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On completeness for Kantorovich-Rubinstein quasi-metrics

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Outline

❖ The classical setting: **complete** metric spaces of probability measures
❖ Extending this to **quasi**-metric spaces through domain theory
❖ **Warning.** There is way too much to be explained here.
  Please forgive me for skipping a lot of details
  (while giving a pretty technical talk altogether, still 😞)
❖ Main reference:

Topology and its Applications 295
The classical setting
A theorem of Prohorov’s

Let \( \mathcal{P}(X) \equiv \{ \text{Borel probability measures on } X \} \)
We give it the \textbf{weak} topology, generated by \([U > r] \equiv \{ \mu \in \mathcal{P}(X) \mid \mu(U) > r \}\),
where \( U \in \mathcal{O}(X), \ r \in \mathbb{R}_+ \)

Recall that a \textbf{Polish space} is a second-countable, completely metrizable space

\textbf{Theorem (Prohorov 1956).} For every Polish space \( X \), \( \mathcal{P}(X) \) is Polish.
A theorem of Prohorov's

**Theorem (Prohorov 1956).** For every Polish space $X$, $P(X)$ is Polish.

- Crux of the argument: given a metric $d$ on $X$,
  - **lift** $d$ to a metric $d_{LP}$ on $P(X)$
  - show that, if $d$ is complete, then $d_{LP}$ is **complete**
  - show that, if $X$ is second-countable, then the open ball topology of $d_{LP}$ **coincides** with the weak topology
  - Prohorov invented, and used the **Levy-Prohorov** metric $d_{LP}$ for that task
The Kantorovich-Rubinstein metric

- **Theorem (Prohorov 1956).** For every Polish space $X$, $P(X)$ is Polish.

- Instead of $d_{LP}$, we may use the 1-bounded Kantorovich-Rubinstein metric $d_{KR}^1$

  $$d_{KR}^1(\mu, \nu) = \sup_h \left| \int h d\mu - \int h d\nu \right|$$

  ... a kind of $L^1$ metric, where $h$ ranges over the 1-bounded 1-Lipschitz maps

- I will present quasi-metric extensions of this result

- We will proceed through domain theory
Quasi-metrics and formal balls
Quasi-metrics

❖ A quasi-metric \(d\) on \(X\) is an asymmetric form of a metric:

- \(d(x, y) = d(y, x)\) [no symmetry required]
- \(d(x, z) \leq d(x, y) + d(y, z)\) [triangular inequality]
- \(d(x, x) = 0\)
- if \(d(x, y) = 0\) and \(d(y, x) = 0\) then \(x = y\)

❖ Specialization ordering \(x \leq y\) iff \(d(x, y) = 0\)

[I’ll tell you later what topology I prefer; for now, think open ball topology]
Fundamental examples of quasi-metrics

- Any **metric** is a quasi-metric
  [with equality as specialization ordering]
- Any **poset** $(X, \leq)$ gives rise to a quasi-metric $d_{\leq}(x, y) = 0$ if $x \leq y$,
  $\infty$ otherwise
  [and its specialization ordering is $\leq$]
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- Any **poset** \((X, \leq)\) gives rise to a quasi-metric
  \[d_{\leq}(x, y) = 0 \text{ if } x \leq y, \quad \infty \text{ otherwise}\]
  [and its specialization ordering is \(\leq\)]

- On \(\mathbb{R}, \mathbb{R}_+, \overline{\mathbb{R}}_+\): \(d_{\mathbb{R}}(s, t) = (s - t)_+\), namely 0 if \(s \leq t\), \(s - t\) otherwise
  [specialization ordering is \(\leq\), but \(d_{\mathbb{R}} \neq d_{\leq}\)]
Fundamental examples of quasi-metrics

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  [specialization ordering is \(\leq\), but \(d_{\mathbb{R}} \neq d_{\leq}\)]

Hence quasi-metrics **unify** classical metric topology and order theory
Completeness

- A **metric** is **complete** iff every Cauchy net converges.
- Similarly, one can define a quasi-metric \( d \) as being (Yoneda-)complete iff every (forward) Cauchy net has a so-called \( d \)-limit.
Completeness

- A metric is complete iff every Cauchy net converges.
- Similarly, one can define a quasi-metric $d$ as being (Yoneda-)complete iff every (forward) Cauchy net has a so-called $d$-limit.
- Instead of using this definition, I will use an equivalent one based on formal balls (Weihrauch&Schreiber81, Edalat&Heckmann98, Kostanek&Waszkiewicz10).
Let \((X, d)\) be a quasi-metric space. A **formal ball** is a pair \((x, r)\) of:

- a point \(x\) of \(X\) [the center]
- a number \(r \in \mathbb{R}_+\) [the radius]

This is **syntax** for an actual (closed) ball.
Formal balls

Let \((X, d)\) be a quasi-metric space. A **formal ball** is a pair \((x, r)\) of:

- a point \(x\) of \(X\) [the **center**]
- a number \(r \in \mathbb{R}_+\) [the **radius**]

This is **syntax** for an actual (closed) ball

Formal balls are ordered by: \((x, r) \leq^d (y, s)\) iff \(d(x, y) \leq r - s\)

[in particular, \(r \geq s\)]

This implies \(B^d_{x, \leq r} \supseteq B^d_{y, \leq s}\) (reverse inclusion of formal balls), but is not equivalent to it
The Kostanek-Waszkiewicz theorem

- There is a **poset** $\mathcal{B}(X, d)$ of **formal balls**, ordered by $(x, r) \leq^{d^+} (y, s)$ iff $d(x, y) \leq r - s$

- We take the following theorem as a definition (Kostanek & Waszkiewicz 10)

  - **Defn.** The quasi-metric space $(X, d)$ is:
    - **complete** iff $\mathcal{B}(X, d)$ is a **dcpo**
    - **continuous complete** iff $\mathcal{B}(X, d)$ is a **continuous dcpo**.
The idea behind the Kostanek-Waszkiewicz theorem

- Consider any monotone net of formal balls \((x_i, r_i)_{i \in I, \subseteq}\) such that \(\inf_{i \in I} r_i = 0\)
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- Consider any monotone net of formal balls \((x_i, r_i)_{i \in I, \subseteq}\) such that \(\inf_{i \in I} r_i = 0\)
- Then \((x_i)_{i \in I, \subseteq}\) is a (forward) Cauchy net whose speed of convergence is controlled by the radii \(r_i\)
  - I call \((x_i, r_i)_{i \in I, \subseteq}\) a Cauchy-weighted net,
  \((x_i)_{i \in I, \subseteq}\) a Cauchy-weightable net
The idea behind the Kostanek-Waszkiewicz theorem

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  — I call \((x_i, r_i)_{i \in I, \subseteq}\) a **Cauchy-weighted net**,\((x_i)_{i \in I, \subseteq}\) a **Cauchy-weightable net**

- A supremum \((x, r)\) of the net \((x_i, r_i)_{i \in I, \subseteq}\) must have \(r = 0\),
  and \(x\) must be the so-called **d-limit** of \((x_i)_{i \in I, \subseteq}\)
  — I will take that as **definition** of a d-limit
Examples of continuous complete quasi-metrics

- \((X, d)\) is [continuous] complete iff \(B(X, d)\) is a [continuous] dcpo
- For \(d\) metric, complete iff complete in the usual sense and this implies \textbf{continuity} (Edalat&Heckmann96)
Examples of continuous complete quasi-metrics

- $(X, d)$ is [continuous] complete iff $B(X, d)$ is a [continuous] dcpo
- For $d$ metric, complete iff complete in the usual sense and this implies continuity (Edalat&Heckmann96)
- For $d=d_{\leq}$ (arising from a poset),
  - $(X, d_{\leq})$ complete iff $(X, \leq)$ dcpo
  - $(X, d_{\leq})$ continuous complete iff $(X, \leq)$ continuous dcpo
Examples of continuous complete quasi-metrics

- \((X, d)\) is [continuous] complete iff \(B(X, d)\) is a [continuous] dcpo
- For \(d\) metric, complete iff complete in the usual sense and this implies \textit{continuity} (Edalat&Heckmann96)
- For \(d=d_\leq\) (arising from a poset),
  - \((X, d_\leq)\) complete iff \((X, \leq)\) dcpo
  - \((X, d_\leq)\) continuous complete iff \((X, \leq)\) continuous dcpo
- Recall \(d_\mathbb{R}(s, t) \triangleq (s - t)_+\): \textit{continuous complete} on \(\mathbb{R}_+\), not even complete on \(\mathbb{R}, \mathbb{R}_+\) (missing \(\infty\))
The $d$-Scott topology

- The usual topology on a quasi-metric space $(X, d)$ is the **open ball** topology.
- Let me instead consider the **$d$-Scott topology**, defined below.
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  - Inject $X$ into $B(X, d)$ by equating $x$ with $(x, 0)$. 

![Diagram showing the $d$-Scott topology](image)
The $d$-Scott topology

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- Let me instead consider the **$d$-Scott topology**, defined below:
  - Inject $X$ into $B(X, d)$ by equating $x$ with $(x, 0)$.
  - Give $B(X, d)$ the Scott topology of $\leq^{d^+}$.
The \(d\)-Scott topology

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  - The \(d\)-Scott topology on \(X\) is the subspace topology induced by the embedding into \(B(X, d)\).
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  - Give $B(X, d)$ the Scott topology of $\leq d^+$.
  - The $d$-Scott topology on $X$ is the subspace topology induced by the embedding into $B(X, d)$.
- **Note.** $d$-Scott=open ball on **metric spaces**
  - $d$-Scott=Scott on **posets**
  - $d_{\mathbb{R}}$-Scott=Scott on $\mathbb{R}, \mathbb{R}_+, \overline{\mathbb{R}}_+$
A nagging point: standardness

- **X, d is standard iff**

  for every directed family \((x_i, r_i)_{i \in I}\) of formal balls,
  
  for every shift \(s \geq -\inf_{i \in I} r_i\),

  \((x_i, r_i)_{i \in I}\) has a supremum \(\Leftrightarrow (x_i, r_i + s)_{i \in I}\) has a supremum.

- It is unfortunate that not all quasi-metric spaces are standard.
A nagging point: standardness

- $X, d$ is **standard** iff
  for every directed family $(x_i, r_i)_{i \in I}$ of formal balls,
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- It is unfortunate that not all quasi-metric spaces are standard

- If $X, d$ is standard, then lots of nice things happen:
  — the radius map $(x, r) \mapsto r$ is **Scott-continuous** from $B(X, d)$ to $\mathbb{R}^o p$
  — $V_\epsilon \triangleq \{(x, r) \in B(X, d) \mid r < \epsilon\}$ is **Scott-open** in $B(X, d)$
  — $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$ in $B(X, d)$
A nagging point: standardness

- **$X, d$ is standard** iff
  
  for every directed family $(x_i, r_i)_{i \in I}$ of formal balls,
  
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- If $X, d$ is standard, then lots of nice things happen:
  
  — the radius map $(x, r) \mapsto r$ is **Scott-continuous** from $B(X, d)$ to $\mathbb{R}^+_{op}$
  
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  — $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$ in $B(X, d)$

- Fortunately: **Thm.** Every complete quasi-metric space is standard.
Lipschitz maps

- $f : X, d \to Y$, $\partial$ is $a$-Lipschitz iff for all $x, y \in X$, $\partial(f(x), f(y)) \leq a \cdot d(x, y)$
- This entails continuity wrt. the underlying open ball topologies, not wrt. the underlying $d$-Scott topologies
Lipschitz maps

- \( f : X, d \to Y, \partial \) is \( a \)-Lipschitz iff for all \( x, y \in X \), \( \partial(f(x), f(y)) \leq a \cdot d(x, y) \)
- This entails continuity wrt. the underlying open ball topologies, \textbf{not} wrt. the underlying \( d \)-Scott topologies
- The domain-theoretic view:
  - let \( B_a(f) \) map \((x, r) \in B(X, d)\) to \((f(x), a \cdot r) \in B(Y, \partial)\)

\textbf{Fact.} \( f \) is \( a \)-Lipschitz iff \( B_a(f) \) is monotonic
Lipschitz maps

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- **Defn.** \( f \) is \( a \)-Lipschitz **continuous** iff \( B_a(f) \) is Scott-continuous
Lipschitz maps

- $f : X, d \to Y, \partial$ is $a$-Lipschitz iff for all $x, y \in X$, $\partial(f(x), f(y)) \leq a \cdot d(x, y)$

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- The domain-theoretic view:
  let $B_a(f)$ map $(x, r) \in B(X, d)$ to $(f(x), a \cdot r) \in B(Y, \partial)$

- **Fact.** $f$ is $a$-Lipschitz iff $B_a(f)$ is monotonic

- **Defn.** $f$ is $a$-Lipschitz continuous iff $B_a(f)$ is Scott-continuous

- Between metric spaces, Lipschitzianity implies continuity
  Between posets, $a$-Lipschitz=monotonic, $a$-Lipschitz continuous=Scott-continuous
Spaces of Lipschitz continuous maps
When the target space is $\mathbb{R}^+$

- Special case $Y = \mathbb{R}^+, \partial = d_{\mathbb{R}}$
- $h : X, d \to \mathbb{R}^+$ is $a$-Lipschitz iff for all $x, y \in X$, $h(x) \leq h(y) + a \cdot d(x, y)$
- $h$ is $a$-Lipschitz continuous iff $h' : B(X, d) \to \mathbb{R} \cup \{\infty\}$,
  
  $h'(x, r) = h(x) - a \cdot r$

  is Scott-continuous  
  [provided $X, d$ is standard]

- I will write $\mathcal{L}_a X$ for the set of $a$-Lipschitz continuous maps from $X$ to $\mathbb{R}^+$
- and also $\mathcal{L}^1_a X$ for those bounded from above by $a$
\[ \mathcal{L}_a X \subseteq \mathcal{L}X \]

- Let \( \mathcal{L}X \equiv \{ \text{continuous maps} : X \to \overline{\mathbb{R}}_+ \} \) / with the Scott topology
  where \( X \) has the \( d \)-Scott topology
  and \( \overline{\mathbb{R}}_+ \) has the \( d_{\mathbb{R}} \)-Scott = Scott topology

- **Fact.** If \( X, d \) is standard, then \( \mathcal{L}_a X \subseteq \mathcal{L}X \).

- **Proof.** For every \( h \in \mathcal{L}_a X, h : X \to B(X, d) \to \overline{\mathbb{R}} \cup \{ \infty \} \)
  \[ x \cong (x,0) \leftrightarrow h'(x,0) \quad [h'(x, r) \triangleq h(x) - a \cdot r] \]
Let $\mathcal{L}X \equiv \{\text{continuous maps : } X \to \mathbb{R}_+\}$ / with the Scott topology where $X$ has the $d$-Scott topology and $\mathbb{R}_+$ has the $d_{\mathbb{R}}$-Scott = Scott topology

**Fact.** If $X, d$ is standard, then $\mathcal{L}_a X \subseteq \mathcal{L}X$.

**Proof.** For every $h \in \mathcal{L}_a X$, $h : X \to \mathcal{B}(X, d) \to \mathbb{R} \cup \{\infty\}$

$x \cong (x, 0) \mapsto h'(x, 0)$ \quad $[h'(x, r) \equiv h(x) - a \cdot r]$

Hence I will equip $\mathcal{L}_a X$ with the **subspace topology** from $\mathcal{L}X$

(this is **not** the Scott topology on $\mathcal{L}_a X$ in general!)
(Assuming $X$ standard.)

\[ \mathcal{L}_a X \subseteq \ldots \subseteq \mathcal{L}_{\infty} X \subseteq \mathcal{L} X \]

\[ \mathcal{L}_{\infty} X \cong \bigcup \mathcal{L}_a X \text{ (Lipschitz cont. maps)} \]

\[ \mathcal{L}_1 X \mathcal{L}_2 X \mathcal{L}_3 X \]

$\mathcal{L} X$ (continuous maps)
Lipschitz approximation

**Thm.** Let $X, d$ be standard. For every $h \in \mathcal{L}X$, for every $a > 0$,
— there is a largest $h^{(a)} \in \mathcal{L}_aX$ below $h$
— $h = \sup_a h^{(a)}$
Continuous valuations
Continuous valuations

- Instead of working with measures, let me consider **continuous valuations**
  = maps $\nu: \mathcal{O}(X) \to \mathbb{R}_+$ that are:
  - **strict**: $\nu(\emptyset) = 0$
  - **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
  - **Scott-continuous**.

- Let $V(X) \equiv \{\text{continuous valuations on } X\}$,
  - $V_{\leq 1}(X) \equiv \{\text{subprobability continuous valuations } \nu \text{ on } X \text{ (i.e., } \nu(X) \leq 1)\}$,
  - $V_1(X) \equiv \{\text{probability continuous valuations } \nu \text{ on } X \text{ (i.e., } \nu(X) = 1)\}$
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\( V_1(X) \equiv \{ \text{probability continuous valuations } \nu \text{ on } X \text{ (i.e., } \nu(X) = 1) \} \)

Theorem. Every continuous valuation on a continuous complete quasi-metric space (with the \( d \)-Scott topology) extends to a (\( \tau \)-smooth) Borel measure.

Linear previsions

For every \( \nu \in \mathbf{V}(X) \), \( G : h \in \mathcal{L}X \to \int h d\nu \) is:

- **linear**: \( G(a \cdot h) = aG(h) \), \( G(h_1 + h_2) = G(h_1) + G(h_2) \)
- **Scott-continuous**.
Linear previsions

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- **linear**: $G(a \cdot h) = aG(h)$, $G(h_1 + h_2) = G(h_1) + G(h_2)$
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what I call a [linear prevision](#)
Linear previsions

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  - linear: $G(a \cdot h) = aG(h), G(h_1 + h_2) = G(h_1) + G(h_2)$
  - Scott-continuous.

Thm (« baby Riesz »). Continuous valuations $\cong$ linear previsions.

Proof sketch. Given any linear prevision $G$, we retrieve $\nu$ by

$$\nu(U) \triangleq G(\chi_U)$$

what I call a linear prevision
Linear $\mathcal{L}$-previsions

- Linear previsions $G$ are defined on the whole of $\mathcal{L}X$
- Given $X$, $d$ standard, the restriction $G_{|\mathcal{L}\infty X}$ is:
  - linear
  - continuous: $\mathcal{L}\infty X \xrightarrow{\text{continuous}} \bar{\mathbb{R}}_+$
Linear $\mathcal{L}$-previsions

- Linear previsions $G$ are defined on the whole of $\mathcal{L}X$.
- Given $X$, $d$ standard, the restriction $G_{|\mathcal{L}_\infty X}$ is:
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what I call a linear $\mathcal{L}$-prevision
Linear \mathcal{L}-previsions

- Linear previsions $G$ are defined on the whole of $\mathcal{L}X$.
- Given $X$, $d$ standard, the restriction $G|_{\mathcal{L}_\infty X}$ is:
  - linear
  - continuous: $\mathcal{L}_\infty X \rightarrow \mathbb{R}_+$

**Prop.** Linear prevision $\cong$ linear $\mathcal{L}$-prevision

**Proof sketch.** Given linear $\mathcal{L}$-prevision $H$, define
$$G(h) \equiv \sup_a H(h^{(a)})$$
[recall $h^{(a)} =$ Lipschitz approximation]

**Note.** Similar results with spaces of **bounded** Lipschitz maps.
The Kantorovich-Rubinstein quasi-metrics
The bounded KR quasi-metric

Recall the classical definition:  

\[ d_{KR}^1(\mu, \nu) \triangleq \sup_h \left| \int h \, d\mu - \int h \, d\nu \right| \]

where \( h \) ranges over the 1-bounded, 1-Lipschitz maps

It is fitting to change this to the quasi-metric setting into:

\[ d_{KR}^1(\mu, \nu) \triangleq \sup_{h \in \mathcal{L}^1_X} \left( \int h \, d\mu - \int h \, d\nu \right)_+ \]
The bounded KR quasi-metric

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It is fitting to change this to the quasi-metric setting into:

\[
\begin{aligned}
&d_{KR}^1(\mu, \nu) \equiv \sup_{h \in \mathcal{L}^1_X} \left( \int h \, d\mu - \int h \, d\nu \right)_+ \\
\end{aligned}
\]

ranging over 1-bounded, 1-Lipschitz continuous maps.
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It is fitting to change this to the quasi-metric setting into:

\[
d_{\text{KR}}^1(\mu, \nu) \equiv \sup_{h \in \mathcal{L}^1(X)} \left( \int h \, d\mu - \int h \, d\nu \right)_+ \]

ranging over 1-bounded, 1-Lipschitz continuous maps

using \( d_\mathbb{R} = (\_ - \_)_+ \) instead of \( |\_ - \_| \)
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where \( h \) ranges over the 1-bounded, 1-Lipschitz maps

It is fitting to change this to the quasi-metric setting into:

\[ d_{\text{KR}}^1(\mu, \nu) \equiv \sup_{h \in \mathcal{L}^1_X} \left( \int h \, d\mu - \int h \, d\nu \right) + \]

Fact. The two definitions are equivalent for continuous valuations on a metric space \((X, d)\)
The unbounded KR quasi-metric

- I will concentrate on the **unbounded** variant:

\[ d_{KR}(\mu, \nu) \equiv \sup_{h \in L_1 X} \left( \int h \, d\mu - \int h \, d\nu \right)_+ \]

ranging over all 1-Lipschitz **continuous** maps
Is $V \cdot (X), d_{KR}$ complete?

- We aim to show that $B(V \cdot (X), d_{KR})$ is a dcpo
  Hence we consider a monotone net $(\nu_i, r_i)_{i \in I, \subseteq}$ with $\nu_i$ continuous valuations

- A formal ball $(\nu, r)$ is an upper bound of that net iff
  $d_{KR}(\nu_i, \nu) \leq r_i - r$ for every $i \in I$
  iff $\int h d\nu_i \leq \int h d\nu + r_i - r$ for all $i \in I, h \in \mathcal{L}_1 X$
Is $\mathbf{V}(X), d_{\text{KR}}$ complete?

- We aim to show that $\mathbf{B}(\mathbf{V}(X), d_{\text{KR}})$ is a dcpo
  - Hence we consider a monotone net $(\nu_i, r_i)_{i \in I, \subseteq}$ with $\nu_i$ continuous valuations

- A formal ball $(\nu, r)$ is an upper bound of that net iff
  \[ d_{\text{KR}}(\nu_i, \nu) \leq r_i - r \text{ for every } i \in I \]
  iff \[ \int h d\nu_i \leq \int h d\nu + r_i - r \text{ for all } i \in I, h \in \mathcal{L}_1 X \]

- This suggests that the least upper bound is given by:
  - $r \equiv \inf_{i \in I} r_i$
  - $\int h d\nu \equiv \sup_{i \in I} \left( \int h d\nu_i - r_i + r \right)$ for every $h \in \mathcal{L}_1 X$
Is $V \cdot (X), d_{KR}$ complete?

- More generally (and multiplying by an arbitrary $a$),
- This suggests that the least upper bound is given by:
  \[ r \equiv \inf_{i \in I} r_i \]
  \[ G(h) \equiv \sup_{i \in I} \left( \int h d\nu_i - a \cdot r_i + a \cdot r \right) \text{ for every } h \in \mathcal{L}_aX \]
Is $V \cdot (X)$, $d_{KR}$ complete?

- More generally (and multiplying by an arbitrary $a$),

- This suggests that the least upper bound is given by:
  - $r \equiv \inf_{i \in I} r_i$
  - $G(h) \equiv \sup_{i \in I} \left( \int h d\nu_i - a \cdot r_i + a \cdot r \right)$ for every $h \in \mathcal{L}_a X$

- **Thm.** $G$ is a well-defined linear map from $\mathcal{L}_\infty X$ to $\mathbb{R}_+$. If $G$ is **continuous**, then:
  - it is a linear $\mathcal{L}$-prevision
  - $G \equiv$ a unique continuous valuation $\nu$, which is the **desired** $d_{KR}$-limit.
Is $V \cdot (X), d_{KR}$ complete?

- More generally (and multiplying by an arbitrary $a$),

- This suggests that the least upper bound is given by:
  - $r \equiv \inf_{i \in I} r_i$

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- **Thm.** $G$ is a well-defined linear map from $\mathcal{L}_\infty X$ to $\mathbb{R}_+$. If $G$ is **continuous**, then:
  - it is a linear $\mathcal{L}$-prevision
  - $G \equiv$ a unique continuous valuation $\nu$, which is the **desired** $d_{KR}$-limit.

We will call this the **naive supremum** of $(\nu_i, r_i)_{i \in I, \sqsubseteq}$.
A frustrating situation

- The only missing thing is to show that the naive supremum
  \[ G(h) = \sup_{i \in I} \left( \int h d\nu_i - a \cdot r_i + a \cdot r \right) \]
  is continuous from \( \mathcal{L}_\infty X \) to \( \overline{\mathbb{R}}_+ \).

- **Fact.** \( G \) restricted to every subspace \( \mathcal{L}_a X \) is continuous.
The only missing thing is to show that the naive supremum

\[ G(h) \equiv \sup_{i \in I} \left( \int h d\nu_i - a \cdot r_i + a \cdot r \right) \]

is continuous from \( \mathcal{L}_\infty X \) to \( \overline{\mathbb{R}}_+ \).

**Fact.** \( G \) restricted to every subspace \( \mathcal{L}_a X \) is continuous.

That is not enough to conclude — unless topology of \( \mathcal{L}_\infty X \) is determined by those of its subspaces \( \mathcal{L}_a X \) (=colimit) [open problem!]
Lipschitz-regular quasi-metric spaces
The assignment $U \mapsto \hat{U}$

Recall that the $d$-Scott topology on $X$ is the subspace topology induced by the embedding $x \in X \mapsto (x, 0) \in B(X, d)$

---

Recall that the $d$-Scott topology on $X$ is the subspace topology induced by the embedding $x \in X \mapsto (x,0) \in B(X,d)$.

For every open subset $U$ of $X$, let $\hat{U}$ be the largest Scott-open subset of $B(X,d)$ such that $U = \hat{U} \cap X$. 

\begin{align*}
\text{JGL (2020) Some topological properties of spaces of Lipschitz continuous maps on quasi-metric spaces. T&A. 282}
\end{align*}
The assignment $U \mapsto \hat{U}$

- Recall that the $d$-Scott topology on $X$ is the subspace topology induced by the embedding $x \in X \mapsto (x,0) \in B(X,d)$.

- For every open subset $U$ of $X$, let $\hat{U}$ be the **largest Scott-open** subset of $B(X,d)$ such that $U = \hat{U} \cap X$.

---

JGL (2020) *Some topological properties of spaces of Lipschitz continuous maps on quasi-metric spaces*. T&A. 282
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- The map $U \mapsto \hat{U}$ is right adjoint to $V \mapsto V \cap X$, hence preserves arbitrary meets

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- The map $U \mapsto \hat{U}$ is right adjoint to $V \mapsto V \cap X$, hence preserves arbitrary meets

- **Defn.** $X, d$ is Lipschitz-regular iff $U \mapsto \hat{U}$ is Scott-continuous
  ($= \text{if } X \text{ is finitarily embedded into } B(X, d)$, see Escardó 98)
Lipschitz-regular spaces

- **Defn.** $X, d$ is **Lipschitz-regular** iff $U \mapsto \hat{U}$ is Scott-continuous.

- **Prop.** If $X, d$ is Lipschitz-regular, then topology of $\mathcal{L}_\infty X$ is determined by those of its subspaces $\mathcal{L}_a X$. 

![Diagram showing $U \mapsto \hat{U}$ and the space $X$ with subspaces and neighborhoods.](image-url)
Lipschitz-regular spaces

- **Defn.** $X, d$ is **Lipschitz-regular** iff $U \mapsto \hat{U}$ is Scott-continuous

- **Prop.** If $X, d$ is Lipschitz-regular, then topology of $\mathcal{L}_\infty X$ is determined by those of its subspaces $\mathcal{L}_a X$.

- **Proof sketch.** The canonical injection $i_a : \mathcal{L}_a X \to \mathcal{L} X$ and the $a$-Lipschitz approximation map $r_a : \mathcal{L} X \to \mathcal{L}_a X$
  \[ h \mapsto h^{(a)} \]
  form an embedding-projection pair.
Lipschitz-regular spaces and completeness

- **Defn.** $X, d$ is **Lipschitz-regular** iff $U \mapsto \hat{U}$ is Scott-continuous

- **Prop.** If $X, d$ is Lipschitz-regular, then topology of $\mathcal{L}_\infty X$ is determined by those of its subspaces $\mathcal{L}_a X$.

- As a corollary,

- **Prop.** If $X, d$ is Lipschitz-regular, then:
  - $V \cdot (X), d_{KR}$ is complete
  - directed suprema of formal balls $(\nu_i, r_i)_{i \in I}$ are **naive suprema**:

\[
G(h) \neq \sup_{i \in I} \left( \int h \nu_i - a \cdot r_i + a \cdot r \right), \text{ for every } h \in \mathcal{L}_a X, a > 0
\]
Is Lipschitz-regularity acceptable?

- Hmm … no.
- If $X, d$ algebraic complete, then
  Lipschitz-regular $\Leftrightarrow$ has relatively compact open balls
- That is a pretty strong property — stronger than local compactness and remember that local compactness is not required in the metric case!
A miracle

- \( \mathcal{B}(X, d) \) itself is a quasi-metric space, with
  \[
d^+((x, r), (y, s)) = \max(d(x, y) - r + s, 0)
\]
  and \( d^+ \)-Scott topology = Scott topology

- **Thm.** For every quasi-metric space \( X, d, \mathcal{B}(X, d), d^+ \)
  is Lipschitz-regular
  [in fact, \( U \mapsto \hat{U} \) preserves all unions].

- Let me only give a sketch of the argument...
  (assuming \( X, d \) standard, which will be enough for our purposes)
Thm. There is a monad \((\mathcal{B}, \eta, \mu)\) on the category of standard quasi-metric spaces where:
- \(\mathcal{B}(f): (x, r) \mapsto (f(x), r)\) [what I wrote \(\mathcal{B}_1\) earlier on]
- \(\eta: x \in X \mapsto (x, 0) \in \mathcal{B}(X, d)\)
- \(\mu: ((x, r), s) \mapsto (x, r + s)\)

Formal ball monads

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Thm. There is a monad \((\mathcal{B}, \eta, \mu)\) on the category of standard quasi-metric spaces where:
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In fact a left KZ-monad:
\(\mathcal{B}\eta \leq \eta \iff \mu \dashv \eta \iff \mathcal{B}\eta \dashv \mu\)
so we know what the \(\mathcal{B}\)-algebras are [but I won’t spell it out here]
Formal ball monads

- **Thm.** There is a monad \((B, \eta, \mu)\) on the category of standard quasi-metric spaces where:
  - \(B(f): (x, r) \mapsto (f(x), r)\) [what I wrote \(B_1\) earlier on]
  - \(\eta: x \in X \mapsto (x, 0) \in B(X, d)\)
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- In fact a left KZ-monad:
  \(B\eta \leq \eta \iff \mu \dashv \eta \iff B\eta \dashv \mu\)
  so we know what the \(B\)-algebras are [but I won’t spell it out here]

- **Prop.** For every \(B\)-algebra \(\alpha: B(X, d) \to X\),
  we have \(\hat{U} = \alpha^{-1}(U)\); in particular, \(X, d\) is Lipschitz-regular.
Formal ball monads

**Thm.** There is a monad \((\mathcal{B}, \eta, \mu)\) on the category of standard quasi-metric spaces where:
- \(B(f): (x, r) \mapsto (f(x), r)\) [what I wrote \(B_1\) earlier on]
- \(\eta: x \in X \mapsto (x, 0) \in B(X, d)\)
- \(\mu: ((x, r), s) \mapsto (x, r + s)\)

**In fact a left KZ-monad:**
\(B\eta \leq \eta \Leftrightarrow \mu \dashv \eta \Leftrightarrow B\eta \dashv \mu\)
so we know what the \(B\)-algebras are [but I won’t spell it out here]

**Prop.** For every \(B\)-algebra \(\alpha: B(X, d) \to X\),
we have \(\hat{U} = \alpha^{-1}(U)\); in particular, \(X, d\) is Lipschitz-regular.

\(B(X, d), d^+\) is the free \(B\)-algebra, hence is Lipschitz-regular.
Back to the completeness theorem
Embedding into the formal ball model

Recall:

**Prop.** If $X, d$ is Lipschitz-regular, then:

— $V_\cdot(X), d_{KR}$ is complete
— directed suprema of formal balls $(\nu_i, r_i)_{i \in I}$ are naive suprema:

$$G(h) \equiv \sup_{i \in I} \left( \int h d\nu_i - a \cdot r_i + a \cdot r \right), \text{ for every } h \in \mathcal{L}_aX, a > 0$$
Embedding into the formal ball model

Recall:

**Prop.** If $X, d$ is Lipschitz-regular, then:
- $\mathbf{V}(X), d_{KR}$ is complete
- directed suprema of formal balls $(\nu_i, r_i)_{i \in I}$ are naive suprema:
  \[ G(h) \equiv \sup_{i \in I} \left( \int h d\nu_i - a \cdot r_i + a \cdot r \right), \text{ for every } h \in \mathcal{L}_a X, a > 0 \]

Since $\mathcal{B}(X, d), d^+$ is always Lipschitz-regular, for every space $X, d$ we have:
- $\mathbf{V}(\mathcal{B}(X, d))$ is complete
- directed suprema of formal balls $(\tilde{\nu}_i, r_i)_{i \in I}$ are naive suprema
  [each $\tilde{\nu}_i$ is a continuous valuation on $\mathcal{B}(X, d)$]
Recap. Directed suprema of formal balls \((\tilde{\nu}_i, r_i)_{i \in I}\) are naive suprema [each \(\tilde{\nu}_i\) is a continuous valuation on \(B(X, d)\)]

Now consider any directed family of formal balls \((\nu_i, r_i)_{i \in I}\) [each \(\nu_i\) a continuous valuation on \(X\)]
Recap. Directed suprema of formal balls \((\tilde{\nu}_i, r_i)_{i \in I}\) are naive suprema [each \(\tilde{\nu}_i\) is a continuous valuation on \(\mathbf{B}(X, d)\)]

Now consider any directed family of formal balls \((\nu_i, r_i)_{i \in I}\) [each \(\nu_i\) a continuous valuation on \(X\)]

Let \(\tilde{\nu}_i = \eta[\nu_i]\), image valuation of \(\nu_i\) by \(\eta: X \rightarrow \mathbf{B}(X, d)\)

\[
\begin{array}{ccc}
(\nu_i, r_i)_{i \in I} & \xrightarrow{\eta} & (\eta[\nu_i], r_i)_{i \in I}
\end{array}
\]
Embedding into the formal ball model

- **Recap.** Directed suprema of formal balls \((\tilde{\nu}_i, r_i)_{i \in I}\) are naive suprema
  [each \(\tilde{\nu}_i\) is a continuous valuation on \(B(X, d)\)]

- Now consider any directed family of formal balls \((\nu_i, r_i)_{i \in I}\)
  [each \(\nu_i\) a continuous valuation on \(X\)]

- Let \(\tilde{\nu}_i \equiv \eta[\nu_i]\), image valuation of \(\nu_i\) by \(\eta: X \to B(X, d)\)

\[
\begin{align*}
(\nu_i, r_i)_{i \in I} & \xrightarrow{\eta} (\eta[\nu_i], r_i)_{i \in I} \\
\tilde{\nu}_i & \quad \text{naive supremum } (\tilde{\nu}, r)
\end{align*}
\]
Embedding into the formal ball model

- **Recap.** Directed suprema of formal balls \((\tilde{\nu}_i, r_i)_{i \in I}\) are naive suprema [each \(\tilde{\nu}_i\) is a continuous valuation on \(B(X, d)\)]

- Now consider any directed family of formal balls \((\nu_i, r_i)_{i \in I}\) [each \(\nu_i\) a continuous valuation on \(X\)]

- Let \(\tilde{\nu}_i \equiv \eta[\nu_i]\), image valuation of \(\nu_i\) by \(\eta : X \to B(X, d)\)

- **Lemma.** If \(\tilde{\nu} = \eta[\nu]\) for some \(\nu \in \mathcal{V}(X)\) then \((\nu, r)\) is the (naive) supremum of \((\nu_i, r_i)_{i \in I}\)

[Diagram showing the embedding process and the lemma statement]
Let $\eta$ be an inclusion of spaces $X \to B$

**Defn.** A continuous valuation $\tilde{\nu} \in \text{V.}(B)$ is **supported on** $X$ if and only if $\tilde{\nu} = \eta[\nu]$ for some $\nu \in \text{V.}(X)$

**Lemma.** $\tilde{\nu} \in \text{V.}(B)$ is supported on $X$ iff for all open subsets $V, W$ of $B$ such that $V \cap X = W \cap X$,

$$\tilde{\nu}(V) = \tilde{\nu}(W)$$

**Proof:** Exercise.
Let $\eta$ be an inclusion of spaces $X \to B$

**Defn.** A continuous valuation $\tilde{\nu} \in V.(B)$ is **supported on** $X$
if and only if $\tilde{\nu} = \eta[\nu]$ for some $\nu \in V.(X)$

**Lemma.** $\tilde{\nu} \in V.(B)$ is supported on $X$ iff
for all open subsets $V, W$ of $B$ such that $V \cap X = W \cap X$,
$\tilde{\nu}(V) = \tilde{\nu}(W)$

**Proof:** Exercise.

Almost there! It remains to check that
the naive supremum $\tilde{\nu} \in V.(B(X, d))$ is supported on $X$
Another source of frustration

- The best we can prove (for now) is that the naive supremum $\tilde{\nu} \in V.(B(X, d))$ is supported on $V_\epsilon = \{(x, r) \mid r < \epsilon\}$ for every $\epsilon > 0$

- Recall that $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$

- Does this imply that $\tilde{\nu}$ is supported on $X$?
The best we can prove (for now) is that the naive supremum $\tilde{\nu} \in V \cdot (B(X, d))$ is supported on $V_\epsilon = \{(x, r) \mid r < \epsilon\}$ for every $\epsilon > 0$.

Recall that $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$.

Does this imply that $\tilde{\nu}$ is supported on $X$?

Yes if $X, d$ is continuous complete and $\tilde{\nu}$ is bounded ($\tilde{\nu}(B(X, d)) < \infty$): see next slide.
If $X, d$ is continuous complete, then $\mathcal{B}(X, d)$ is a **continuous dcpo**
and $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$, hence Borel, in it.

**Thm.** Every continuous valuation (e.g., $\tilde{\nu}$) on a continuous dcpo
(or even a locally compact sober space) extends to a **Borel measure**.

---

Invoking some measure theory

If \( X, d \) is continuous complete, then \( B(X, d) \) is a **continuous dcpo** and \( X = \bigcap_{n \in \mathbb{N}} V_{1/2^n} \) is \( G_\delta \), hence Borel, in it.

**Thm.** Every continuous valuation (e.g., \( \tilde{\nu} \)) on a continuous dcpo (or even a locally compact sober space) extends to a **Borel measure**.

Since \( \tilde{\nu} \) supported on \( V_\varepsilon \), for every open subset \( V \) of \( B(X, d) \),

\[
\tilde{\nu}(V) = \tilde{\nu}(V \cap V_\varepsilon)
\]

[\( V \) and \( V \cap V_\varepsilon \) have the same intersection with \( V_\varepsilon \)].

---

If $X, d$ is continuous complete, then $\mathcal{B}(X, d)$ is a continuous dcpo and $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$, hence Borel, in it.

**Thm.** Every continuous valuation (e.g., $\tilde{\nu}$) on a continuous dcpo (or even a locally compact sober space) extends to a Borel measure.

Since $\tilde{\nu}$ supported on $V_{\epsilon'}$ for every open subset $V$ of $\mathcal{B}(X, d)$,

$$\tilde{\nu}(V) = \tilde{\nu}(V \cap V_{\epsilon'})$$  \hspace{1cm} [V and $V \cap V_{\epsilon'}$ have the same intersection with $V_{\epsilon'}$]

Then, if $\tilde{\nu}$ is bounded,

$$\tilde{\nu}(V) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V \cap V_{1/2^n}) = \tilde{\nu}(\bigcap_{n \in \mathbb{N}} V \cap V_{1/2^n}) = \tilde{\nu}(V \cap X)$$
Invoking some measure theory

- If $X, d$ is continuous complete, then $B(X, d)$ is a **continuous dcpo**
  and $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$, hence Borel, in it.

- **Thm.** Every continuous valuation (e.g., $\tilde{\nu}$) on a continuous dcpo
  (or even a locally compact sober space) extends to a **Borel measure**.

- Since $\tilde{\nu}$ supported on $V_{c'}$ for every open subset $V$ of $B(X, d)$,
  $\tilde{\nu}(V) = \tilde{\nu}(V \cap V_{c'})$ [\(V\) and $V \cap V_{c'}$ have the same intersection with $V_{c'}$]

- Then, if $\tilde{\nu}$ is **bounded**, $\tilde{\nu}(V) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V \cap V_{1/2^n}) = \tilde{\nu}(\bigcap_{n \in \mathbb{N}} V \cap V_{1/2^n}) = \tilde{\nu}(V \cap X)$

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**References**

Invoking some measure theory

- If $X, d$ is continuous complete, then $B(X, d)$ is a **continuous dcpo** and $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$, hence Borel, in it.

- **Thm.** Every continuous valuation (e.g., $\tilde{\nu}$) on a continuous dcpo (or even a locally compact sober space) extends to a **Borel measure**.

- Since $\tilde{\nu}$ supported on $V_e$, for every $V \in B(X, d)$,
  \[ \tilde{\nu}(V) = \tilde{\nu}(V \cap V_e) \quad [V \cap V_e \text{ intersection with } V_e] \]

- Then, if $\tilde{\nu}$ is **bounded**, then $\tilde{\nu}(V) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V \cap V_{1/2^n}) = \tilde{\nu}(\bigcap_{n \in \mathbb{N}} V \cap V_{1/2^n}) = \tilde{\nu}(V \cap X)

**Notes:**

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A bounded measure commutes with infs of countable chains.

The inf of a constant sequence.
Invoking some measure theory

- If $X, d$ is continuous complete, then $B(X, d)$ is a **continuous dcpo** and $X = \bigcap_{n \in \mathbb{N}} V_{1/2^n}$ is $G_\delta$, hence Borel, in it.

- **Thm.** Every continuous valuation (e.g., $\tilde{\nu}$) on a continuous dcpo (or even a locally compact sober space) extends to a **Borel measure**.

- Since $\tilde{\nu}$ supported on $V_\varepsilon$, for every $V \subseteq X$, $d$, $\tilde{\nu}(V) = \tilde{\nu}(V \cap V_\varepsilon)$ [if $\cap V_\varepsilon$ is closed intersection with $V_\varepsilon$].

- Then, if $\tilde{\nu}$ is **bounded**, $\tilde{\nu}(V) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V \cap V_{1/2^n}) = \tilde{\nu}(\bigcap_{n \in \mathbb{N}} V \cap V_{1/2^n}) = \tilde{\nu}(V \cap X)$

- The inf of a **constant** sequence

- A bounded measure commutes with infs of countable chains

- $V \cap X$ is Borel, and $\tilde{\nu}$ is a measure

---

**References:**

Invoking some measure theory

- If \( X, d \) is continuous complete, then \( \mathcal{B}(X, d) \) is a **continuous dcpo** and \( X = \bigcap_{n \in \mathbb{N}} V_{1/2^n} \) is \( G_\delta \), hence Borel, in it.

- **Thm.** Every continuous valuation (e.g., \( \tilde{\nu} \)) on a continuous dcpo (or even a locally compact sober space) extends to a **Borel measure**.

- Since \( \tilde{\nu} \) supported on \( V_\epsilon \), for every \( V \),

\[
\tilde{\nu}(V) = \tilde{\nu}(V \cap V_\epsilon)
\]

- Then, if \( \tilde{\nu} \) is **bounded**, \( \tilde{\nu}(V) = \inf_{n \in \mathbb{N}} \tilde{\nu}(V \cap V_{1/2^n}) = \tilde{\nu}(\bigcap_{n \in \mathbb{N}} V \cap V_{1/2^n}) = \tilde{\nu}(V \cap X) \)

- In particular, if \( V \cap X = W \cap X \), then \( \tilde{\nu}(V) = \tilde{\nu}(W) \): \( \tilde{\nu} \) is **supported on** \( X \).

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**Bibliography**

We are done!

❖ Summing up:

❖ **Thm.** For every continuous complete quasi-metric space $X, d,$ $V_1(X)$ and $V_{\leq 1}(X)$ are **complete** under the $d_{KR}$ quasi-metric.
(And directed suprema of formal balls are computed as naive suprema.)
Final remarks (a long list... )
We are done!

- Summing up:
  - **Thm.** For every **continuous complete** quasi-metric space $X, d,$ $V_1(X)$ and $V_{\leq 1}(X)$ are **complete** under the $d_{KR}$ quasi-metric. (And directed suprema of formal balls are computed as naive suprema.)
We are done!

- Summing up:

- **Thm.** For every *continuous complete* quasi-metric space $X, d, V_1(X)$ and $V_{\leq 1}(X)$ are *complete* under the $d_{KR}$ quasi-metric. (And directed suprema of formal balls are computed as naive suprema.)
We are done!

- Summing up:

- **Thm.** For every **continuous complete** quasi-metric space \( X, d \), \( V_1(X) \) and \( V_{\leq 1}(X) \) are **complete** under the \( d_{KR} \) quasi-metric.
  (And directed suprema of formal balls are computed as naive suprema.)

- What about \( V(X) \) (unbounded valuations)? — open problem
Summing up:

**Thm.** For every **continuous complete** quasi-metric space $X, d$, $V_1(X)$ and $V_{≤1}(X)$ are **complete** under the $d_{KR}$ quasi-metric. (And directed suprema of formal balls are computed as naive suprema.)

- What about $V(X)$ (unbounded valuations)? — **open problem**

- In fact, $V_{≤1}(X)$ is even **continuous complete**
  as well as $V_1(X)$ if $X, d$ has a so-called root
  [would need another talk]

  Goes through preservation of **algebraic** completeness,
  using the remarkable fact that for $X, d$ continuous complete,
  $\mathcal{L}_d X$ is **stably compact**, and topology=compact-open=pointwise

---

**We are done!**

Are we, really?

---

JGL (2020) *Some topological properties of spaces of Lipschitz continuous maps on quasi-metric spaces.* T&A. 282
Are we done yet?

- Using the bounded version \( d_{KR}^1 \), we obtain:

  Thm. For every continuous complete quasi-metric space \( X, d \), \( V_1(X) \) and \( V_{\leq 1}(X) \) are **continuous complete** under \( d_{KR}^1 \).
  (And directed suprema of formal balls are computed as naive suprema.)
Are we done yet?

- Using the **bounded** version $d^1_{KR}$, we obtain:

  Thm. For every **continuous complete** quasi-metric space $X, d$, $V_1(X)$ and $V_{\leq 1}(X)$ are **continuous complete** under $d^1_{KR}$.
  (And directed suprema of formal balls are computed as naive suprema.)

- If $X, d$ is **algebraic** complete, then so are $V_1(X)$ and $V_{\leq 1}(X)$, too.
Are we done yet?

- Using the **bounded** version $d_{KR}^1$, we obtain:

  - **Thm.** For every **continuous complete** quasi-metric space $X$, $d$, $V_1(X)$ and $V_{\leq 1}(X)$ are **continuous complete** under $d_{KR}^1$.
  (And directed suprema of formal balls are computed as naive suprema.)

- If $X$, $d$ is **algebraic** complete, then so are $V_1(X)$ and $V_{\leq 1}(X)$, too.

- When $X$ is an algebraic dcpo, $d_{KR}^1$ is Sünderrauf (1998)’s sup quasi-metric, and we retrieve his result that $V_{\leq 1}(X)$ is algebraic complete in that case.
The weak topology

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- **Thm.** For every continuous complete quasi-metric space $X, d$, $d_{KR}^1$-Scott topology = weak topology on $V_1(X)$ and $V_{\leq 1}(X)$. 
The weak topology

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  **Thm.** For every continuous complete quasi-metric space $X, d,$
  $d_{KR}^1$-Scott topology = weak topology on $V_1(X)$ and $V_{\leq 1}(X)$.

- **Not** true for $d_{KR}$-Scott topology, even when $d$ metric (Kravchenko 2006).
Beyond continuous valuations: previsions

- In general, $d_{KR}$ makes sense on any space of functionals: $\mathcal{L}X \to \overline{\mathbb{R}}_+$, not just linear previsions (=continuous valuations)

- **Defn.** A prevision is any Scott-continuous map $F: \mathcal{L}X \to \overline{\mathbb{R}}_+$ satisfying $F(a \cdot h) = a \cdot F(h)$

- **Defn.** $d_{KR}(F, F') \triangleq \sup_{h \in \mathcal{L}_1X} (F(h) - F'(h))_+$
In general, $d_{KR}$ makes sense on any space of functionals: $\mathcal{L}X \to \mathbb{R}_+$, not just linear previsions (=continuous valuations).

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**Defn.** $d_{KR}(F, F') \equiv \sup_{h \in \mathcal{L}_1X} (F(h) - F'(h))_+$. (Note the absence of brackets for $\mathcal{L}_1X$.)

We have similar theorems for discrete/sublinear/superlinear previsions.
Beyond continuous valuations: previsions

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- In particular, **discrete** previsions $\cong$ Hoare/Smyth hyperspaces, with asymmetric variants of the Pompeiu-Hausdorff quasi-metric

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Any questions? ... meanwhile, a few references


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