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Continuous R-valuations

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Continuous valuations

What's so special about $\overline{\mathbb{R}}_+$ here?

$\Rightarrow$ We will replace $\overline{\mathbb{R}}_+$ by **Abelian d-rags**

- **Continuous valuations** on a topological space X

  = maps $\nu : \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ that are:

  — **strict**: $\nu(\emptyset) = 0$

  — **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$

  — **Scott-continuous**.

- Continuous valuations ≅ measures.

http://www.andrej.com/mathematicians/large/Danos_Vincent.jpg

Rags

- **Defn.** A **rag** is $(R, 0, +, 1, \times)$ such that:
  — $(R, 0, +)$ Abelian monoid
  — $(R, 1, \times)$ monoid
  — $\times$ distributes over $+$
- Similar but weaker than a **semi-ring** (or **rig**):
  we do **not** require $0 \times r = r \times 0 = 0$

D-rags

- **Defn.** A **rag** is $(R, 0, +, 1, \times)$ such that:
  — $(R, 0, +)$ Abelian monoid
  — $(R, 1, \times)$ monoid
  — $\times$ distributes over $+$
- A **d-rag** is a rag with a **dcpo** structure
  such that $+, \times$ are **Scott-continuous**
- An **Abelian d-rag** is one whose $\times$ is commutative

Fundamental examples

- **Example 1.** $\overline{\mathbb{R}}_+$

  Note: actually a rig, and $0 = \perp$ here

- **Example 2.**

  $\mathbf{I}\mathbb{R}^*_+ \stackrel{\text{def}}{=} \{\text{intervals } [a, b], 0 \leq a \leq b \leq \infty\}$, reverse inclusion

  $0 \stackrel{\text{def}}{=} [0, 0]$ … **different** from $\perp = [0,\infty]$

- with the obvious (componentwise) operations

  except $[a,0] \times [b,\infty] \stackrel{\text{def}}{=} [ab,\infty]$

  (while $[0,c] \times [\infty, d] \stackrel{\text{def}}{=} [0,cd]$)

required for $\times$ to be Scott-continuous
(and then causes it **not** to be a rig)

Continuous R-valuations: the wrong approach

- Given an Abelian d-rag R,
  the **obvious** definition of an R-valued continuous valuation would be:
  $\nu\colon \mathcal{O}(X) \to R$ that are:
  — **strict**: $\nu(\emptyset) = ?$
  — **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
  — **Scott-continuous**.

0? (needed for algebraic reasoning)

⊥? (needed to define integral as supremum of elementary sums)

- Instead, we define continuous R-valuations

  as the desired **integration functionals** $h \mapsto \int h d\nu$, **directly**

  (and we will write them simply as ν)

Continuous R-valuations

- **Defn.** A **continuous R-valuation** on X is a **Scott-continuous, linear** map ν from $[X \to R]$ to R

with pointwise ordering

$\nu(a \times h) = a \times \nu(h) \;\; (a \in R)$
$\nu(h_1 + h_2) = \nu(h_1 + h_2)$

- **Note.** with $R = \overline{\mathbb{R}}_+$, we retrieve the usual notion of continuous valuation
with $R = \mathbf{I}\mathbb{R}^*_+$, we get something akin to (but subtly different from) the interval-valued integrals of

📖 Abbas Edalat (2009) *A computable approach to measure and integration theory*. Inf. Comp. 207:642–659

Monads of continuous R-valuations

- **Thm.** Fix an Abelian d-rag R.

  There is a strong monad $(V^R, \eta, \_^\dagger, t)$ on **Dcpo** (or on **Top**) where

  — $V^R(X) \stackrel{\text{def}}{=}$ dcpo of continuous R-valuations, ordered pointwise

  — $\eta \colon x \in X \mapsto \delta_x$, where $\delta_x(h) \stackrel{\text{def}}{=} h(x)$ [**Dirac** R-valuation]

  — for every $f \colon X \to V^R(Y)$, $f^\dagger \colon V^R(X) \to V^R(Y)$ is defined by

  $$f^\dagger(\nu) \stackrel{\text{def}}{=} k \in V^R(Y) \mapsto \nu(x \in X \mapsto f(x)(k))$$

  — for all $x \in X, \nu \in V^R(Y)$, $t(x, \nu) \stackrel{\text{def}}{=} k \in [X \times Y \to R] \mapsto \nu(y \mapsto k(x, y))$

A word of warning, and a subtle point

- **Warning.** Not directly useful for semantics
  of programs with interval-valued probabilities:
  if μ is a (representable) measure on [0,1],
  then $\mu([0,\frac{1}{2}[)$ is not Scott-**co**continuous
  i.e., if $[a, b] = \mu([0,\frac{1}{2}[)$ and $a = b$
  then $[a, b]$ is **not continuous** as a function of μ

📖 Klaus Weihrauch (1999) *Computability on the probability measures on the Borel sets of the unit interval.* TCS 219:421– 437

- In practice, the semantics of any non-trivial **loop**/**recursive** function
  using a monad of continuous R-valuations with $R = \mathbf{I}\mathbb{R}_+^*$
  (implemented in RealPCF, say)
  will be an **imprecise** interval of the form $[a, \infty]$

Commutative monads of continuous R-valuations

- In semantics, we wish our probability monads to be **commutative**
  (« $x \leftarrow$ random; $y \leftarrow$ random » should be equivalent to « $y \leftarrow$ random; $x \leftarrow$ random »)
- **Defn.** An **elementary *R*-valuation** is a finite **non-empty** linear combination
  $$\sum_{i=1}^{n} a_i \times \delta_{x_i} \quad (\text{with } a_i \in R)$$
  (Note the similarity with **simple valuations**)
- **Defn.** The dcpo $V^R_m(X)$ of **minimal *R*-**valuations
  is the inductive closure of the set of elementary *R*-valuations inside $V^R(X)$
  (= smallest subdcpo =take elementary *R*-valuations, their directed suprema, then again directed suprema, etc.)
- **Thm.** The monad $(V^R_m, \eta, \_^\dagger, t)$ is **commutative** on **Dcpo**.

But are minimal *R*-valuations enough to represent, say, Lebesgue measure?

The Lebesgue R-valuation on [0,1] is minimal (on $\mathbf{I}\mathbb{R}$)

- Let $R \stackrel{\text{def}}{=} \mathbf{I}\mathbb{R}^*_+$.
  How do we model drawing a real number uniformly in [0, 1]?
- Let $\overline{\lambda}_n \stackrel{\text{def}}{=} \sum_{i=1}^{2^n} \left[\frac{1}{2^n}, \frac{1}{2^n}\right] \times \delta_{\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]}$: an **elementary** ***R*-valuation** on $\mathbf{I}\mathbb{R}$
- The directed supremum $\overline{\lambda} \stackrel{\text{def}}{=} \sup_n^{\uparrow} \overline{\lambda}_n$ is the **Lebesgue** ***R*-valuation** on [0, 1]
  … and is **minimal** by definition
- This is essentially Edalat's interval-valued Riemann integration operator

📖 Abbas Edalat (2009) *A computable approach to measure and integration theory*. Inf. Comp. 207:642–659

How does $\overline{\lambda}$ model Lebesgue measure, really?

- Say that $k\colon X \to \mathbb{IR}$ **approximates** $f\colon X \to \mathbb{R}$
  iff $f(x) \in k(x)$ for every $x \in X$
  i.e., $k^-(x) \leq f(x) \leq k^+(x)$ where $k = [k^-, k^+]$
- $\nu \in V^R(X)$ **approximates** a Borel measure μ on X
  iff for all k, f such that k approximates f, we have $\int f d\mu \in \nu(k)$
  i.e., $\nu^-(k) \leq \int f d\mu \leq \nu^+(k)$
- **Thm.** $\overline{\lambda}$ is the **largest** (=most precise) continuous R-valuation
  that approximates Lebesgue measure on $[0, 1] \subseteq \mathbb{IR}$.

On largest continuous R-valuations approximating a measure

- For every τ-smooth measure μ on X, and every usc map $h\colon X \to \overline{\mathbb{R}}_+$,

  let $\int^+ h d\mu \stackrel{\text{def}}{=} \int h d\mu$ if h is **μ-bounded** (namely if $h < \infty$ on some compact sat. support of μ)

  ∞ otherwise

  needed to make $h \mapsto \int^+ h d\mu$ Scott-**cocontinuous**

  (commute with filtered infs)

- Then $\tilde{\mu}(k) \stackrel{\text{def}}{=} \left[\int k^- d\mu, \int^+ k^+ d\mu\right]$ defines a continuous R-valuation

- **Thm.** If μ is non-zero and bounded and X is a T_2 patch-compact subset of a stably compact, 2nd countable space, then:

  — μ is τ-smooth

  — $\tilde{\mu}$ is the **largest** continuous R-valuation that **approximates** μ.

Summary

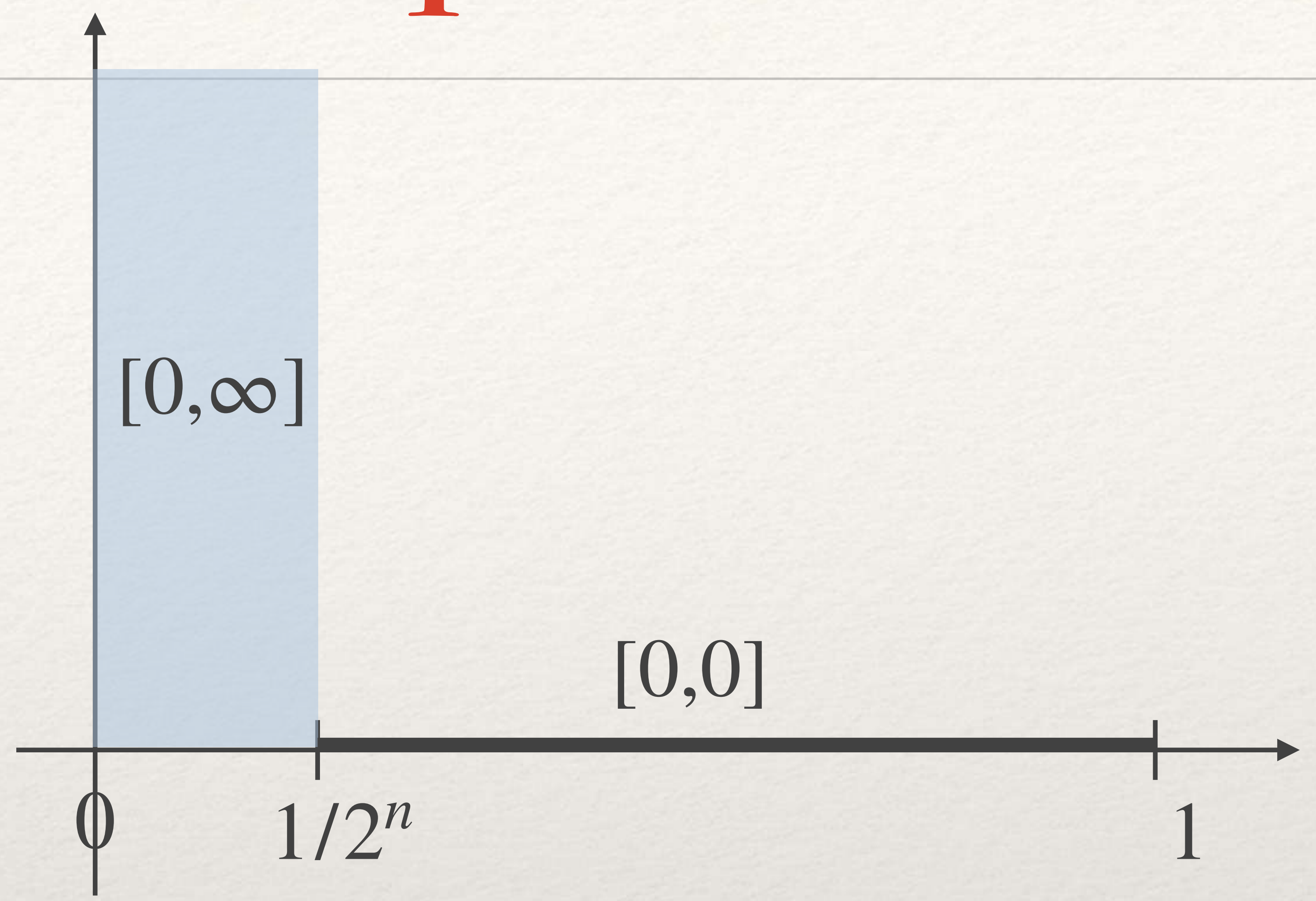
- We can extend continuous valuations to continuous R-valuations
  where R is any **Abelian d-rag**
- When $R = \overline{\mathbb{R}}_+$, generalizes ordinary continuous valuations
  When $R = \mathbf{I}\mathbb{R}^*_+$, we retrieve something close to
  Edalat's interval-valued integration operators
- We obtain **commutative** monads of **minimal** R-valuations
- Under some assumptions, there is a **largest** (=most precise) continuous $\mathbf{I}\mathbb{R}^*_+$-valuation approximating a given non-zero bounded measure
- That largest continuous $\mathbf{I}\mathbb{R}^*_+$-valuation is **minimal**
  in the case of Lebesgue measure on $[0, 1] \subseteq \mathbf{I}\mathbb{R}$

Oops, it seems you've got too far

… or have you?

I knew you would ask that question

- Let $h_n : [0,1] \to \mathbf{I}\mathbb{R}^{\star}_{+}$
- (Note: Edalat integrates with values in $\mathbf{I}\mathbb{R}$, not $\mathbf{I}\mathbb{R}^{\star}_{+}$)



- With the obvious variant of Edalat's integral,

$$\int h_n d\lambda = \frac{1}{2^n} \times [0,\infty] + \left(1 - \frac{1}{2^n}\right) \times [0,0] = [0,\infty]$$

- With $h \mathrel{\hat{=}} \sup_n^{\uparrow} h_n$, $\int h d\lambda = [0,0] \neq \sup_n^{\uparrow} \int h_n d\lambda$
- Hence that obvious variant is **not continuous** — we repair this by using $\int^{+}$.