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Continuous R-valuations

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Continuous valuations

- ❖ **Continuous valuations** on a topological space X
= maps $\nu: \mathcal{O}(X) \rightarrow \overline{\mathbb{R}}_+$ that are:
 - **strict**: $\nu(\emptyset) = 0$
 - **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
 - **Scott-continuous**.
- ❖ Continuous valuations \cong measures.

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\Rightarrow We will replace $\overline{\mathbb{R}}_+$
by **Abelian d-rags**

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Rags

- ❖ **Defn.** A **rag** is $(R, 0, +, 1, \times)$ such that:
 - $(R, 0, +)$ Abelian monoid
 - $(R, 1, \times)$ monoid
 - \times distributes over $+$
- ❖ Similar but weaker than a **semi-ring** (or **rig**):
 - we do **not** require $0 \times r = r \times 0 = 0$

D-rags

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 - $(R, 1, \times)$ monoid
 - \times distributes over $+$
- ❖ A **d-rag** is a rag with a **dcpo** structure such that $+, \times$ are **Scott-continuous**
- ❖ An **Abelian d-rag** is one whose \times is commutative

Fundamental examples

❖ **Example 1.** $\overline{\mathbb{R}}_+$

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❖ with the obvious (componentwise) operations

except $[a, 0] \times [b, \infty] \stackrel{\text{def}}{=} [ab, \infty]$

(while $[0, c] \times [\infty, d] \stackrel{\text{def}}{=} [0, cd]$)

required for \times to be
Scott-continuous
(and then causes it **not** to be a rig)

Continuous R -valuations: the wrong approach

- ❖ Given an Abelian d-rag R ,
the **obvious** definition of an R -valued continuous valuation would be:
 $\nu: \mathcal{O}(X) \rightarrow R$ that are:
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 - **Scott-continuous**. \perp ? (needed to define integral as supremum of elementary sums)
- ❖ Instead, we define continuous R -valuations
as the desired **integration functionals** $h \mapsto \int h d\nu$, **directly**
(and we will write them simply as ν)

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- ❖ **Note.** with $R = \overline{\mathbb{R}}_+$, we retrieve the usual notion of continuous valuation
with $R = \mathbf{IR}_+^*$, we get something akin to (but subtly different from)
the interval-valued integrals of

Monads of continuous R-valuations

❖ **Thm.** Fix an Abelian d-rag R .

There is a strong monad $(V^R, \eta, _^\dagger, t)$ on **Dcpo** (or on **Top**) where

— $V^R(X) \stackrel{\text{def}}{=} \text{dcpo of continuous } R\text{-valuations, ordered pointwise}$

— $\eta: x \in X \mapsto \delta_x$, where $\delta_x(h) \stackrel{\text{def}}{=} h(x)$ [Dirac R -valuation]


— for every $f: X \rightarrow V^R(Y)$, $f^\dagger: V^R(X) \rightarrow V^R(Y)$ is defined by

$$f^\dagger(\nu) \stackrel{\text{def}}{=} k \in V^R(Y) \mapsto \nu(x \in X \mapsto f(x)(k))$$

— for all $x \in X, \nu \in V^R(Y)$, $t(x, \nu) \stackrel{\text{def}}{=} k \in [X \times Y \rightarrow R] \mapsto \nu(y \mapsto k(x, y))$

A word of warning, and a subtle point

- ❖ **Warning.** Not directly useful for semantics of programs with interval-valued probabilities:
if μ is a (representable) measure on $[0,1]$,
then $\mu([0, \frac{1}{2}[)$ is not Scott-continuous
i.e., if $[a, b] = \mu([0, \frac{1}{2}[)$ and $a = b$
then $[a, b]$ is **not continuous** as a function of μ

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- ❖ In practice, the semantics of any non-trivial **loop** / **recursive** function using a monad of continuous R -valuations with $R = \mathbf{IR}_+^*$ (implemented in RealPCF, say)

will be an **imprecise** interval of the form $[a, \infty]$

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Commutative monads of continuous R -valuations

- ❖ In semantics, we wish our probability monads to be **commutative**
(« $x \leftarrow \text{random}; y \leftarrow \text{random}$ » should be equivalent to « $y \leftarrow \text{random}; x \leftarrow \text{random}$ »)

- ❖ **Defn.** An elementary R -valuation is a finite **non-empty** linear combination

$$\sum_{i=1}^n a_i \times \delta_{x_i} \quad (\text{with } a_i \in R)$$

(Note the similarity with **simple valuations**)

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But are minimal R -valuations
enough to represent, say,
Lebesgue measure?

The Lebesgue R -valuation on $[0, 1]$ is minimal (on \mathbf{IR})

❖ Let $R \stackrel{\text{def}}{=} \mathbf{IR}_+^*$.

How do we model drawing a real number uniformly in $[0, 1]$?

❖ Let $\bar{\lambda}_n \stackrel{\text{def}}{=} \sum_{i=1}^{2^n} \left[\frac{1}{2^n}, \frac{1}{2^n} \right] \times \delta_{\left[\frac{i-1}{2^n}, \frac{i}{2^n} \right]}$: an **elementary R -valuation** on \mathbf{IR}

❖ The directed supremum $\bar{\lambda} \stackrel{\text{def}}{=} \sup_n \bar{\lambda}_n$ is the **Lebesgue R -valuation** on $[0, 1]$
... and is **minimal** by definition

❖ This is essentially Edalat's interval-valued Riemann integration operator

How does $\bar{\lambda}$ model Lebesgue measure, really?

- ❖ Say that $k: X \rightarrow \mathbf{IR}$ approximates $f: X \rightarrow \mathbb{R}$
iff $f(x) \in k(x)$ for every $x \in X$
i.e., $k^-(x) \leq f(x) \leq k^+(x)$ where $k = [k^-, k^+]$
- ❖ $\nu \in V^R(X)$ approximates a Borel measure μ on X
iff for all k, f such that k approximates f , we have $\int f d\mu \in \nu(k)$
i.e., $\nu^-(k) \leq \int f d\mu \leq \nu^+(k)$
- ❖ **Thm.** $\bar{\lambda}$ is the largest (=most precise) continuous R -valuation that approximates Lebesgue measure on $[0, 1] \subseteq \mathbf{IR}$.

On largest continuous R -valuations approximating a measure

❖ For every τ -smooth measure μ on X , and every usc map $h: X \rightarrow \overline{\mathbb{R}}_+$,

$$\text{let } \int^+ h d\mu \stackrel{\text{def}}{=} \begin{cases} \int h d\mu & \text{if } h \text{ is } \mu\text{-bounded} \\ \infty & \text{otherwise} \end{cases} \quad (\text{namely if } h < \infty \text{ on some compact sat. support of } \mu)$$

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- ❖ **Thm.** If μ is non-zero and bounded and X is a T_2 patch-compact subset of a stably compact, 2nd countable space, then:
 - μ is τ -smooth
 - $\tilde{\mu}$ is the **largest** continuous R -valuation that **approximates** μ .

Summary

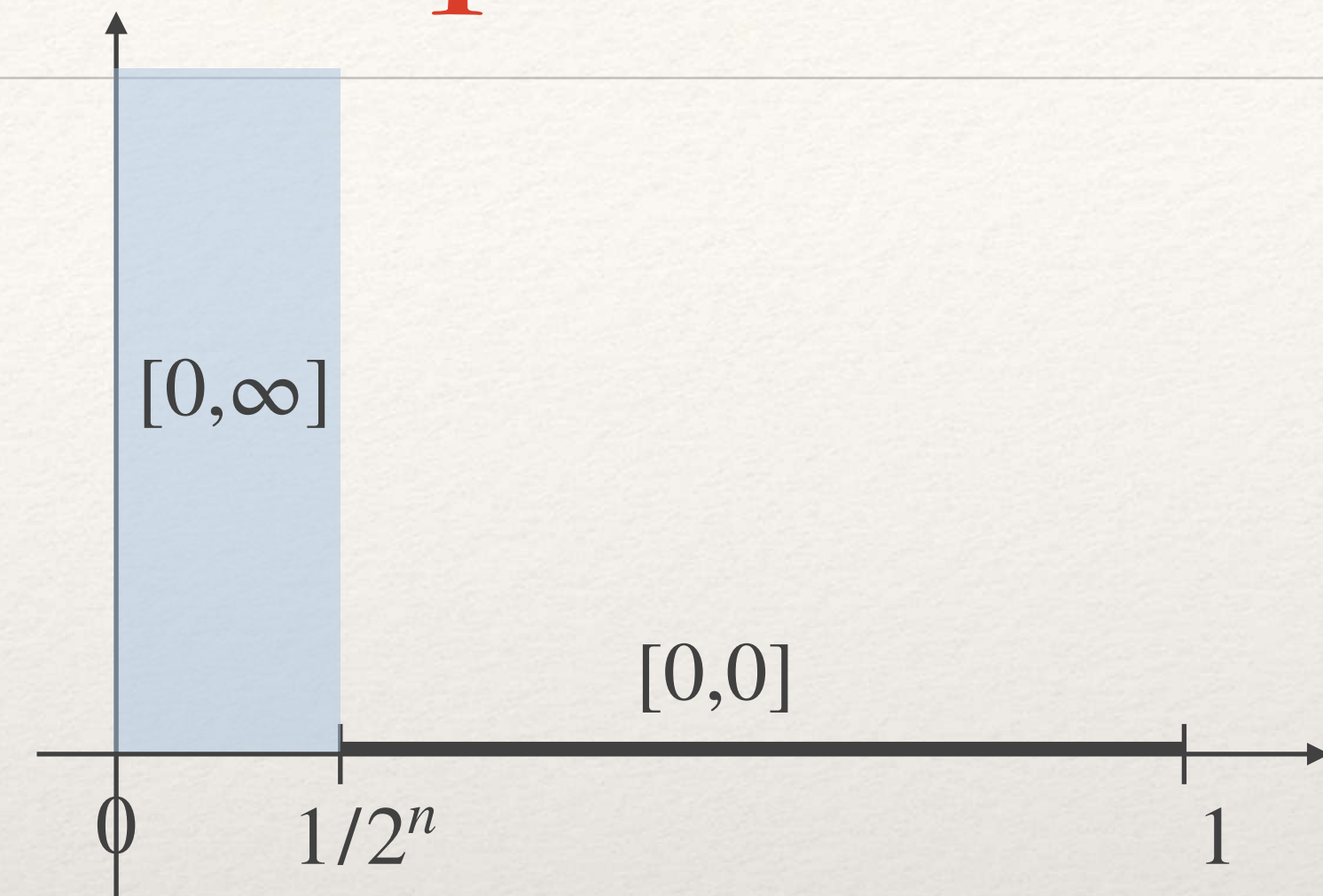
- ❖ We can extend continuous valuations to continuous R -valuations where R is any **Abelian d-rag**
- ❖ When $R = \overline{\mathbb{R}}_+$, generalizes ordinary continuous valuations
When $R = \mathbf{IR}_+^*$, we retrieve something close to
Edalat's interval-valued integration operators
- ❖ We obtain **commutative** monads of **minimal** R -valuations
- ❖ Under some assumptions, there is a **largest** (=most precise) continuous \mathbf{IR}_+^* -valuation approximating a given non-zero bounded measure
- ❖ That largest continuous \mathbf{IR}_+^* -valuation is **minimal**
in the case of Lebesgue measure on $[0, 1] \subseteq \mathbf{IR}$

Oops, it seems you've got too far
... or have you?

I knew you would ask that question

Let $h_n: [0,1] \rightarrow \mathbf{IR}_+^*$

❖ (Note: Edalat integrates with values in \mathbf{IR} ,
not \mathbf{IR}_+^*)



❖ With the obvious variant of Edalat's integral,

$$\int h_n d\lambda = \frac{1}{2^n} \times [0, \infty] + \left(1 - \frac{1}{2^n}\right) \times [0, 0] = [0, \infty]$$

❖ With $h \hat{=} \sup_n^\uparrow h_n$ $\int h d\lambda = [0, 0] \neq \sup_n^\uparrow \int h_n d\lambda$

❖ Hence that obvious variant is **not continuous** — we repair this by using \int^+ .