Continuous R-valuations
Continuous valuations

- Continuous valuations on a topological space $X$ are maps $\nu : \mathcal{O}(X) \to \mathbb{R}_+$ that are:
  - strict: $\nu(\emptyset) = 0$
  - modular: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
  - Scott-continuous.

- Continuous valuations $\cong$ measures.

⇒ We will replace $\mathbb{R}_+$ by Abelian d-rags

What’s so special about $\mathbb{R}_+$ here?

http://www.andrej.com/mathematicians/large/Danos_Vincent.jpg
\textbf{Rags}

\begin{itemize}
\item **Defn.** A \textit{rag} is \((R,0, + ,1,\times)\) such that:
  \begin{itemize}
  \item \((R,0,+ )\) Abelian monoid
  \item \((R,1,\times)\) monoid
  \item \(\times\) distributes over +
  \end{itemize}

\item Similar but weaker than a \textit{semi-ring} (or \textit{rig}): we do not require \(0 \times r = r \times 0 = 0\)
\end{itemize}
D-rags

- **Defn.** A *rag* is \((R,0, + , 1,\times)\) such that:
  - \((R,0,+)\) Abelian monoid
  - \((R,1,\times)\) monoid
  - \(\times\) distributes over +

- A **d-rag** is a rag with a *dcpo* structure such that +, \(\times\) are **Scott-continuous**

- An **Abelian d-rag** is one whose \(\times\) is commutative
Fundamental examples

Example 1. \( \overline{\mathbb{R}_+} \)

Note: actually a rig, and \( 0 = \perp \) here

Example 2.

\( \mathbb{I}_+^{*} \cong \{ \text{intervals } [a, b], 0 \leq a \leq b \leq \infty \}, \) reverse inclusion

\( 0 \cong [0, 0] \quad \text{... different from } \perp = [0, \infty] \)

\( \text{with the obvious (componentwise) operations} \)

\( \text{except } [a, 0] \times [b, \infty] \cong [ab, \infty] \)

\( (\text{while } [0, c] \times [\infty, d] \cong [0, cd]) \)

required for \( \times \) to be Scott-continuous

(and then causes it not to be a rig)
Continuous R-valuations: the wrong approach

- Given an Abelian d-rag $R$,
  the **obvious** definition of an $R$-valued continuous valuation would be:
  $\nu : \mathcal{O}(X) \rightarrow R$ that are:
  - **strict**: $\nu(\emptyset) = ?$
  - **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
  - **Scott-continuous**.

- Instead, we define continuous $R$-valuations
  as the desired **integration functionals** $h \mapsto \int h d\nu$, **directly**
  (and we will write them simply as $\nu$)
Continuous R-valuations

- **Defn.** A **continuous R-valuation** on \( X \) is a **Scott-continuous, linear** map \( \nu \) from \([X \to R]\) to \( R\)

  - with pointwise ordering
  - \( \nu(a \times h) = a \times \nu(h) \quad (a \in R) \)
  - \( \nu(h_1 + h_2) = \nu(h_1 + h_2) \)

- **Note.** with \( R = \overline{\mathbb{R}}_+ \), we retrieve the usual notion of continuous valuation

  - with \( R = \mathbb{IR}_+^* \), we get something akin to (but subtly different from)

  the interval-valued integrals of

Monads of continuous $R$-valuations

**Thm.** Fix an Abelian d-rag $R$.

There is a strong monad $(V^R, \eta, \_ \uparrow t)$ on \textbf{Dcpo} (or on \textbf{Top}) where

- $V^R(X) \equiv \text{dcpo of continuous } R\text{-valuations, ordered pointwise}$
- $\eta: x \in X \mapsto \delta_x$, where $\delta_x(h) \equiv h(x)$  \hspace{1cm} [Dirac $R$-valuation]
- for every $f: X \to V^R(Y), f^\uparrow: V^R(X) \to V^R(Y)$ is defined by
  \[ f^\uparrow(\nu) \equiv k \in V^R(Y) \mapsto \nu(x \in X \mapsto f(x)(k)) \]
- for all $x \in X, \nu \in V^R(Y), t(x, \nu) \equiv k \in [X \times Y \to R] \mapsto \nu(y \mapsto k(x, y))$
A word of warning, and a subtle point

- **Warning.** Not directly useful for semantics of programs with interval-valued probabilities: if $\mu$ is a (representable) measure on $[0,1]$, then $\mu([0,\frac{1}{2}])$ is not Scott-cocontinuous, i.e., if $[a, b] = \mu([0,\frac{1}{2}])$ and $a = b$ then $[a, b]$ is **not continuous** as a function of $\mu$.

- In practice, the semantics of any non-trivial loop/recursive function using a monad of continuous $R$-valuations with $R = \mathbb{I}\mathbb{R}_+$ (implemented in RealPCF, say) will be an **imprecise** interval of the form $[a, \infty]$. 

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Klaus Weihrauch (1999) *Computability on the probability measures on the Borel sets of the unit interval.* TCS 219:421–437
Commutative monads of continuous $R$-valuations

- In semantics, we wish our probability monads to be **commutative**
  (« $x \leftarrow$ random; $y \leftarrow$ random » should be equivalent to « $y \leftarrow$ random; $x \leftarrow$ random »)

- **Defn.** An **elementary $R$-valuation** is a finite **non-empty** linear combination
  \[
  \sum_{i=1}^{n} a_i \times \delta_{x_i} \quad (\text{with } a_i \in R)
  \]
  (Note the similarity with **simple valuations**)

- **Defn.** The dcpo $V^R_m(X)$ of **minimal $R$-valuations**
  is the inductive closure of the set of elementary $R$-valuations inside $V^R(X)$
  (= smallest subdcpo =take elementary $R$-valuations, their directed suprema, then again directed suprema, etc.)

- **Thm.** The monad $(V^R_m, \eta, \_\uparrow, t)$ is **commutative** on **Dcpo**.

But are minimal $R$-valuations enough to represent, say, Lebesgue measure?
The Lebesgue $R$-valuation on $[0,1]$ is minimal (on $\mathbb{I}\mathbb{R}$)

- Let $R \equiv \mathbb{I}\mathbb{R}_+^*$. How do we model drawing a real number uniformly in $[0,1]$?

- Let $\lambda_n \equiv \sum_{i=1}^{2^n} \left[ \frac{1}{2^n}, \frac{1}{2^n} \right] \times \delta\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$: an elementary $R$-valuation on $\mathbb{I}\mathbb{R}$

- The directed supremum $\bar{\lambda} \equiv \sup_n \lambda_n$ is the Lebesgue $R$-valuation on $[0,1]$ ... and is minimal by definition

- This is essentially Edalat’s interval-valued Riemann integration operator

How does $\lambda$ model Lebesgue measure, really?

- Say that $k: X \to \mathbb{I}\mathbb{R}$ approximates $f: X \to \mathbb{R}$ iff $f(x) \in k(x)$ for every $x \in X$
  
  i.e., $k^-(x) \leq f(x) \leq k^+(x)$ where $k = [k^-, k^+]$

- $\nu \in V^R(X)$ approximates a Borel measure $\mu$ on $X$
  
  iff for all $k, f$ such that $k$ approximates $f$, we have $\int f d\mu \in \nu(k)$
  
  i.e., $\nu^-(k) \leq \int f d\mu \leq \nu^+(k)$

- **Thm.** $\lambda$ is the largest (=most precise) continuous $R$-valuation that approximates Lebesgue measure on $[0, 1] \subseteq \mathbb{I}\mathbb{R}$. 
On largest continuous $R$-valuations approximating a measure

- For every $\tau$-smooth measure $\mu$ on $X$, and every usc map $h : X \to \overline{\mathbb{R}}_+$,
  let
  \[
  \int h d\mu \equiv \int h d\mu \text{ if } h \text{ is } \mu\text{-bounded} \quad \text{(namely if } h < \infty \text{ on some compact sat. support of } \mu) \\
  \infty \text{ otherwise}
  \]

- Then $\tilde{\mu}(k) \equiv \left[ \int k^- d\mu, \int k^+ d\mu \right]$ defines a continuous $R$-valuation

- **Thm.** If $\mu$ is non-zero and bounded and $X$ is a $T_2$ patch-compact subset of a stably compact, 2nd countable space, then:
  - $\mu$ is $\tau$-smooth
  - $\tilde{\mu}$ is the **largest** continuous $R$-valuation that approximates $\mu$. 
Summary

- We can extend continuous valuations to continuous $R$-valuations where $R$ is any Abelian d-rag.
- When $R = \mathbb{R}_+$, generalizes ordinary continuous valuations. When $R = I\mathbb{R}^*_+$, we retrieve something close to Edalat’s interval-valued integration operators.
- We obtain commutative monads of minimal $R$-valuations.
- Under some assumptions, there is a largest (=most precise) continuous $I\mathbb{R}^*_+$-valuation approximating a given non-zero bounded measure.
- That largest continuous $I\mathbb{R}^*_+$-valuation is minimal in the case of Lebesgue measure on $[0, 1] \subseteq I\mathbb{R}$. 
Oops, it seems you’ve got too far
... or have you?
I knew you would ask that question

- Let $h_n : [0,1] \to \mathbb{I}\mathbb{R}^*_+$

- (Note: Edalat integrates with values in $\mathbb{I}\mathbb{R}$, not $\mathbb{I}\mathbb{R}^*_+$)

- With the obvious variant of Edalat’s integral,

\[
\int h_n \, d\lambda = \frac{1}{2^n} \times [0, \infty] + \left(1 - \frac{1}{2^n}\right) \times [0,0] = [0,\infty]
\]

- With $h \doteq \sup_n h_n$, \[
\int h \, d\lambda = [0,0] \neq \sup_n \int h_n \, d\lambda
\]

- Hence that obvious variant is not continuous — we repair this by using $\int$.  