Laboratoire
Méthodes
Formelles

Jean Goubault-Larrecq

On completeness for Kantorovich-Rubinstein quasi-metrics

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université PARIS-SACLAY
cnrs
école normale supérieure paris—saclay

Outline

- The classical setting: **complete** metric spaces of probability measures
- Extending this to **quasi-**metric spaces through domain theory
- **Warning.** There is way too much to be explained here.
  Please forgive me for skipping a lot of details
  (while giving a pretty technical talk altogether, still ☹️)
- Main reference:

📖 JGL (2021) *Kantorovich-Rubinstein quasi-metrics I: spaces of measures and of continuous valuations.* Topology and its Applications 295

The classical setting

A theorem of Prohorov's

- Let $\mathbf{P}(X) \stackrel{\text{def}}{=} \{\text{Borel probability measures on } X\}$
  We give it the **weak** topology, generated by $[U > r] \stackrel{\text{def}}{=} \{\mu \in \mathbf{P}(X) \mid \mu(U) > r\}$,
  where $U \in \mathcal{O}(X), r \in \mathbb{R}_+$
- Recall that a **Polish space** is a second-countable, completely metrizable space
- **Theorem (Prohorov 1956).** For every Polish space X, $\mathbf{P}(X)$ is Polish.

A theorem of Prohorov's

- **Theorem (Prohorov 1956).** For every Polish space X, $\mathbf{P}(X)$ is Polish.
- Crux of the argument: given a metric d on X,
  - **lift** d to a metric d_{LP} on $\mathbf{P}(X)$
  - show that, if d is complete, then d_{LP} is **complete**
  - show that, if X is second-countable, then the open ball topology of d_{LP} **coincides** with the weak topology
- Prohorov invented, and used the **Levy-Prohorov** metric d_{LP} for that task

The Kantorovich-Rubinstein metric

- **Theorem (Prohorov 1956).** For every Polish space X, $\mathbf{P}(X)$ is Polish.
- Instead of d_{LP}, we may use the **1-bounded Kantorovich-Rubinstein** metric d^1_{KR}

$$d^1_{\mathrm{KR}}(\mu, \nu) \stackrel{\text{def}}{=} \sup_h \left| \int h d\mu - \int h d\nu \right|$$

… a kind of L^1 metric, where h ranges over the 1-bounded 1-Lipschitz maps

- I will present **quasi-metric** extensions of this result
- We will proceed through **domain theory**

Quasi-metrics and formal balls

Quasi-metrics

- A **quasi-metric** d on X is an **asymmetric form** of a metric:
  - ~~$d(x, y)=d(y, x)$~~ [no symmetry required]
  - $d(x, z) \leq d(x, y) + d(y, z)$ [triangular inequality]
  - $d(x, x)=0$
  - if $d(x, y)=0$ **and** $d(y, x)=0$ then $x=y$
- **Specialization ordering** $x \leq y$ iff $d(x, y)=0$

  [I'll tell you later what topology I prefer; for now, think open ball topology]

Fundamental examples of quasi-metrics

- Any **metric** is a quasi-metric
  [with equality as specialization ordering]

- Any **poset** $(X, \leq)$ gives rise to a quasi-metric $d_\leq(x, y) \stackrel{\text{def}}{=} 0$ if $x \leq y$,
  ∞ otherwise
  [and its specialization ordering is $\leq$]

- On $\mathbb{R}$, $\mathbb{R}_+$, $\overline{\mathbb{R}}_+$: $d_\mathbb{R}(s, t) \stackrel{\text{def}}{=} (s - t)_+$, namely 0 if $s \leq t$, $s - t$ otherwise
  [specialization ordering is $\leq$, but $d_\mathbb{R} \neq d_\leq$]

Hence quasi-metrics **unify** classical metric topology and order theory

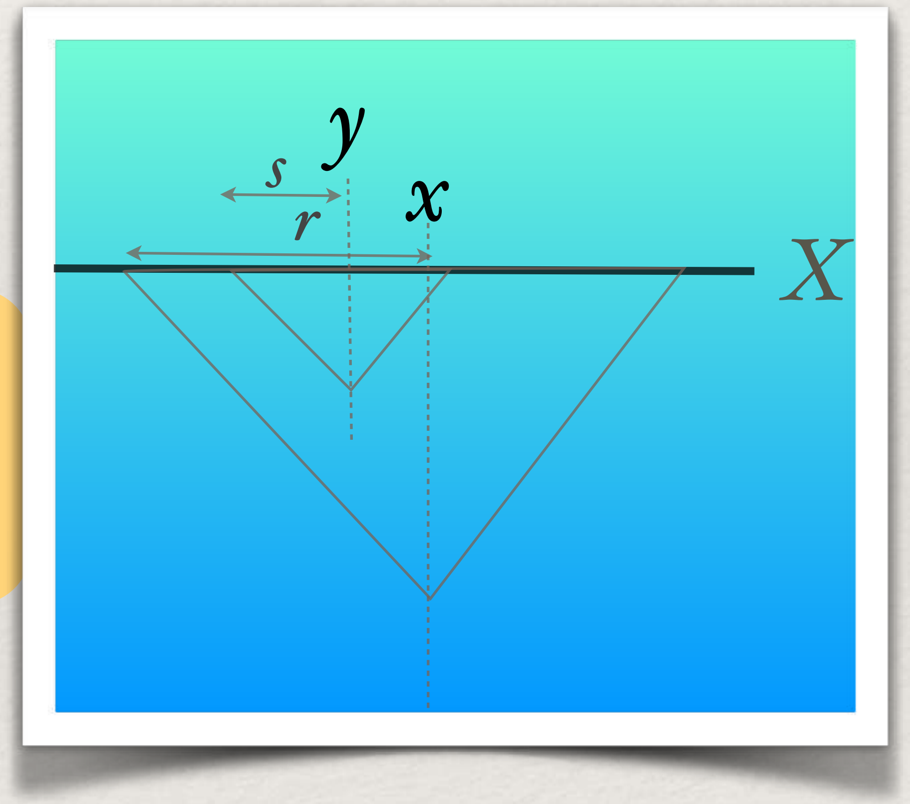
Completeness

- A **metric** is **complete** iff every Cauchy net converges
- Similarly, one can define a quasi-metric *d* as being (Yoneda-)complete
  iff every (forward) Cauchy net has a so-called *d*-limit
- Instead of using this definition, I will use an equivalent one
  based on **formal balls**
  (Weihrauch&Schreiber81,
  Edalat&Heckmann98,
  Kostanek&Waszkiewicz10)

Formal balls

- Let (X, d) be a quasi-metric space. A **formal ball** is a pair (x, r) of:
  — a point x of X [the **center**]
  — a number $r \in \mathbb{R}_+$ [the **radius**]

- This is **syntax** for an actual (closed) ball



- Formal balls are ordered by: $(x, r) \leq^{d^+} (y, s)$ iff $d(x, y) \leq r - s$
  [in particular, $r \geq s$]

- This implies $B^d_{x,\leq r} \supseteq B^d_{y,\leq s}$ (reverse inclusion of formal balls),
  but is not equivalent to it

The Kostanek-Waszkiewicz theorem

- There is a **poset** $\mathbf{B}(X, d)$ of **formal balls**,

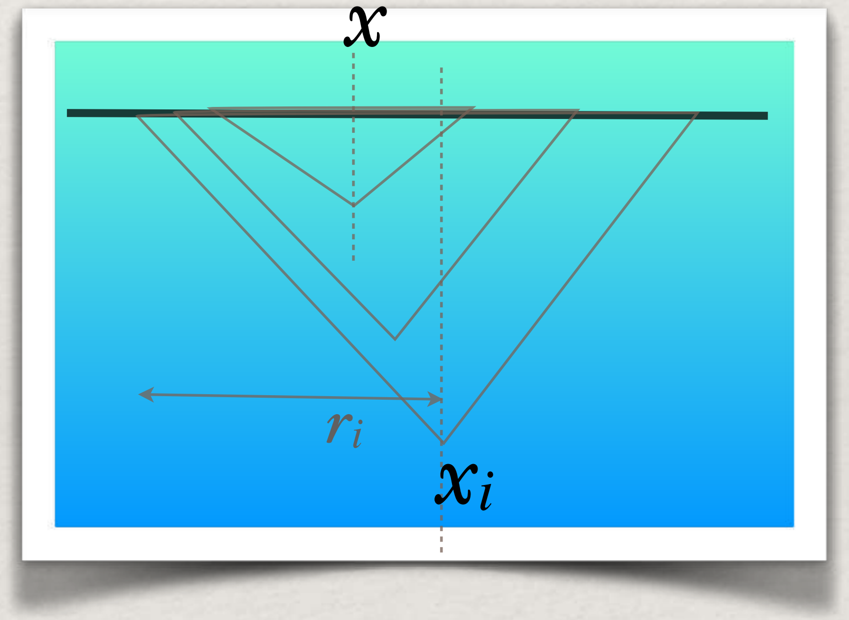
  ordered by $(x, r) \leq^{d^+} (y, s)$ iff $d(x, y) \leq r - s$

- We take the following theorem as a definition (Kostanek&Waszkiewicz10)

- **Defn.** The quasi-metric space (X, d) is:
  - **complete** iff $\mathbf{B}(X, d)$ is a **dcpo**
  - **continuous complete** iff $\mathbf{B}(X, d)$ is a **continuous dcpo**.

The idea behind the Kostanek-Waszkiewicz theorem

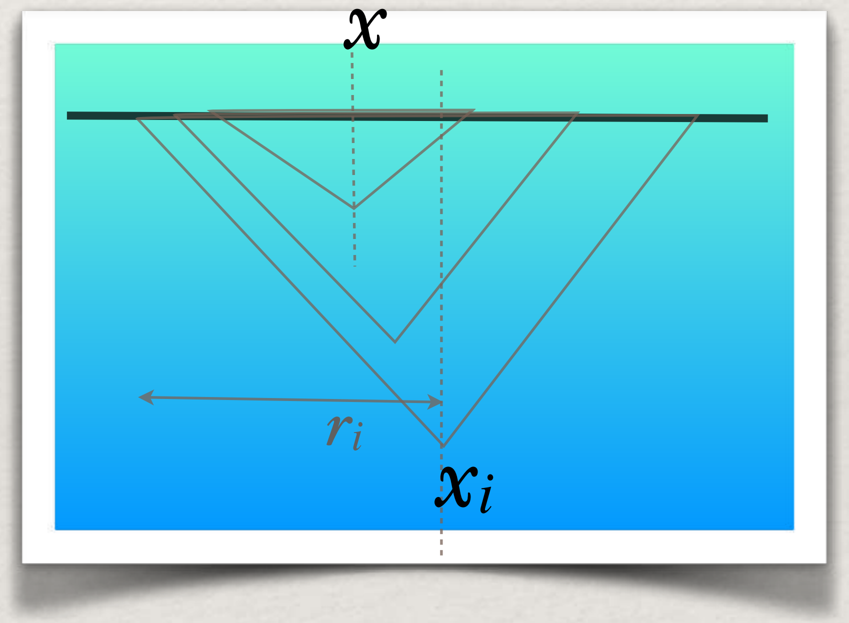
- Consider any monotone net of formal balls $(x_i, r_i)_{i \in I, \sqsubseteq}$ such that $\inf_{i \in I} r_i = 0$
- Then $(x_i)_{i \in I, \sqsubseteq}$ is a (forward) Cauchy net
  whose speed of convergence is controlled by the radii r_i
  — I call $(x_i, r_i)_{i \in I, \sqsubseteq}$ a **Cauchy-weighted net,**
  $(x_i)_{i \in I, \sqsubseteq}$ a **Cauchy-weightable net**



- A supremum (x, r) of the net $(x_i, r_i)_{i \in I, \sqsubseteq}$ must have $r = 0$,
  and x must be the so-called ***d*-limit** of $(x_i)_{i \in I, \sqsubseteq}$
  — I will take that as **definition** of a d-limit

Examples of continuous complete quasi-metrics

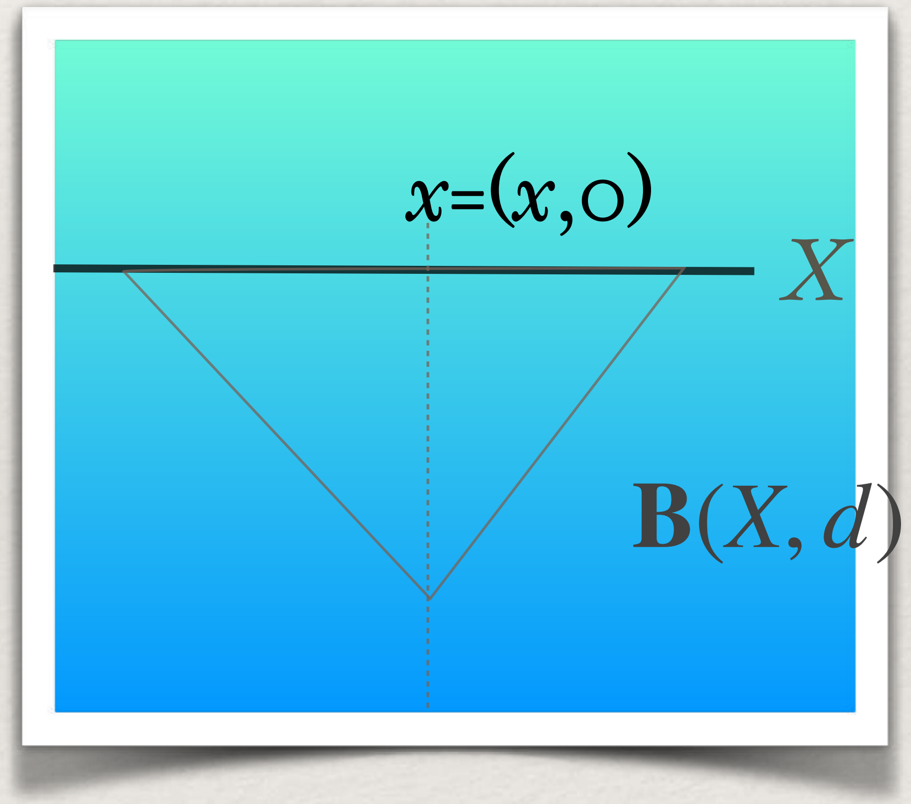
- (X, d) is [continuous] complete iff $\mathbf{B}(X, d)$ is a [continuous] dcpo
- For d **metric**, complete iff complete in the usual sense
  and this implies **continuity** (Edalat&Heckmann96)
- For $d = d_\leq$ (arising from a **poset**),
  — $(X, d_\leq)$ **complete** iff $(X, \leq)$ **dcpo**
  — $(X, d_\leq)$ **continuous complete** iff $(X, \leq)$ **continuous dcpo**



- Recall $d_{\mathbb{R}}(s, t) \stackrel{\text{def}}{=} (s - t)_+$: **continuous complete** on $\overline{\mathbb{R}}_+$,
  not even complete on $\mathbb{R}$, $\mathbb{R}_+$ (missing ∞)

The d-Scott topology

- The usual topology on a quasi-metric space (X, d) is the **open ball** topology
- Let me instead consider the ***d*-Scott topology**, defined below
  - Inject X into $\mathbf{B}(X, d)$ by equating x with $(x, 0)$
  - Give $\mathbf{B}(X, d)$ the Scott topology of $\leq^{d^+}$
  - The d-Scott topology on X is the subspace topology induced by the embedding into $\mathbf{B}(X, d)$



- **Note.** d-Scott=open ball on **metric spaces**
  d-Scott=Scott on **posets**
  $d_{\mathbb{R}}$-Scott=Scott on $\mathbb{R}, \mathbb{R}_+, \overline{\mathbb{R}}_+$

A nagging point: standardness

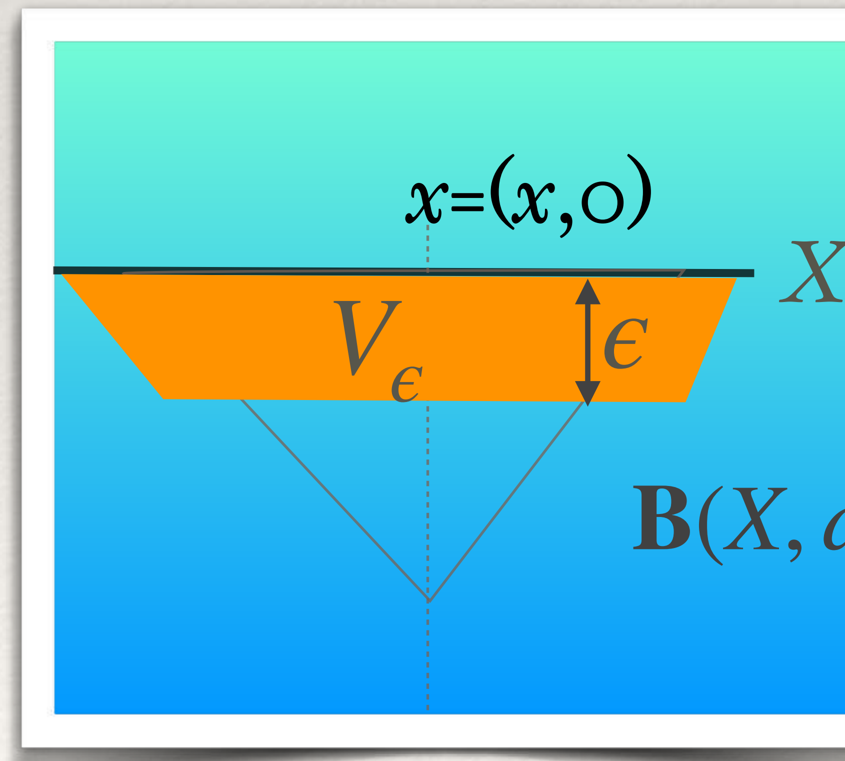
📖 JGL & K.M. Ng (2017) *A few notes on formal balls*. LMCS 13(4:18)1–34

- X, d is **standard** iff

  for every directed family $(x_i, r_i)_{i \in I}$ of formal balls,
  for every **shift** $s \geq -\inf_{i \in I} r_i$,
  $(x_i, r_i)_{i \in I}$ has a supremum $\Leftrightarrow$ $(x_i, r_i + s)_{i \in I}$ has a supremum

- It is unfortunate that not all quasi-metric spaces are standard

- If X, d is standard, then lots of nice things happen:
  — the radius map $(x, r) \mapsto r$ is **Scott-continuous** from $\mathbf{B}(X, d)$ to $\overline{\mathbb{R}}_+^{op}$
  — $V_\epsilon \stackrel{\text{def}}{=} \{(x, r) \in \mathbf{B}(X, d) \mid r < \epsilon\}$ is **Scott-open** in $\mathbf{B}(X, d)$
  — $X = \bigcap^{\downarrow}_{n \in \mathbb{N}} V_{1/2^n}$ is G_δ in $\mathbf{B}(X, d)$



- Fortunately: **Thm.** Every complete quasi-metric space is standard.

Lipschitz maps

- $f : X, d \to Y, \partial$ is **a-Lipschitz** iff for all $x, y \in X$, $\partial(f(x), f(y)) \leq a \,.\, d(x, y)$
- This entails continuity wrt. the underlying open ball topologies,
  not wrt. the underlying d-Scott topologies
- The domain-theoretic view:
  let $\mathbf{B}_a(f)$ map $(x, r) \in \mathbf{B}(X, d)$ to $(f(x), a \,.\, r) \in \mathbf{B}(Y, \partial)$
- **Fact.** f is a-Lipschitz iff $\mathbf{B}_a(f)$ is monotonic
- **Defn.** f is **a-Lipschitz continuous** iff $\mathbf{B}_a(f)$ is Scott-continuous
- Between **metric** spaces, Lipschitzianity implies continuity
  Between posets, a-Lipschitz=monotonic, a-Lipschitz continuous=Scott-continuous

Spaces of Lipschitz continuous maps

When the target space is $\overline{\mathbb{R}}_+$

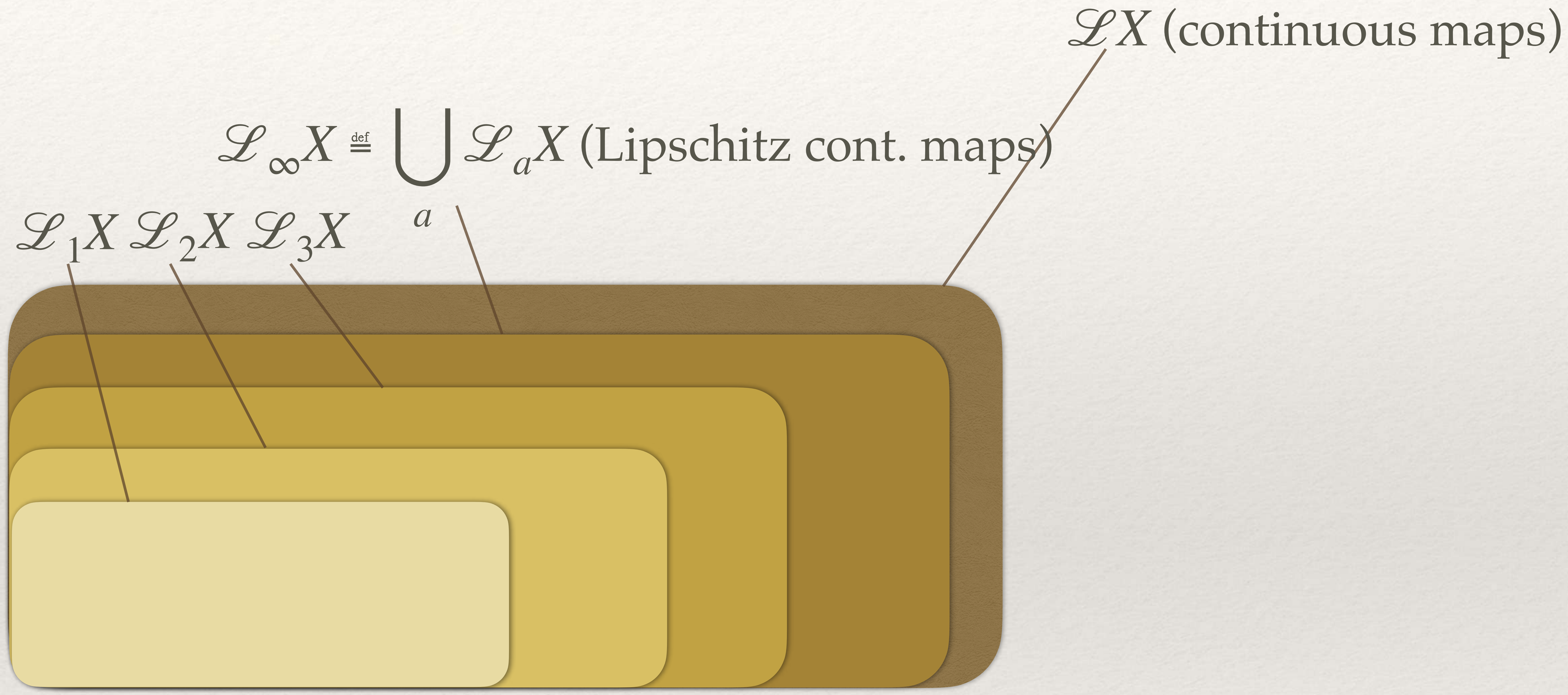
- Special case $Y = \overline{\mathbb{R}}_+$, $\partial = d_{\mathbb{R}}$
- $h : X, d \to \overline{\mathbb{R}}_+$ is ***a*-Lipschitz** iff for all $x, y \in X$, $h(x) \leq h(y) + a \,.\, d(x, y)$
- h is ***a*-Lipschitz continuous** iff $h' : \mathbf{B}(X, d) \to \mathbb{R} \cup \{\infty\}$,
  $$h'(x, r) \stackrel{\text{def}}{=} h(x) - a \,.\, r$$
  is **Scott-continuous** [provided X, d is standard]
- I will write $\mathscr{L}_a X$ for the set of ***a*-Lipschitz continuous** maps from X to $\overline{\mathbb{R}}_+$
- and also $\mathscr{L}^1_a X$ for those bounded from above by a

$\mathscr{L}_a X \subseteq \mathscr{L} X$

- Let $\mathscr{L}X \stackrel{\text{def}}{=} \{\text{continuous maps} : X \to \overline{\mathbb{R}}_+\}$ / with the Scott topology
  where X has the *d*-Scott topology
  and $\overline{\mathbb{R}}_+$ has the $d_{\mathbb{R}}$-Scott = Scott topology
- **Fact.** If X, d is standard, then $\mathscr{L}_a X \subseteq \mathscr{L} X$.
- *Proof.* For every $h \in \mathscr{L}_a X$, $h \colon X \to \mathbf{B}(X, d) \to \overline{\mathbb{R}} \cup \{\infty\}$
  $$x \;\cong\; (x,0) \;\mapsto\; h'(x,0) \qquad [h'(x,r) \stackrel{\text{def}}{=} h(x) - a\,.\,r]$$
- Hence I will equip $\mathscr{L}_a X$ with the **subspace topology** from $\mathscr{L}X$
  (this is **not** the Scott topology on $\mathscr{L}_a X$ in general!)

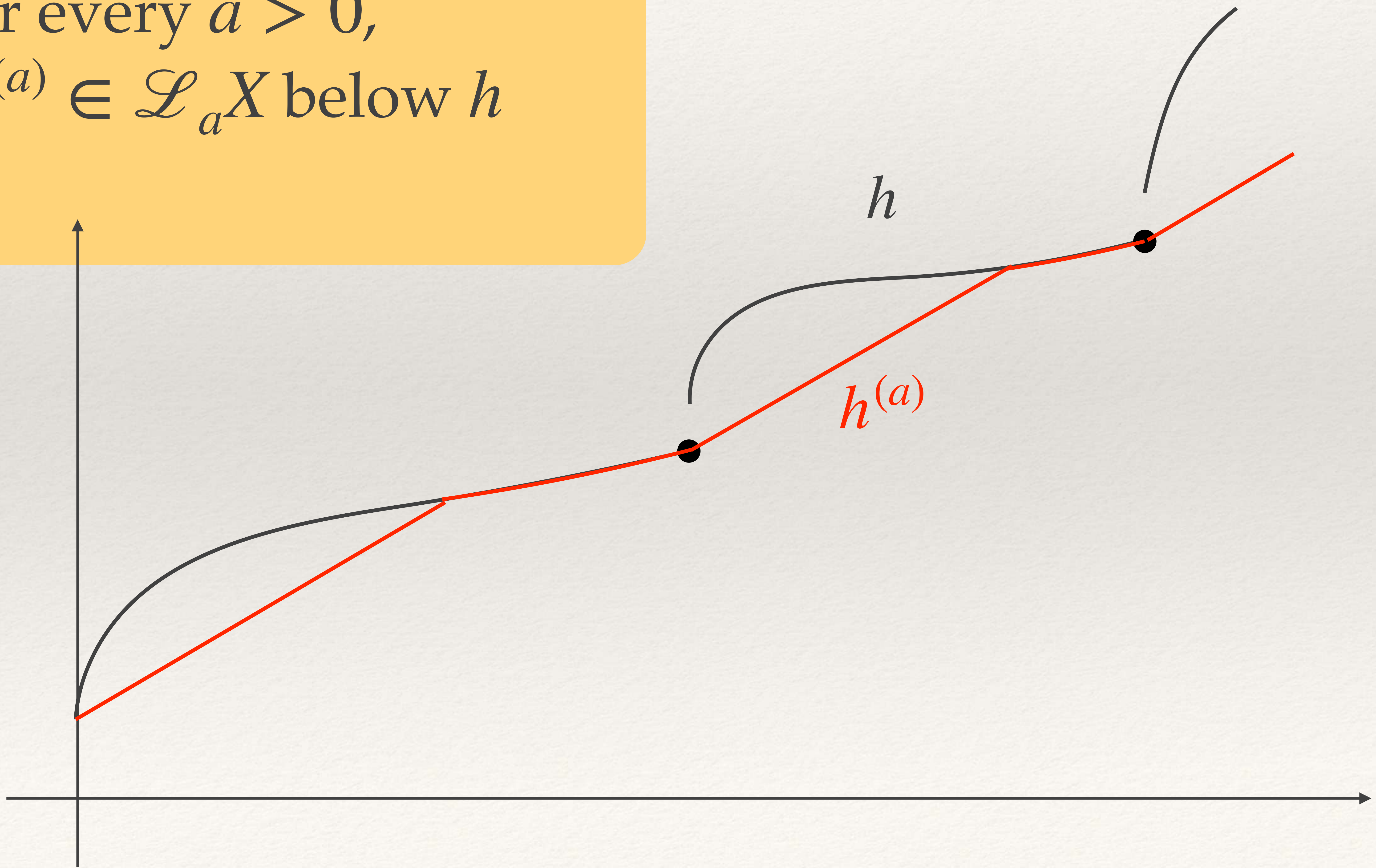
$\mathscr{L}_a X \subseteq \ldots \subseteq \mathscr{L}_\infty X \subseteq \mathscr{L} X$

(Assuming X standard.)



Lipschitz approximation

- **Thm.** Let X, d be standard.
  For every $h \in \mathscr{L}X$, for every $a > 0$,
  — there is a **largest** $h^{(a)} \in \mathscr{L}_a X$ below h
  — $h = \sup_a^{\uparrow} h^{(a)}$



Continuous valuations

Continuous valuations

- Instead of working with measures, let me consider **continuous valuations** = maps $\nu \colon \mathcal{O}(X) \to \overline{\mathbb{R}}_+$ that are:
  — **strict**: $\nu(\emptyset) = 0$
  — **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
  — **Scott-continuous**.

- Let $\mathbf{V}(X) \stackrel{\text{def}}{=} \{$continuous valuations on $X\}$,
  $\mathbf{V}_{\leq 1}(X) \stackrel{\text{def}}{=} \{$**subprobability** continuous valuations ν on X (i.e., $\nu(X) \leq 1)\}$,
  $\mathbf{V}_1(X) \stackrel{\text{def}}{=} \{$**probability** continuous valuations ν on X (i.e., $\nu(X) = 1)\}$

$\mathbf{V}_\bullet(X)$, if I don't want to be more specific

- **Theorem.** Every continuous valuation on a continuous complete quasi-metric space (with the d-Scott topology) extends to a (τ-smooth) Borel measure.

📖 M. de Brecht, JGL, X. Jia, Zh. Lyu (2019). *Domain-complete and LCS-complete spaces*. ISDT'19

Linear previsions

what I call a **linear prevision**

- For every $\nu \in \mathbf{V}(X)$, $G\colon h \in \mathscr{L}X \to \int h d\nu$ is:
  — **linear**: $G(a \,.\, h) = aG(h)$, $G(h_1 + h_2) = G(h_1) + G(h_2)$
  — **Scott-continuous**.

- **Thm (« baby Riesz »).** Continuous valuations ≅ linear previsions.

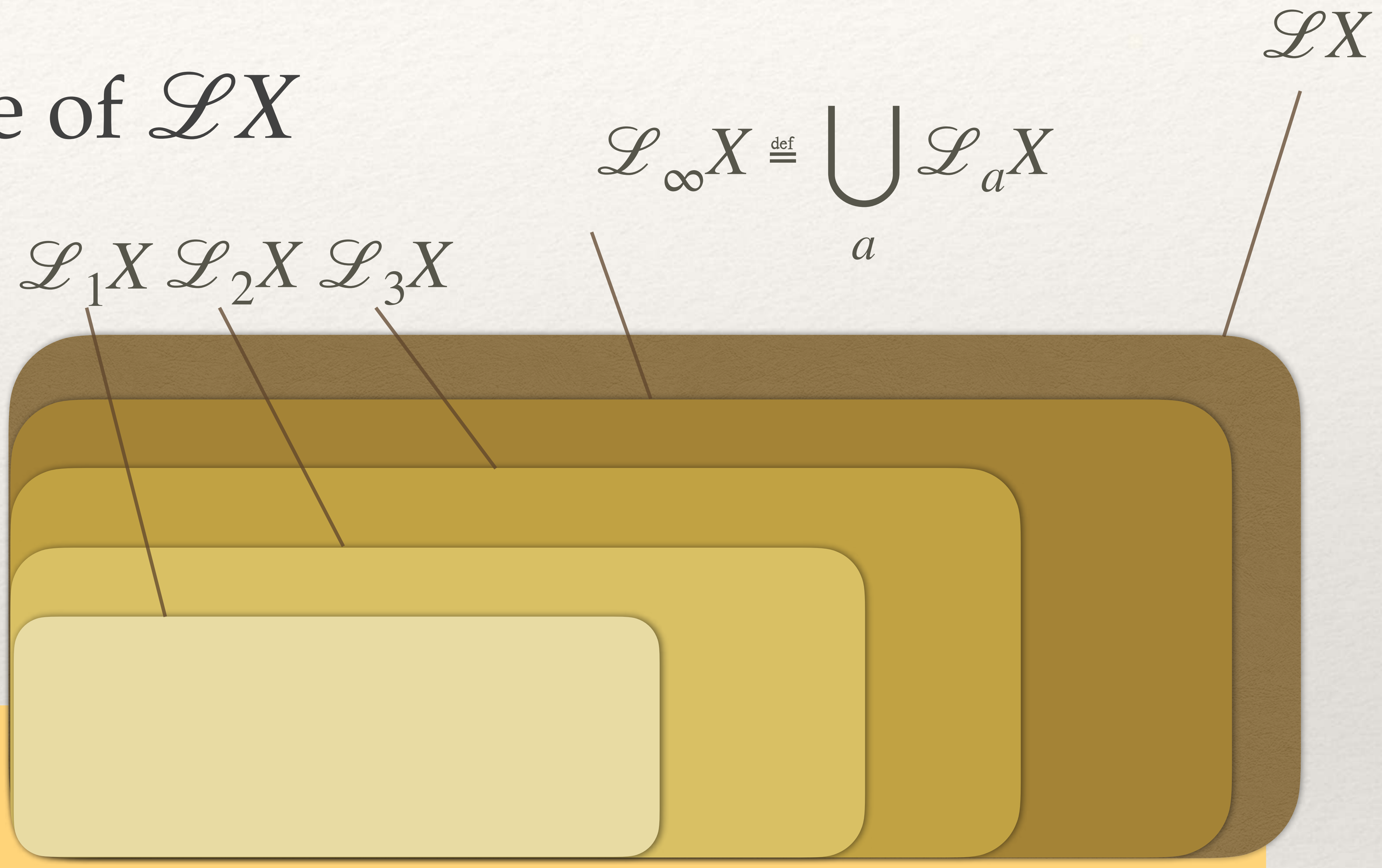
- *Proof sketch.* Given any linear prevision G, we retrieve ν by

$$\nu(U) \stackrel{\text{def}}{=} G(\chi_U)$$

Linear $\mathscr{L}$-previsions

- Linear previsions G are defined on the whole of $\mathscr{L}X$
- Given X, d standard, the restriction $G_{|\mathscr{L}_\infty X}$ is:
  — **linear**
  — **continuous** : $\mathscr{L}_\infty X \to \overline{\mathbb{R}}_+$

what I call a **linear $\mathscr{L}$-prevision**



- **Prop.** Linear prevision ≅ linear $\mathscr{L}$-prevision
- *Proof sketch.* Given linear $\mathscr{L}$-prevision H, define

$$G(h) \stackrel{\text{def}}{=} \sup_a^{\uparrow} H(h^{(a)}) \qquad [\text{recall } h^{(a)} = \text{Lipschitz approximation}]$$

- **Note.** Similar results with spaces of **bounded** Lipschitz maps.

The Kantorovich-Rubinstein **quasi**-metrics

The bounded KR quasi-metric

📖 JGL (2021) *Kantorovich-Rubinstein quasi-metrics I: spaces of measures and of continuous valuations*. T&A. 295

- Recall the classical definition: $d^1_{KR}(\mu, \nu) \stackrel{\text{def}}{=} \sup_h \left| \int h d\mu - \int h d\nu \right|$

  where h ranges over the 1-bounded, 1-Lipschitz maps

- It is fitting to change this to the quasi-metric setting into:

$$d^1_{KR}(\mu, \nu) \stackrel{\text{def}}{=} \sup_{h \in \mathscr{L}^1_1 X} \left(\int h d\mu - \int h d\nu \right)_+$$

ranging over 1-bounded, 1-Lipschitz **continuous** maps

using $d_{\mathbb{R}} = (\_ - \_)_+$ instead of $|\_ - \_|$

- **Fact.** The two definitions are equivalent

  for continuous valuations on a **metric** space (X, d)

The unbounded KR quasi-metric

- I will concentrate on the **unbounded** variant:

$$d_{\mathrm{KR}}(\mu,\nu) \stackrel{\text{def}}{=} \sup_{h \in \mathscr{L}_1 X} \left(\int h d\mu - \int h d\nu \right)_+$$

ranging over **all**
1-Lipschitz **continuous**
maps

Is $\mathbf{V}_\bullet(X), d_{KR}$ complete?

- We aim to show that $\mathbf{B}(\mathbf{V}_\bullet(X), d_{KR})$ is a dcpo
  Hence we consider a monotone net $(\nu_i, r_i)_{i\in I,\sqsubseteq}$ with ν_i continuous valuations

- A formal ball (ν, r) is an upper bound of that net iff

$$d_{KR}(\nu_i, \nu) \le r_i - r \quad \text{for every } i \in I$$

iff
$$\int h d\nu_i \le \int h d\nu + r_i - r \quad \text{for all } i \in I, h \in \mathscr{L}_1 X$$

$$d_{KR}(\mu, \nu) \stackrel{\text{def}}{=} \sup_{h \in \mathscr{L}_1 X} \left(\int h d\mu - \int h d\nu \right)_+$$

- This suggests that the least upper bound is given by:

— $r \stackrel{\text{def}}{=} \inf_{i \in I} r_i$

— $\int h d\nu \stackrel{\text{def}}{=} \sup_{i \in I} \left(\int h d\nu_i - r_i + r \right)$ for every $h \in \mathscr{L}_1 X$

Is $\mathbf{V}_\bullet(\mathcal{X}), d_{\mathrm{KR}}$ complete?

- More generally (and multiplying by an arbitrary a),
- This suggests that the least upper bound is given by:
— $r \stackrel{\text{def}}{=} \inf_{i \in I} r_i$
— $G(h) \stackrel{\text{def}}{=} \sup_{i \in I} \left(\int h d\nu_i - a \,.\, r_i + a \,.\, r \right)$ for every $h \in \mathscr{L}_a X$

We will call this the **naive supremum** of $(\nu_i, r_i)_{i \in I, \sqsubseteq}$

- **Thm.** G is a well-defined linear map from $\mathscr{L}_\infty X$ to $\overline{\mathbb{R}}_+$.
If G is **continuous**, then:
— it is a linear $\mathscr{L}$-prevision
— $G \cong$ a unique continuous valuation ν, which is the **desired d_{KR}-limit.**

A frustrating situation

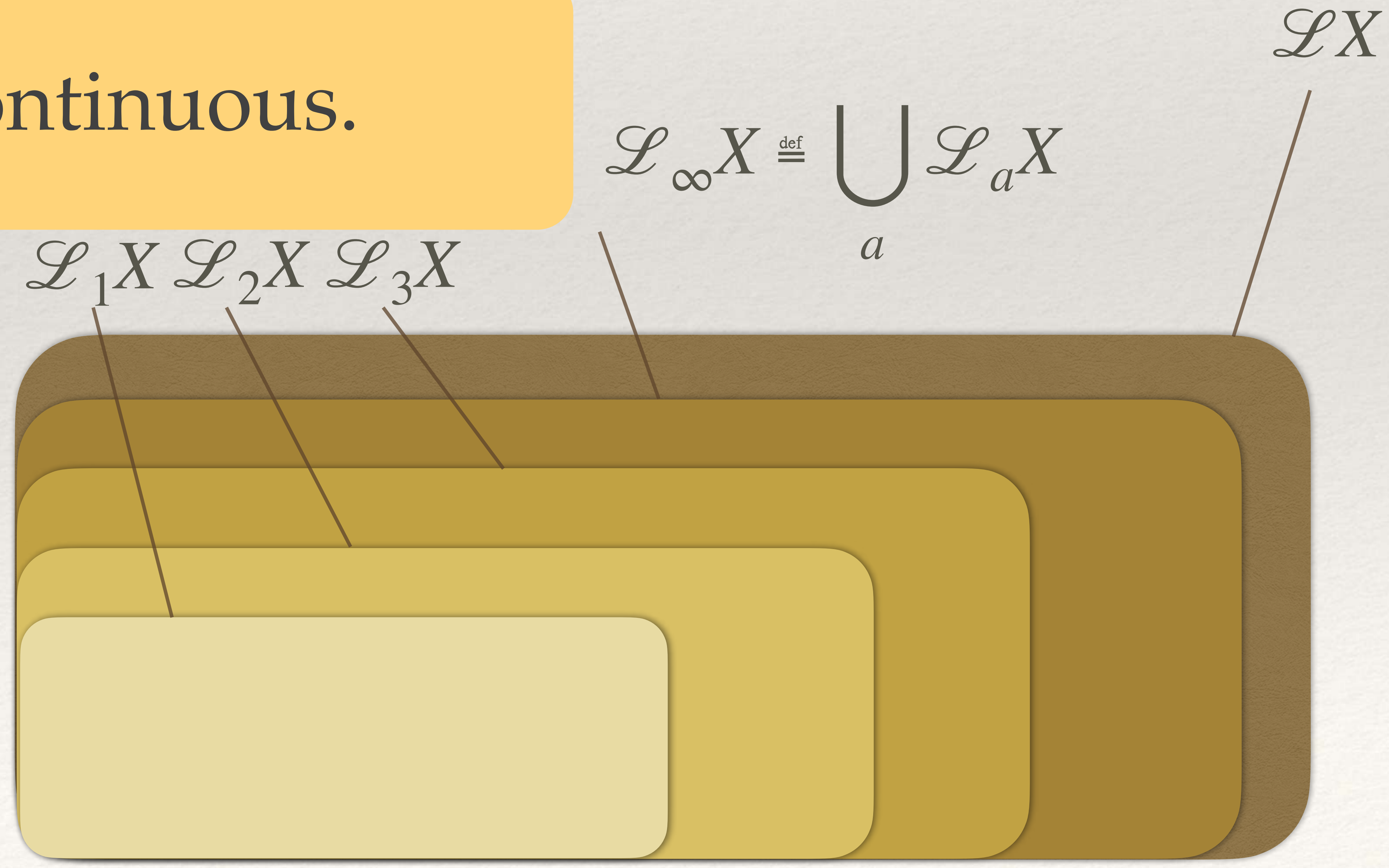
- The only missing thing is to show that the naive supremum

$$G(h) \stackrel{\text{def}}{=} \sup_{i \in I} \left(\int h d\nu_i - a \,.\, r_i + a \,.\, r \right)$$

is **continuous from** $\mathscr{L}_\infty X$ to $\overline{\mathbb{R}}_+$.

- **Fact.** G restricted to every subspace $\mathscr{L}_a X$ **is** continuous.

- That is not enough to conclude
— unless topology of $\mathscr{L}_\infty X$ is **determined** by those of its subspaces $\mathscr{L}_a X$ (=colimit) [open problem!]

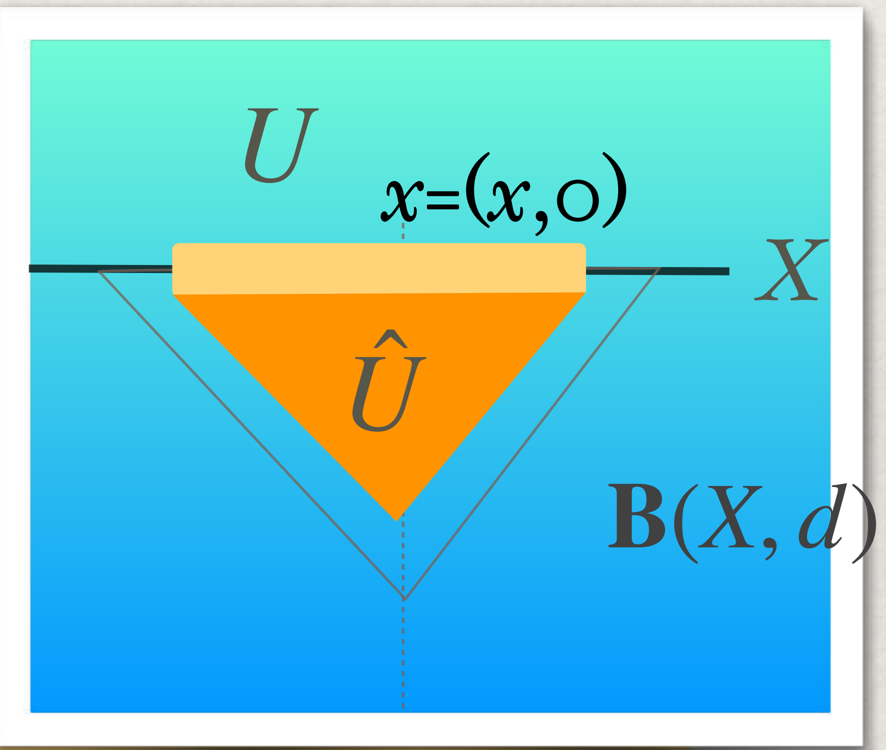


Lipschitz-regular quasi-metric spaces

The assignment $U \mapsto \hat{U}$

📖 JGL (2020) *Some topological properties of spaces of Lipschitz continuous maps on quasi-metric spaces*. T&A. 282

- Recall that the d-Scott topology on X is the subspace topology induced by the embedding $x \in X \mapsto (x,0) \in \mathbf{B}(X, d)$
- For every open subset U of X, let $\hat{U}$ be the **largest Scott-open** subset of $\mathbf{B}(X, d)$ such that $U = \hat{U} \cap X$
- The map $U \mapsto \hat{U}$ is right adjoint to $V \mapsto V \cap X$, hence preserves arbitrary meets

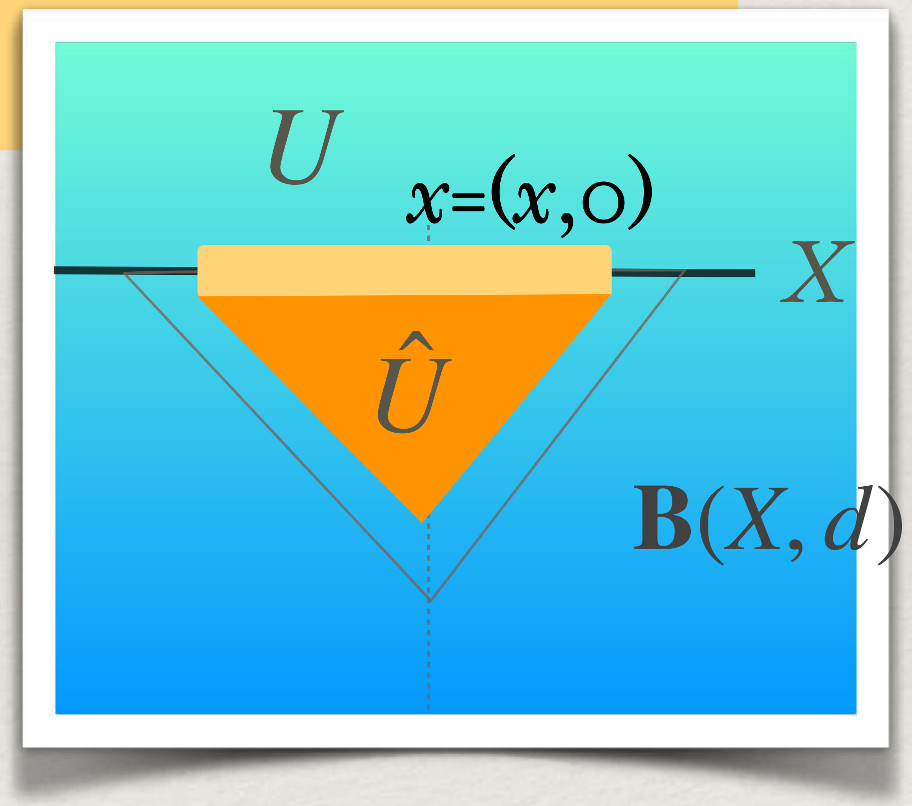


- **Defn.** X, d is **Lipschitz-regular** iff $U \mapsto \hat{U}$ is Scott-continuous (= if X is **finitarily embedded** into $\mathbf{B}(X, d)$, see Escardó 98)

📖 M. H. Escardó (1998) *Properly injective spaces and function spaces*. T&A 89 (1–2).

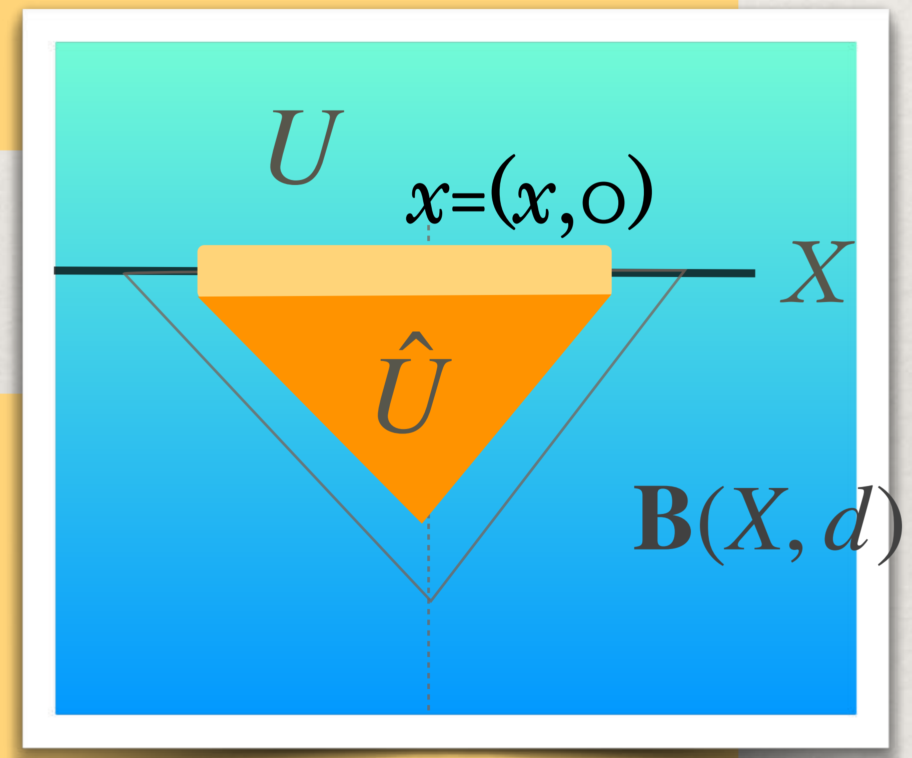
Lipschitz-regular spaces

- **Defn.** X, d is **Lipschitz-regular** iff $U \mapsto \hat{U}$ is Scott-continuous
- **Prop.** If X, d is Lipschitz-regular, then topology of $\mathscr{L}_\infty X$ is determined by those of its subspaces $\mathscr{L}_a X$.
- *Proof sketch.* The canonical injection $i_a \colon \mathscr{L}_a X \to \mathscr{L} X$ and the a-Lipschitz approximation map $r_a \colon \mathscr{L} X \to \mathscr{L}_a X$, $h \mapsto h^{(a)}$ form an **embedding-projection pair**.



Lipschitz-regular spaces and completeness

- **Defn.** X, d is **Lipschitz-regular** iff $U \mapsto \hat{U}$ is Scott-continuous
- **Prop.** If X, d is Lipschitz-regular, then topology of $\mathscr{L}_\infty X$ is determined by those of its subspaces $\mathscr{L}_a X$.
- As a corollary,
- **Prop.** If X, d is Lipschitz-regular, then:
  — $\mathbf{V}_\bullet(X)$, d_{KR} is complete
  — directed suprema of formal balls $(\nu_i, r_i)_{i \in I}$ are **naive suprema**:
  $$G(h) \stackrel{\text{def}}{=} \sup_{i \in I} \left(\int h d\nu_i - a \,.\, r_i + a \,.\, r \right), \text{ for every } h \in \mathscr{L}_a X, a > 0$$



Is Lipschitz-regularity acceptable?

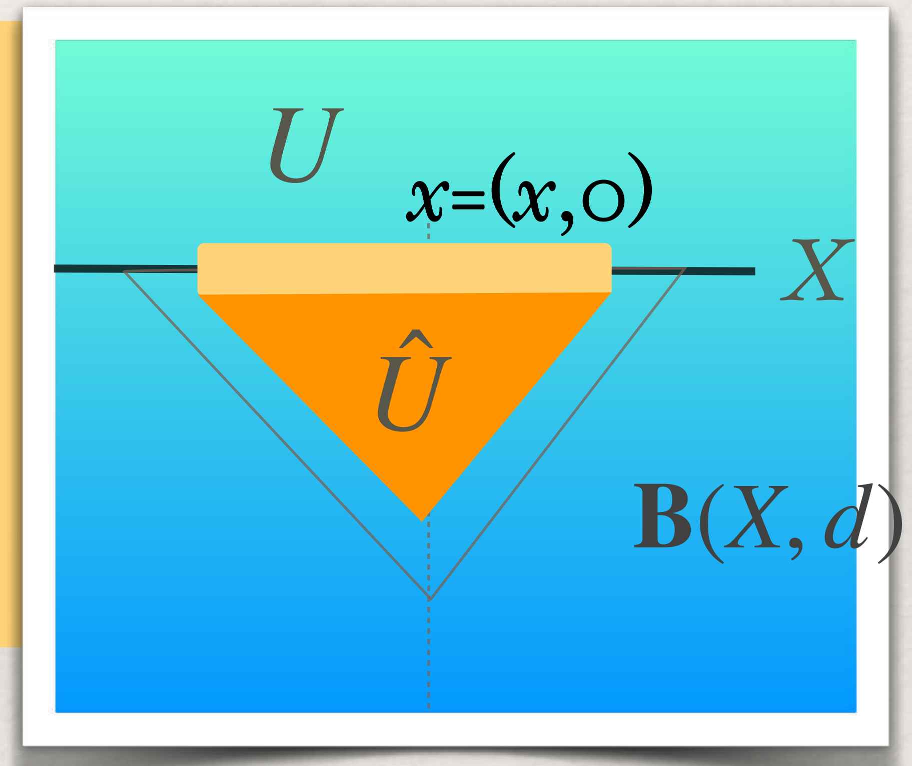
- *Hmm … no.*
- If X, d algebraic complete, then

  Lipschitz-regular ⇔ has relatively compact open balls
- That is a pretty strong property — stronger than local compactness and remember that local compactness is not required in the metric case!

A miracle

- $\mathbf{B}(X, d)$ itself is a quasi-metric space, with
  $$d^+((x, r), (y, s)) \stackrel{\text{def}}{=} \max(d(x, y) - r + s, 0)$$
  and d^+-Scott topology = Scott topology

- **Thm.** For every quasi-metric space X, d, $\mathbf{B}(X, d), d^+$ is Lipschitz-regular [in fact, $U \mapsto \hat{U}$ preserves **all** unions].

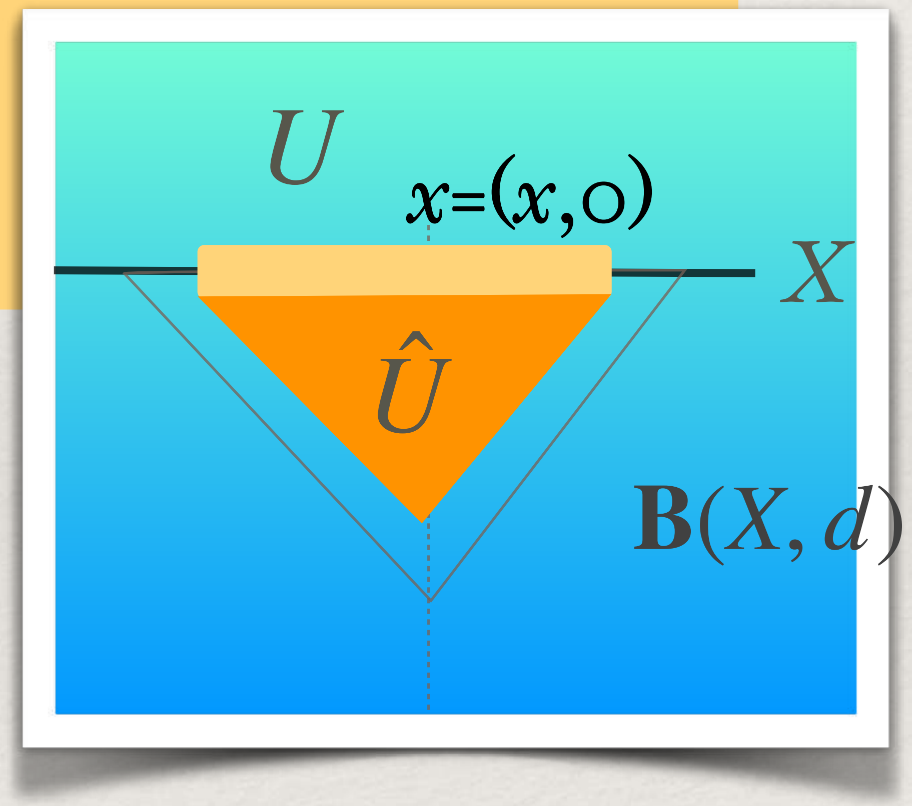


- Let me only give a sketch of the argument…
  (assuming X, d standard, which will be enough for our purposes)

Formal ball monads

📖 JGL (2019) *Formal ball monads*. Topology and its Applications 263:372-391

- **Thm.** There is a **monad** $(\mathbf{B}, \eta, \mu)$ on the category of standard quasi-metric spaces where:
  — $\mathbf{B}(f) : (x, r) \mapsto (f(x), r)$ [what I wrote $\mathbf{B}_1$ earlier on]
  — $\eta : x \in X \mapsto (x,0) \in \mathbf{B}(X, d)$
  — $\mu : ((x, r), s) \mapsto (x, r + s)$



- In fact a **left KZ-monad**:
  $\mathbf{B}\eta \leq \eta \Leftrightarrow \mu \dashv \eta \Leftrightarrow \mathbf{B}\eta \dashv \mu$
  so we know what the **B**-algebras are [but I won't spell it out here]

📖 M. H. Escardó (1998) *Properly injective spaces and function spaces*. T&A 89 (1–2).

- **Prop.** For every **B**-algebra $\alpha : \mathbf{B}(X, d) \to X$,
  we have $\hat{U} = \alpha^{-1}(U)$; in particular, X, d is Lipschitz-regular.

- $\mathbf{B}(X, d), d^+$ is the **free B**-algebra, hence is Lipschitz-regular.

Back to the completeness theorem

Embedding into the formal ball model

- Recall:

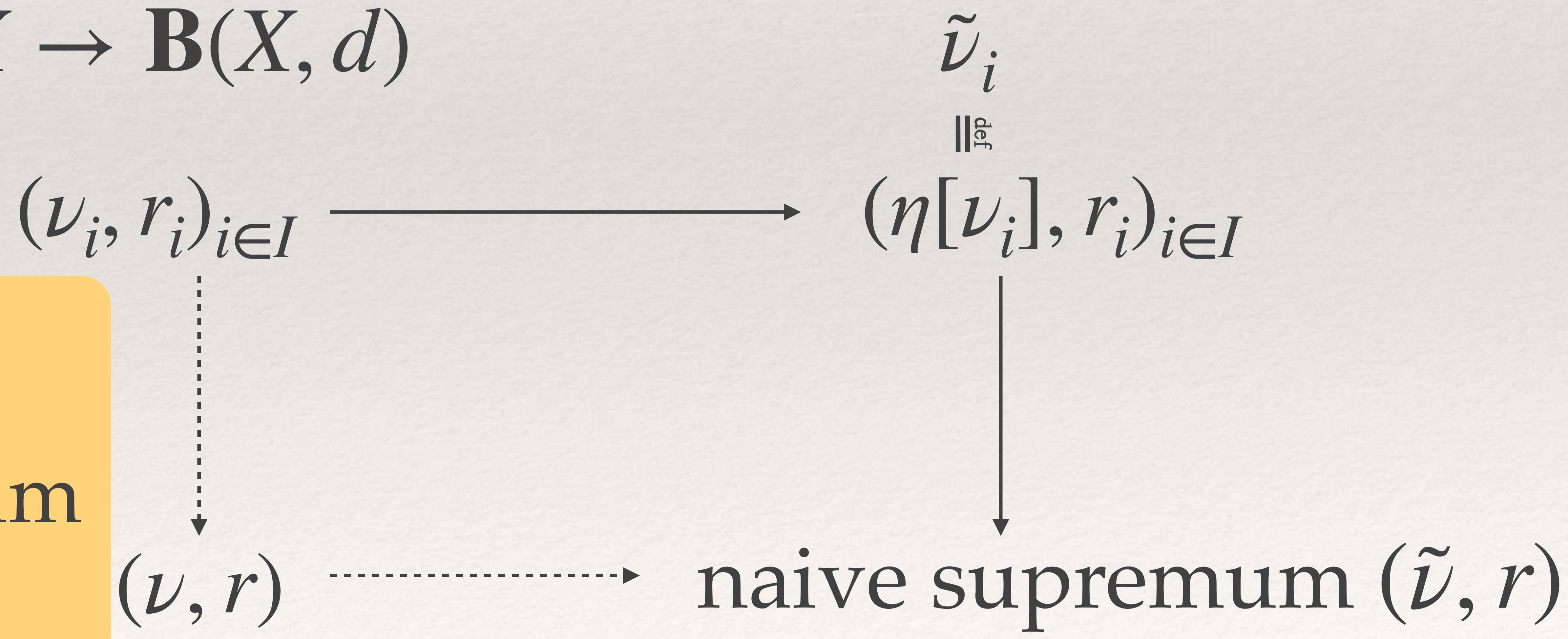
Prop. If X, d is Lipschitz-regular, then:
— $\mathbf{V}_\bullet(X), d_{\mathrm{KR}}$ is complete
— directed suprema of formal balls $(\nu_i, r_i)_{i\in I}$ are naive suprema:

$$G(h) \stackrel{\text{def}}{=} \sup_{i\in I} \left(\int h d\nu_i - a \, . \, r_i + a \, . \, r \right), \text{ for every } h \in \mathscr{L}_a X, a > 0$$

- Since $\mathbf{B}(X, d), d^+$ is always Lipschitz-regular, for every space X, d we have:
— $\mathbf{V}_\bullet(\mathbf{B}(X, d))$ is complete
— directed suprema of formal balls $(\tilde{\nu}_i, r_i)_{i\in I}$ are naive suprema
[each $\tilde{\nu}_i$ is a continuous valuation on $\mathbf{B}(X, d)$]

Embedding into the formal ball model

- **Recap.** Directed suprema of formal balls $(\tilde{\nu}_i, r_i)_{i \in I}$ are naive suprema
  [each $\tilde{\nu}_i$ is a continuous valuation on $\mathbf{B}(X, d)$]
- Now consider any directed family of formal balls $(\nu_i, r_i)_{i \in I}$
  [each ν_i a continuous valuation on X]
- Let $\tilde{\nu}_i \stackrel{\text{def}}{=} \eta[\nu_i]$, image valuation of ν_i by $\eta \colon X \to \mathbf{B}(X, d)$
- **Lemma.** If $\tilde{\nu} = \eta[\nu]$ for some $\nu \in \mathbf{V}_\bullet(X)$ then (ν, r) is the (naive) supremum of $(\nu_i, r_i)_{i \in I}$



Supports

- Let η be an inclusion of spaces $X \to B$

  Defn. A continuous valuation $\tilde{\nu} \in \mathbf{V}_{\bullet}(B)$ is **supported on** X if and only if $\tilde{\nu} = \eta[\nu]$ for some $\nu \in \mathbf{V}_{\bullet}(X)$

- **Lemma.** $\tilde{\nu} \in \mathbf{V}_{\bullet}(B)$ is supported on X iff for all open subsets V, W of B such that $V \cap X = W \cap X$,

  $$\tilde{\nu}(V) = \tilde{\nu}(W)$$

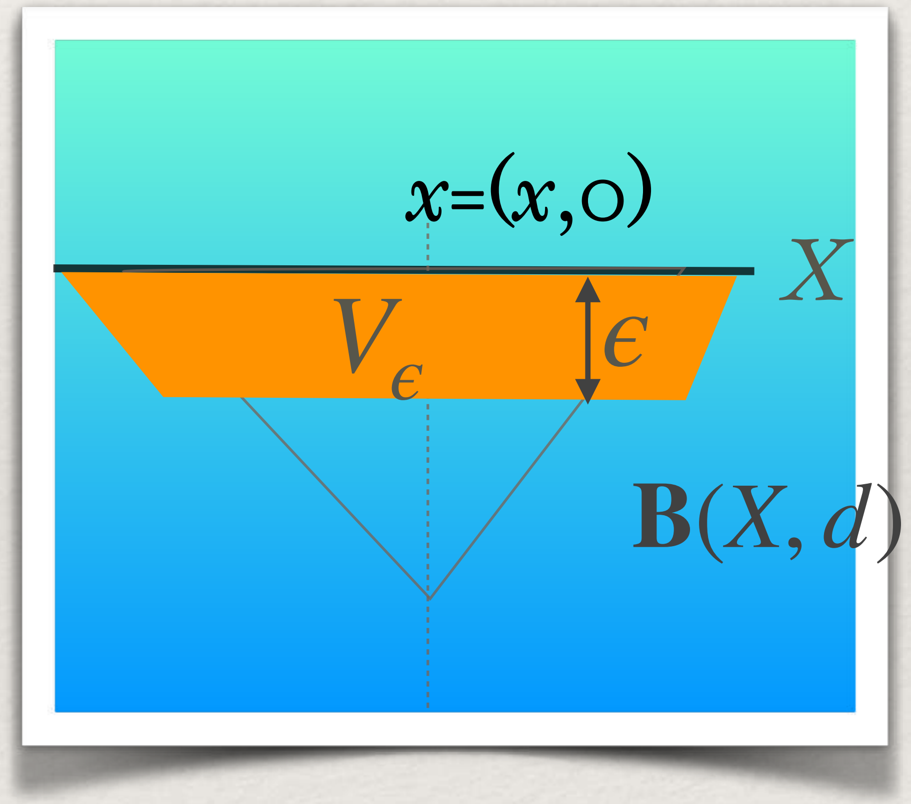
- *Proof:* Exercise.

- Almost there! It remains to check that

  the naive supremum $\tilde{\nu} \in \mathbf{V}_{\bullet}(\mathbf{B}(X, d))$ is supported on X

Another source of frustration

- The best we can prove (for now) is that
  the naive supremum $\tilde{\nu} \in \mathbf{V}_\bullet(\mathbf{B}(X,d))$ is supported on $V_\epsilon = \{(x,r) \mid r < \epsilon\}$
  for every $\epsilon > 0$
- Recall that $X = \bigcap^{\downarrow}_{n \in \mathbb{N}} V_{1/2^n}$
- Does this imply that $\tilde{\nu}$ is supported on X?
- **Yes** if X, d is **continuous complete**
  and $\tilde{\nu}$ is **bounded** ($\tilde{\nu}(\mathbf{B}(X,d)) < \infty$): see next slide

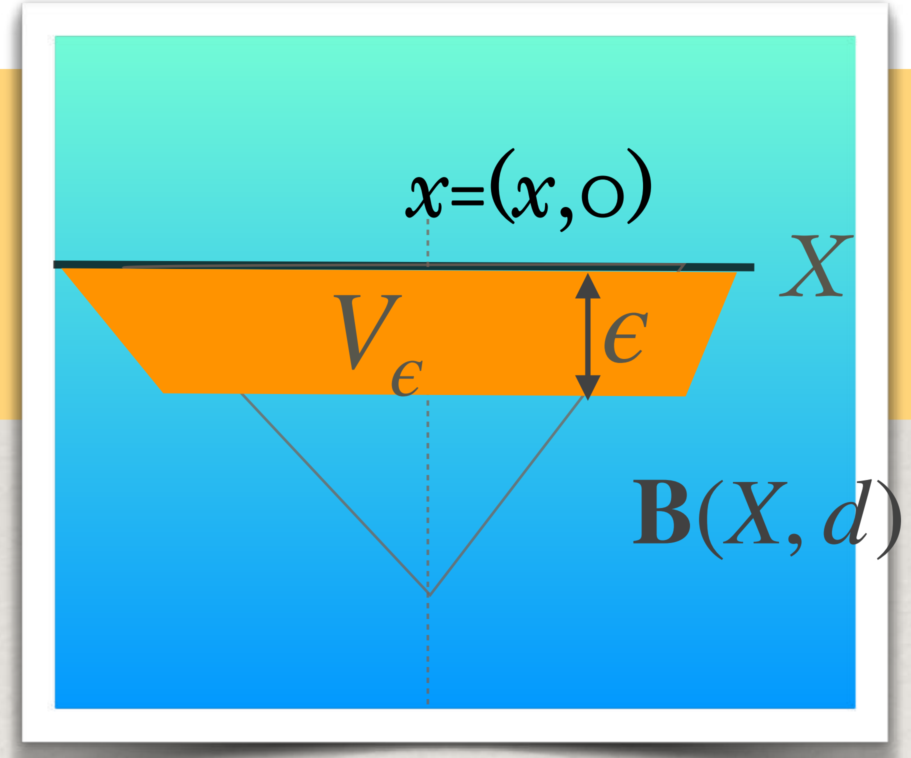


Invoking some measure theory

- If X, d is continuous complete, then $\mathbf{B}(X, d)$ is a **continuous dcpo** and $X = \bigcap^{\downarrow}_{n \in \mathbb{N}} V_{1/2^n}$ is G_δ, hence Borel, in it.

📖 J. D. Lawson (1982) *Valuations on continuous lattices*. Math. Arbeitspapiere 27:204–225
📖 M. Alvarez-Manilla (2000) *Measure theoretic results for continuous valuations on partially ordered spaces*. Ph.D. thesis, Imperial College, London
📖 K. Keimel and J. Lawson (2005) *Measure extension theorems for T_0 spaces*. T&A 149(1–3):57–83

- **Thm.** Every continuous valuation (e.g., $\tilde{\nu}$) on a continuous dcpo (or even a locally compact sober space) extends to a **Borel measure.**



- Since $\tilde{\nu}$ supported on V_ϵ, for ev … $d)$,
  $\tilde{\nu}(V) = \tilde{\nu}(V \cap V_\epsilon)$ [V … ntersection with V_ϵ]

a bounded measure commutes with infs of countable chains

- Then, if $\tilde{\nu}$ is **bounded**, $\tilde{\nu}(V) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V \cap V_{1/2^n}) = \tilde{\nu}(\bigcap_{n \in \mathbb{N}} V \cap V_{1/2^n}) = \tilde{\nu}(V \cap X)$

the inf of a **constant** sequence

$V \cap X$ is Borel, and $\tilde{\nu}$ is a measure

- In particular, if $V \cap X = W \cap X$, then $\tilde{\nu}(V) = \tilde{\nu}(W)$: $\tilde{\nu}$ is **supported on** X.

We are done!

- Summing up:
- **Thm.** For every continuous complete quasi-metric space X, d, $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$ are **complete** under the d_{KR} quasi-metric.
(And directed suprema of formal balls are computed as naive suprema.)

Final remarks (a long list…)

We are done!

Are we, really?

- Summing up:
- **Thm.** For every **continuous complete** quasi-metric space X, d,
  $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$ are **complete** under the d_{KR} quasi-metric.
  (And directed suprema of formal balls are computed as naive suprema.)
- What about $\mathbf{V}(X)$ (unbounded valuations)? — *open problem*
- In fact, $\mathbf{V}_{\leq 1}(X)$ is even **continuous** complete
  as well as $\mathbf{V}_1(X)$ if X, d has a so-called root
  [would need another talk]
  Goes through preservation of **algebraic** completeness,
  using the remarkable fact that for X, d continuous complete,
  $\mathscr{L}_a X$ is **stably compact**, and topology=compact-open=pointwise

📖 JGL (2020) *Some topological properties of spaces of Lipschitz continuous maps on quasi-metric spaces.* T&A. 282

Are we done yet?

- Using the **bounded** version d^1_{KR}, we obtain:
- **Thm.** For every **continuous complete** quasi-metric space X, d, $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$ are **continuous complete** under d^1_{KR}.
  (And directed suprema of formal balls are computed as naive suprema.)
- If X, d is **algebraic** complete, then so are $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$, too.
- When X is an algebraic dcpo, d^1_{KR} is Sünderhauf (1998)'s sup quasi-metric, and we retrieve his result that $\mathbf{V}_{\leq 1}(X)$ is algebraic complete in that case.

The weak topology

- Using the **bounded** version d^1_{KR}, we obtain the
- **Thm.** For every continuous complete quasi-metric space X, d,
  d^1_{KR}-Scott topology = weak topology on $\mathbf{V}_1(X)$ and $\mathbf{V}_{\leq 1}(X)$.
- **Not** true for d_{KR}-Scott topology, even when d metric (Kravchenko 2006).

Beyond continuous valuations: previsions

- In general, d_{KR} makes sense on any space of functionals $: \mathscr{L}X \to \overline{\mathbb{R}}_+$, not just **linear** previsions (=continuous valuations)
- **Defn.** A **prevision** is any Scott-continuous map $F : \mathscr{L}X \to \overline{\mathbb{R}}_+$ satisfying $F(a\,.\,h) = a\,.\,F(h)$
- **Defn.** $d_{\mathrm{KR}}(F, F') \stackrel{\text{def}}{=} \sup_{h \in \mathscr{L}_1 X} (F(h) - F'(h))_+$
- We have similar theorems for discrete / sublinear / superlinear previsions
- In particular, **discrete** previsions ≅ Hoare / Smyth hyperspaces, with asymmetric variants of the Pompeiu-Hausdorff quasi-metric

📖 JGL (2022) *Kantorovich-Rubinstein quasi-metrics II: hyperspaces and powerdomains*. Topology and its Applications 305

Any questions? …meanwhile, a few references

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C. 📖 M. de Brecht, JGL, X. Jia, Zh. Lyu (2019). *Domain-complete and LCS-complete spaces*. ISDT'19

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G. 📖 JGL (2022) *Kantorovich-Rubinstein quasi-metrics II: hyperspaces and powerdomains*. Topology and its Applications 305

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