretical result that a poset that has all suprema also has all infima

The category $\mathbf{C}_{\mathcal{C}}$

- ❖ An object X of \mathbf{C} is \mathcal{C} -generated iff $X = \mathcal{C}X$
- ❖ Let $\mathbf{C}_{\mathcal{C}}$ be the full subcategory of \mathbf{C} consisting of \mathcal{C} -generated objects
- ❖ There is a functor $\mathcal{C} : \mathbf{Map}_{\mathcal{C}} \rightarrow \mathbf{C}_{\mathcal{C}}$
... on morphisms, $\mathcal{C}(f)$ is the unique lift of $f : |X| \rightarrow |Y|$ to $\mathcal{C}X \rightarrow \mathcal{C}Y$ (exercise)

\mathcal{C} -generated spaces

- ❖ For every topological space X , define $\mathcal{C}X$ as X , **retopologized** with the finest topology that makes all the \mathcal{C} -probes $k : C \rightarrow X$ continuous
... this is a finer topology than the original topology on X
- ❖ A topological space X is \mathcal{C} -generated iff $X = \mathcal{C}X$
- ❖ Let $\mathbf{Top}_{\mathcal{C}}$ be the full subcategory of \mathbf{Top} consisting of \mathcal{C} -generated spaces
- ❖ There is a functor $\mathcal{C} : \mathbf{Map}_{\mathcal{C}} \rightarrow \mathbf{Top}_{\mathcal{C}}$
we have just defined it on objects; on morphisms, $\mathcal{C}(f) = f$

$\mathbf{C}_{\mathcal{C}}$ is coreflective in \mathbf{C}

- ❖ There is a functor $\mathcal{C} : \mathbf{Map}_{\mathcal{C}} \rightarrow \mathbf{C}_{\mathcal{C}}$
... and an inclusion functor
$$I : \mathbf{C}_{\mathcal{C}} \rightarrow \mathbf{Map}_{\mathcal{C}}$$

- ❖ **Lemma.** \mathcal{C} and I together define an equivalence of categories between $\mathbf{C}_{\mathcal{C}}$ and $\mathbf{Map}_{\mathcal{C}}$.

- ❖ **Corollary [JGL 14].** If $|_ |$ is topological, $_ |$ has discrete terminal objects, and if \mathcal{C} is strongly productive, then $\mathbf{C}_{\mathcal{C}}$ is a Cartesian-closed, coreflective full subcategory of \mathbf{C} .
... the right adjoint to the inclusion functor is defined just like \mathcal{C}

Coreflective CCCs of topological spaces

- ❖ There is also an inclusion functor $I : \mathbf{Top}_{\mathcal{C}} \rightarrow \mathbf{Map}_{\mathcal{C}}$

- ❖ **Lemma.** \mathcal{C} and I together define an equivalence of categories between $\mathbf{Top}_{\mathcal{C}}$ and $\mathbf{Map}_{\mathcal{C}}$.

i.e., there are isos $X \rightarrow \mathcal{C}IX$ (in $\mathbf{Top}_{\mathcal{C}}$) for every \mathcal{C} -generated space
and $X \rightarrow I\mathcal{C}X$ (in $\mathbf{Map}_{\mathcal{C}}$) for every space X

- ❖ **Corollary [ELS 04].** If \mathcal{C} is a strongly productive class of topological spaces, then $\mathbf{Top}_{\mathcal{C}}$ is a Cartesian-closed, coreflective full subcategory of \mathbf{Top} .

... the right adjoint to the inclusion functor is essential \mathcal{C}

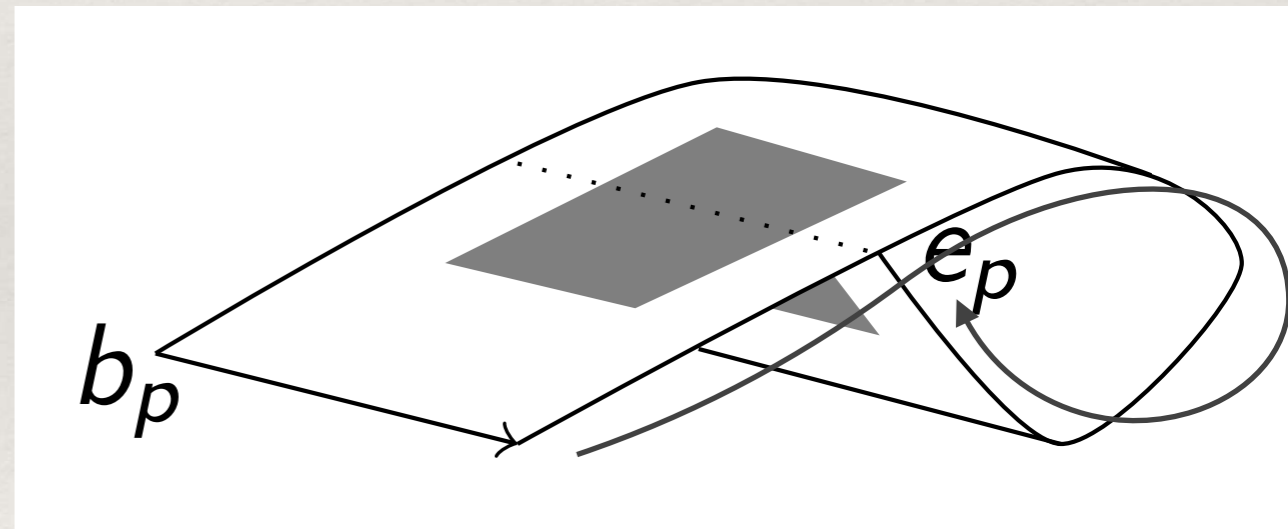
\mathcal{C} -generation by colimits

- ❖ Also, we have:
- ❖ **Proposition** [JGL 14]. Under the previous assumptions, $\mathbf{C}_{\mathcal{C}}$ is complete, cocomplete, and the objects of $\mathbf{C}_{\mathcal{C}}$ (the \mathcal{C} -generated objects of \mathbf{C}) are exactly the colimits of objects of \mathcal{C} , taken in \mathbf{C} .
- ❖ Generalizes a similar result in [Escardó, Lawson, Simpson 04].

An application to streams

Directed spaces

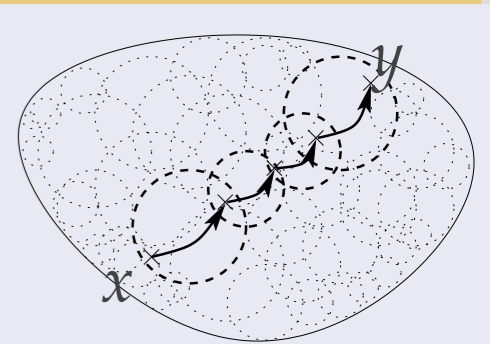
- ❖ How do we model spaces with **local directions of time** (which may be cyclic, globally) as used in **directed** algebraic topology?
- ❖ Now two standard proposals:
d-spaces [Grandis 09],
streams [Krishnan 08],
related by an adjunction [Haucourt 09]



Krishnan's streams

- ❖ A **stream** [Krishnan 08] is a **cosheaf** of preorders on a topological space X , namely:
 - for each open set U , a preordering \sqsubseteq_U on U
 - if $U \subseteq V$ then $x \sqsubseteq_U y$ implies $x \sqsubseteq_V y$
 - if $U = \bigcup_{i \in I} U_i$ and $x \sqsubseteq_U y$ then there is a chain

$$x = x_0 \sqsubseteq_{U_{i_1}} x_1 \sqsubseteq_{U_{i_2}} \cdots \sqsubseteq_{U_{i_n}} x_n = y$$



- ❖ A **stream morphism** is a continuous map that is **locally monotonic**: $x \sqsubseteq_{f^{-1}(V)} y \implies f(x) \sqsubseteq_V f(y)$

The category **Str** is topological

- ❖ Streams and their morphisms form a category **Str**
- ❖ The forgetful functor **Str** \rightarrow **Top** is **topological**
(stated in [Krishnan 08], see [JGL 14] for a proof)
- ❖ hence so is the forgetful functor **Str** \rightarrow **Set**
- ❖ This has **discrete terminal objects**: on a one-element set, only one topology, only one choice of \sqsubseteq_U for any of the two open sets U
- ❖ Hence our construction applies...

Exponentiable streams

- ❖ **Theorem [JGL 14].** The exponentiable streams are exactly those whose underlying topological space is **core-compact**.
- ❖ We recall that the core-compact spaces are those that are exponentiable in **Top**.
- ❖ Hence...

Cartesian-closed categories of streams

- ❖ **Theorem [JGL 14].** Let \mathcal{C} be any class of streams that is strongly productive (= closed under binary products, consisting of core-compact streams). Then the full subcategory $\mathbf{Str}_{\mathcal{C}}$ of \mathcal{C} -generated streams (=colimits of streams in \mathcal{C}) is Cartesian-closed.
- ❖ E.g., if $\mathcal{C} = \{\text{compact Hausdorff streams}\}$,
(and further restricting to Hausdorff \mathcal{C} -generated streams),
we get Krishnan's CCC of **compactly-flowing streams**

Conclusion

Conclusion, and open questions

- ❖ **Topological functors:** effectively generalize topology... enough to lift (apparently) purely topological arguments such as [Lawson, Escardó, Simpson 04]
- ❖ **Q1.** I only obtained that $\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed for $|_|\ : \mathbf{C} \rightarrow \mathbf{D}$ where $\mathbf{D} = \mathbf{Set}$: what about **other** Cartesian-closed categories \mathbf{D} ?
(Seems like laziness on my part.)
- ❖ **Q2.** Is the restriction to having **discrete terminal objects** necessary?
- ❖ **Q3.** Other **applications**, beyond streams?
(The case of d-spaces should be easy, but look outside directed spaces, too.)

Main references

- ❖ Jíří Adámek, Horst Herrlich, George E. Strecker. *Abstract and Concrete Categories. The Joy of Cats*. John Wiley and Sons, 1990. [Online edition](#), 2004.
- ❖ Jean Goubault-Larrecq. *Exponentiable streams and prestreams*. [Applied Categorical Structures](#) 22, 515–549, 2014. The published version, available from [Springer Link](#), contains two mistakes, which are repaired in the [HAL report](#).
- ❖ Martín Escardó, Jimmie Lawson, and Alex Simpson. *Comparing Cartesian Closed Categories of (Core) Compactly Generated Spaces*. [Topology and Its Applications](#), 143(1–3):105–146, 2004.