Laboratoire Méthodes Formelles

université PARIS-SACLAY

cnrs

école normale supérieure paris—saclay

with some parts in collaboration with
Xiaodong Jia (贾晓东),
Clément Theron

Jean Goubault-Larrecq

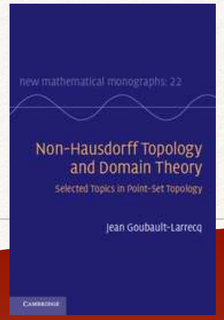
A journey through the semantics of higher-order probabilistic languages, domain theory, and topology

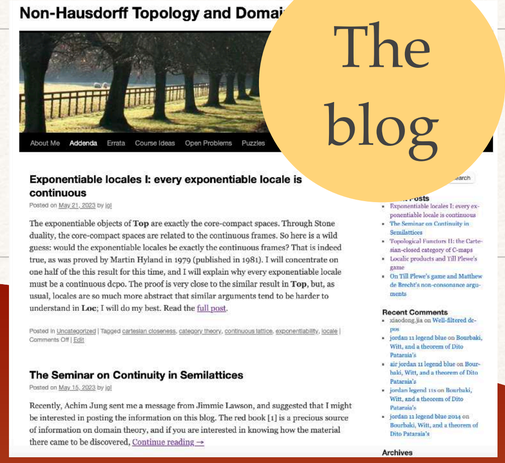
Etiolles 2023

Topology and Applications

The purpose of this talk

- I was asked the following in May 2023:
  « Would you accept to make a long presentation on a topic related to topology in computer science (like a tutorial, or an introduction « for dummies »)? »
  « est-ce que tu accepterais de faire une présentation longue sur un sujet lié à la topologie en informatique (type tutoriel, ou intro "pour les nuls ») ?»
- Let me first give you an overview of my main activities in topology at LMF





The blog

Directed alg topolo

Tru concurr

Non-Hausdorff topology

Noetherian spaces

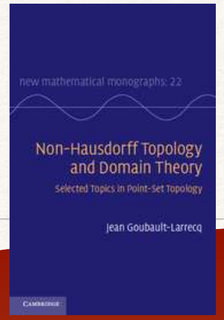
Quasi-metric spaces

Domain theory

Topological measure theory

Verification f infinite-state systems

Semantics of programming languages



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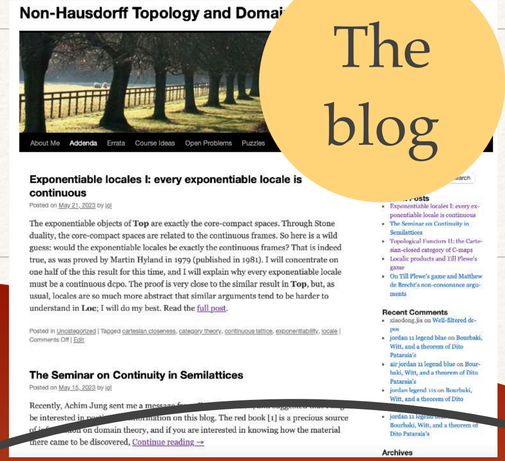

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Topology

What topology is, in this talk

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  — every (arbitrary) union of open sets is open
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- A map $f : X \to Y$ is **continuous** iff
  for every open subset V of Y, $f^{-1}(V)$ is open in X

Domain theory

- An approach to the semantics of programming languages initiated by Dana S. Scott

Theoretical Computer Science 121 (1993) 411–440
Elsevier

A type-theoretical alternative to ISWIM, CUCH, OWHY

Dana S. Scott
Carnegie-Mellon University, Pittsburgh, PA, USA, and RISC-Linz, Austria

Abstract

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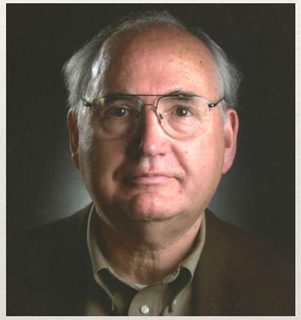

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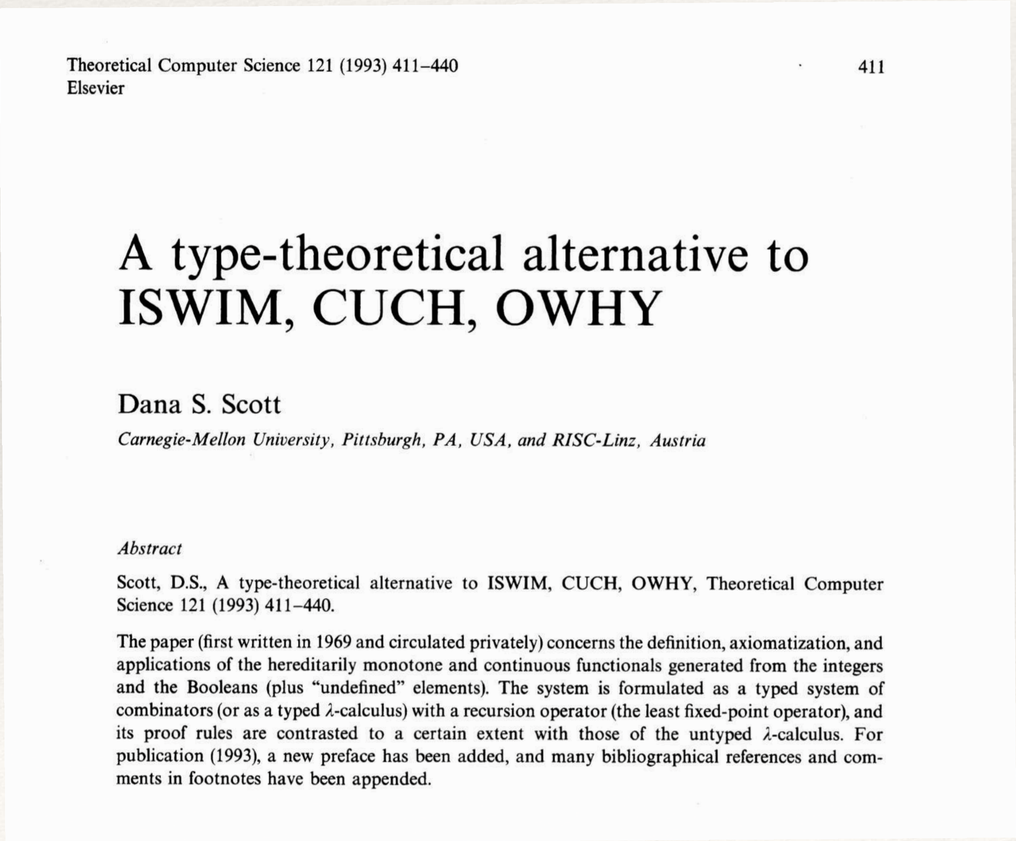

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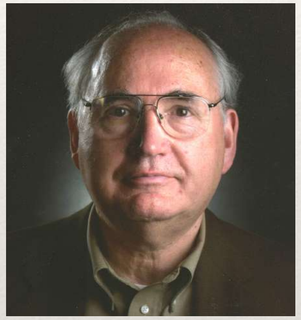

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- based on **partial orders**,
- with some particular **topology**

Dcpos, partial values, and the order of information

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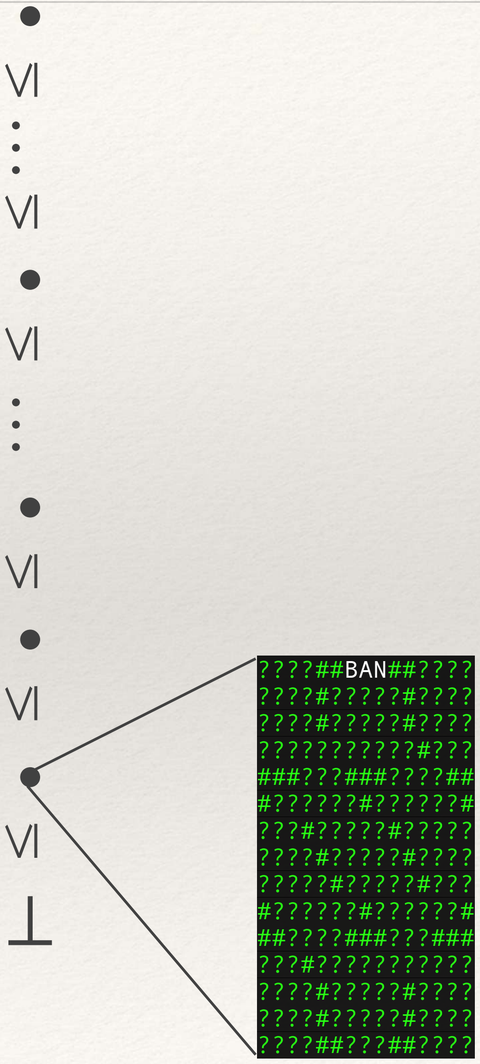

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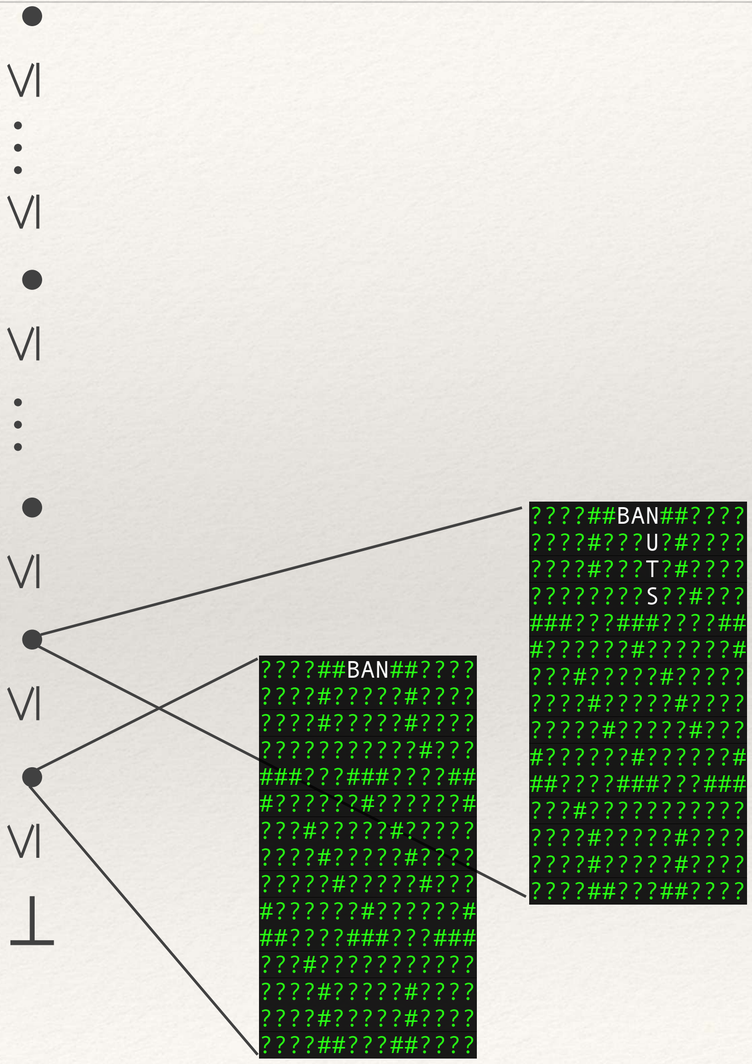

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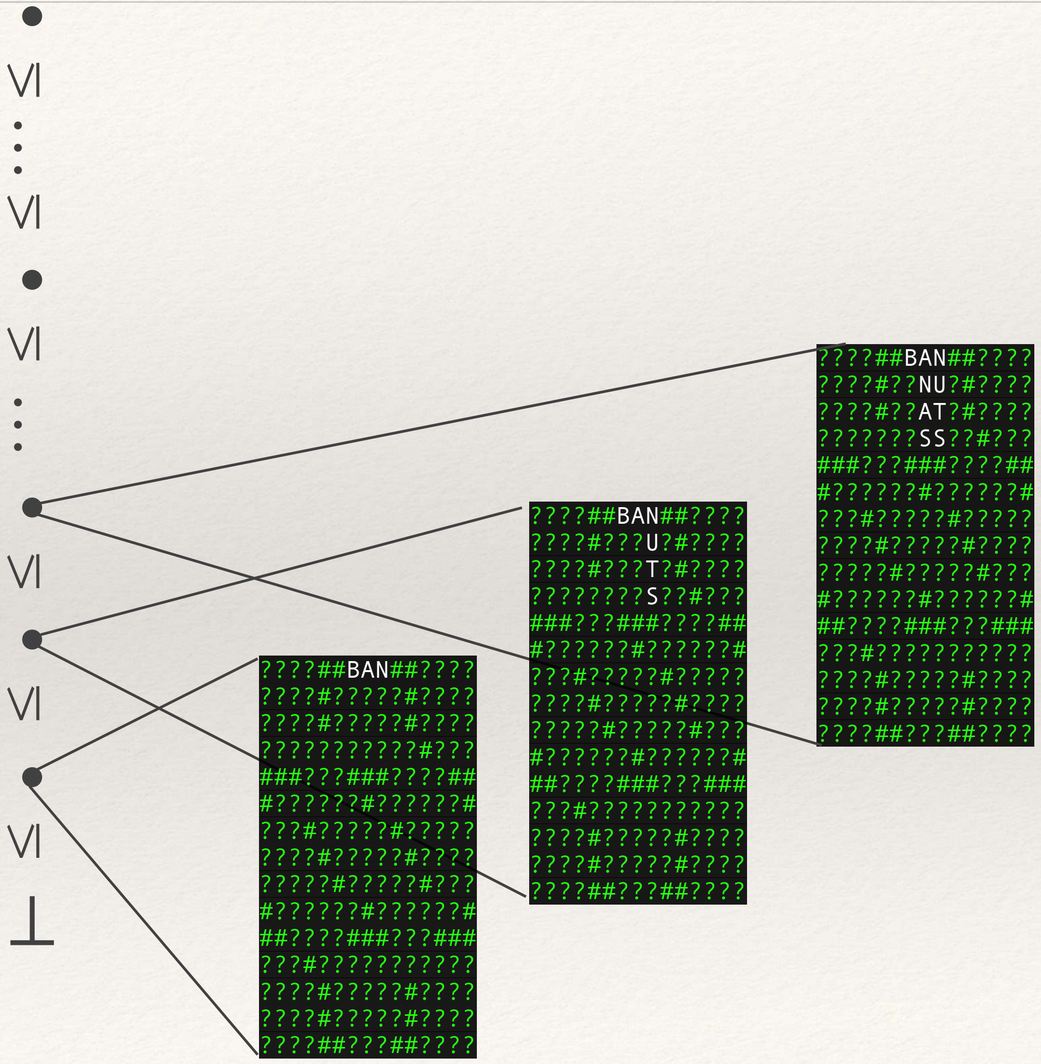

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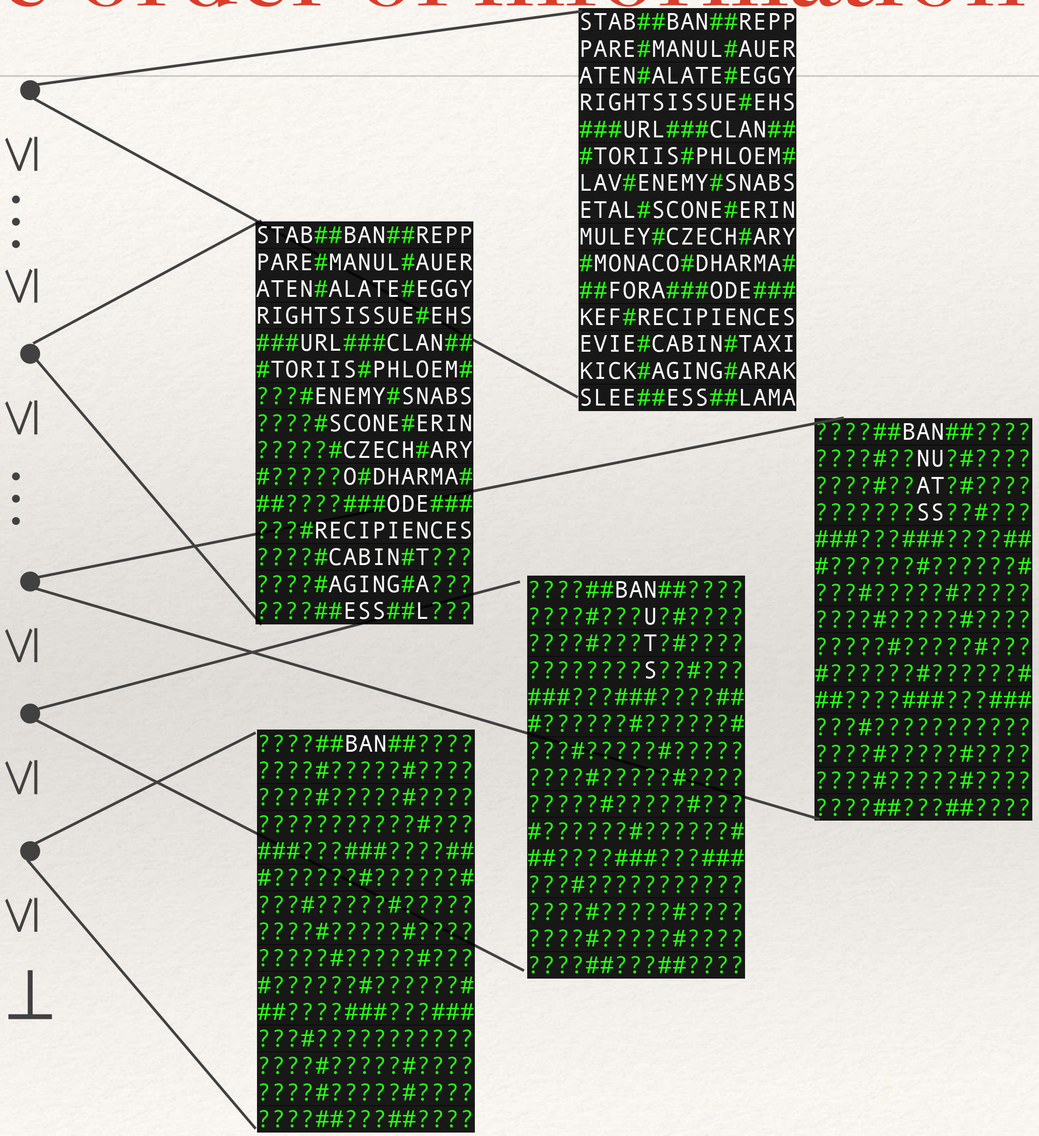

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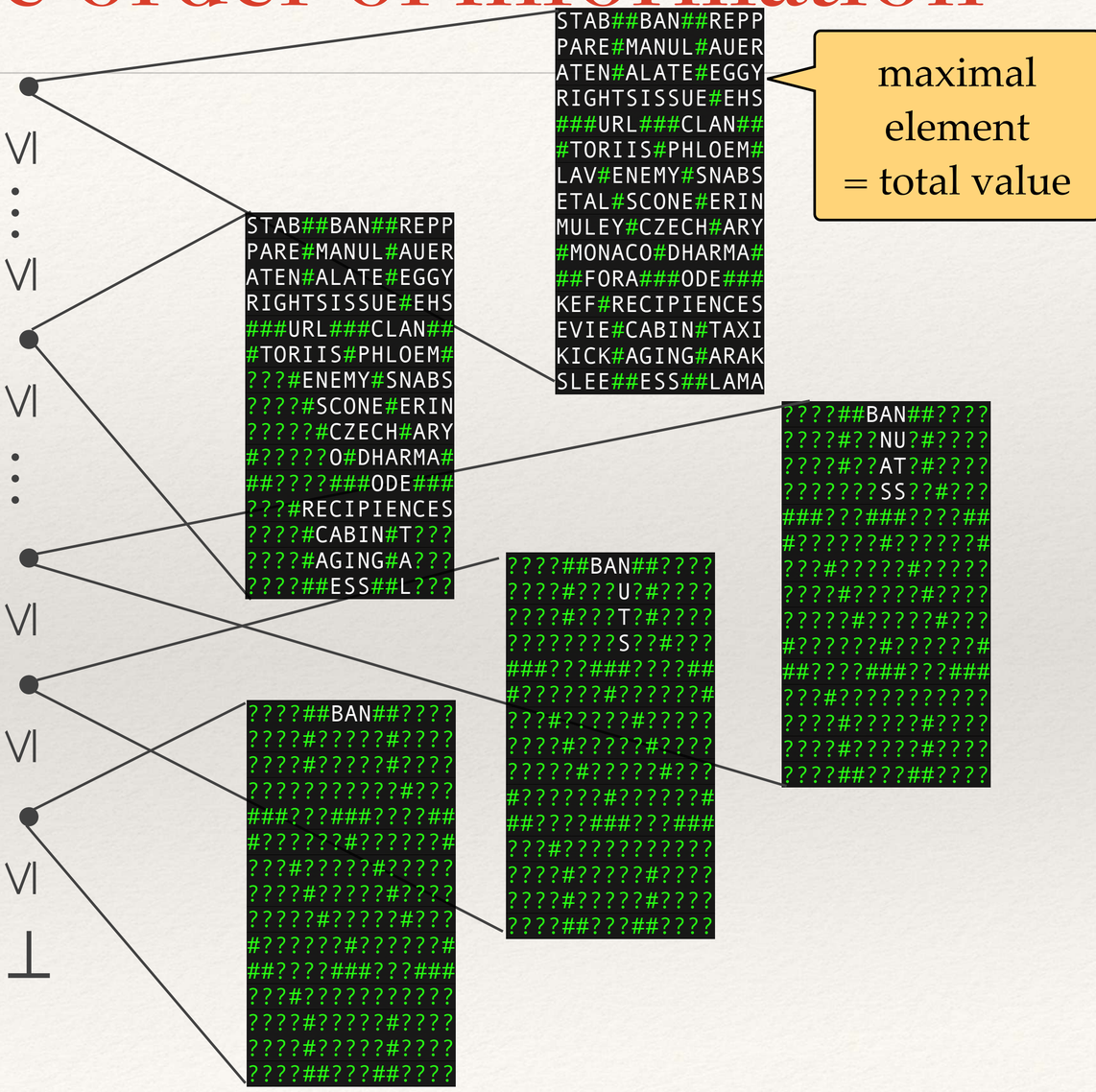

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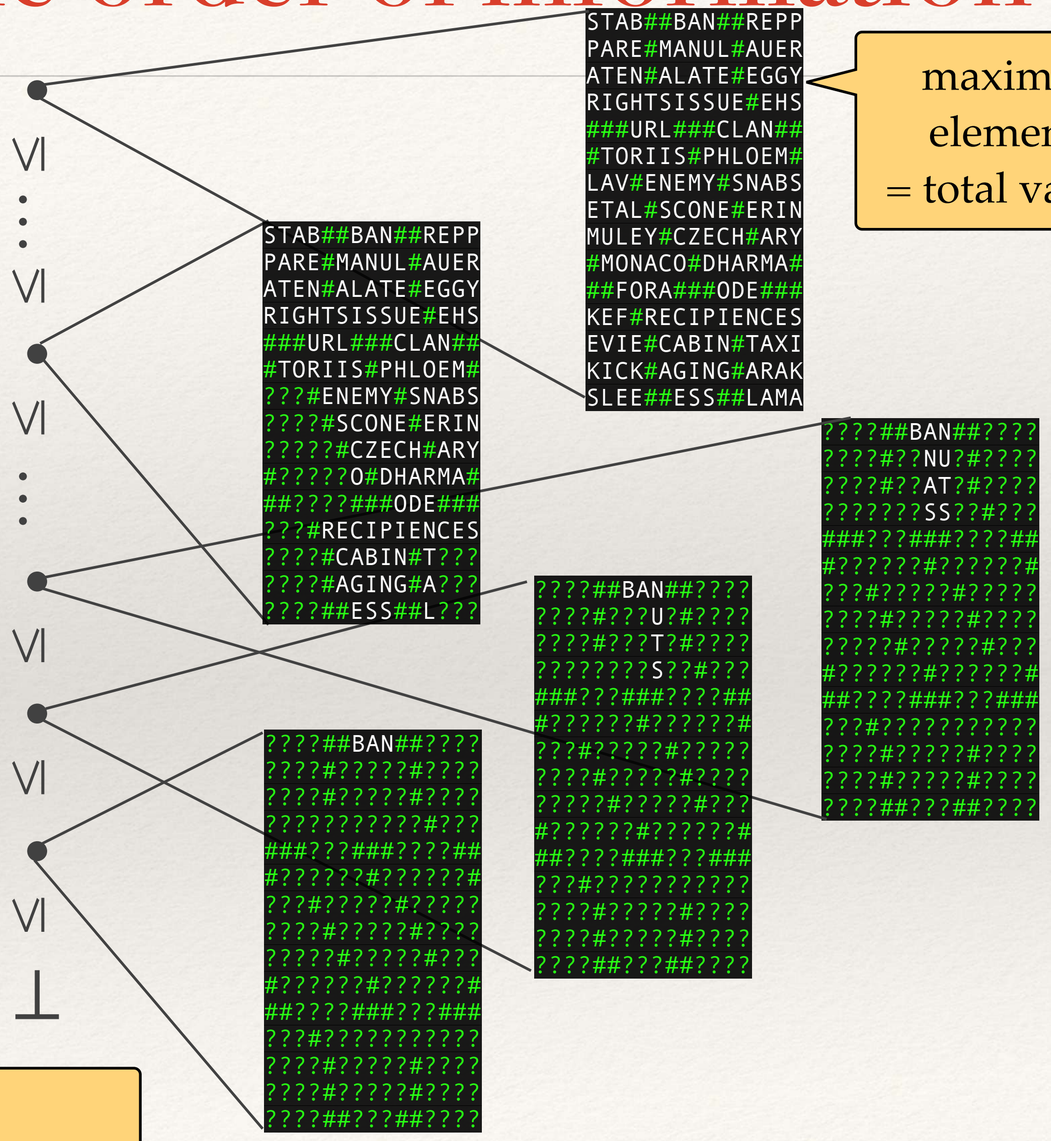

well, directed families, really:
D is **directed** iff $\neq \emptyset$ and every pair of elements of D has an upper bound in D

Scott-continuous maps, least fixed points

- A map $f : X \to Y$ between dcpos is **Scott-continuous** iff:
  — monotonic: $x \leq x' \Longrightarrow f(x) \leq f(x')$
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- **Theorem 2.** $[X \to Y] \triangleq \{f : X \to Y \text{ Scott-continuous}\}$ is a dcpo, where $f \leq g$ iff for every $x \in X$, $f(x) \leq g(x)$; it is pointed if Y is.

Will serve to interpret **higher-order** types

PCF

https://www.research.ed.ac.uk/files-asset/17539241/gdp.jpg?w=320&f=webp

- Consider the higher-order, functional programming language PCF [Plotkin 77]

$M, N, P, \ldots ::= x, y, z, \ldots$	variables
$\mid MN$	application
$\mid \lambda x_\sigma . M$	abstraction
$\mid \mathbf{rec}(M)$	recursion
$\mid 0 \mid 1 \mid 2 \mid \ldots$	natural numbers
$\mid \mathbf{s}(M)$	successor
$\mid \mathbf{p}(M)$	predecessor
$\mid \mathbf{if}\ M = 0\ \mathbf{then}\ N\ \mathbf{else}\ P$	conditional

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elementary operations on type **nat**

Types

- PCF terms are **typed**: $\sigma, \tau, \ldots ::= \mathbf{nat} \mid \sigma \to \tau$

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$$\frac{M : \tau \to \tau}{\mathbf{rec}(M) : \tau}$$

$$\frac{}{0 : \mathbf{nat}} \quad \frac{}{1 : \mathbf{nat}} \quad \ldots$$

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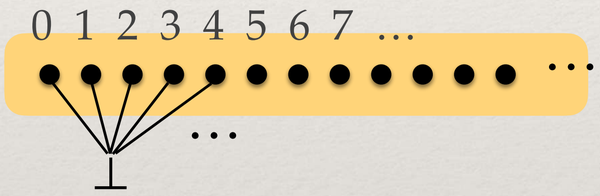

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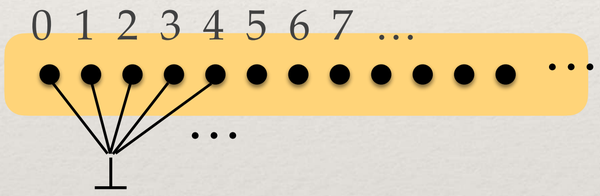

  - $[\![\sigma \to \tau]\!] \triangleq [[\![\sigma]\!] \to [\![\tau]\!]]$
    — space of Scott-continuous maps from $[\![\sigma]\!]$ to $[\![\tau]\!]$

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A denotational semantics for PCF

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Lambda-calculus:

- $[\![x]\!]\rho \mathrel{\hat{=}} \rho(x)$
- $[\![MN]\!]\rho \mathrel{\hat{=}} [\![M]\!]\rho([\![N]\!]\rho)$
- $[\![\lambda x\,.\,M]\!]\rho \mathrel{\hat{=}} (V \mapsto [\![M]\!](\rho[x \mapsto V]))$
- $[\![\mathbf{rec}(M)]\!]\rho \mathrel{\hat{=}} \mathit{lfp}([\![M]\!]\rho)$

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On **nat**:

- $[\![0]\!]\rho \mathrel{\hat{=}} 0$, $[\![1]\!]\rho \mathrel{\hat{=}} 1$, etc.
- $[\![\mathbf{s}(M)]\!]\rho \mathrel{\hat{=}} [\![M]\!]\rho{+}1$,
  $[\![\mathbf{p}(M)]\!]\rho \mathrel{\hat{=}} [\![M]\!]\rho{-}1$ if $[\![M]\!]\rho{\neq}0, \bot$
  $\bot$ otherwise
- $[\![\mathbf{if}\ M = 0\ \mathbf{then}\ N\ \mathbf{else}\ P]\!]\rho \mathrel{\hat{=}}$
  $[\![N]\!]\rho$ if $[\![M]\!]\rho{=}0$
  $[\![P]\!]\rho$ if $[\![M]\!]\rho{\neq}0, \bot$
  $\bot$ if $[\![M]\!]\rho{=}\bot$

Lambda-calculus:

- $[\![x]\!]\rho \mathrel{\hat{=}} \rho(x)$
- $[\![MN]\!]\rho \mathrel{\hat{=}} [\![M]\!]\rho([\![N]\!]\rho)$
- $[\![\lambda x\,.\,M]\!]\rho \mathrel{\hat{=}} (V \mapsto [\![M]\!](\rho[x \mapsto V]))$
- $[\![\mathbf{rec}(M)]\!]\rho \mathrel{\hat{=}} \mathit{lfp}([\![M]\!]\rho)$

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Theorem 1. On a pointed dcpo X, every Scott-continuous map $f : X \to X$ has a least fixed point.

- Expressions have **transparent** semantics
  (functions are functions, application is application, etc.)
- **compositional** semantics: $[\![M]\!]\rho$ defined from the semantics of immediate **subterms** of M
- No **execution** mechanism involved

An operational semantics for PCF

- An **abstract machine** (à la Krivine) = a transition relation between **configurations** C, M

Contexts $C ::= \_ \mid C[\_N] \mid C[\mathbf{s}(\_)] \mid C[\mathbf{p}(\_)] \mid \mid C[\mathbf{if}\ \_ = 0\ \mathbf{then}\ N\ \mathbf{else}\ P]$

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<u>Exploration rules</u> (looking for redexes)

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\begin{aligned}
C, MN &\to C[\_N], M \\
C, \mathbf{s}(M) &\to C[\mathbf{s}(\_)], M \\
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$$

<u>Computation rules</u>

$$
\begin{aligned}
C[\_N], \lambda x\,.\,M &\to C, M[x := N] \\
C[\mathbf{s}(\_)], n &\to C, n+1 \\
C[\mathbf{p}(\_)], n+1 &\to C, n \\
C[\mathbf{if}\ \_ = 0\ \mathbf{then}\ N\ \mathbf{else}\ P], 0 &\to C, N \\
C[\mathbf{if}\ \_ = 0\ \mathbf{then}\ N\ \mathbf{else}\ P], n+1 &\to C, P \\
C, \mathbf{rec}(M) &\to C, M(\mathbf{rec}(M))
\end{aligned}
$$

… i.e., an interpreter

```
type term = V of string                    (* variables *)
          | A of term * term               (* applications *)
          | L of string * term             (* abstractions *)
          | R of term                      (* rec(M) *)
          | C of int                       (* 0, 1, 2, ... *)
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let rec exec (c:ctx) (t:term) =
    match t with
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```
and compute (c:ctx) (t:term) =
    match c, t with
      Ac(c',n), L(x,m)          ->   exec c' (subst m x n)
    | Sc(c'), I(i)              ->   exec c' (I(i+1))
    | Pc(c'), I(i) when i>0     ->   exec c' (I(i-1))
    | Ic(c',n,p), I(0)          ->   exec c' n
    | Ic(c',n,p), I(i) when i>0 ->   exec c' p
    | _, R(m)                   ->   exec c (A(m,t))
    | Ec, result                ->   result (* finished! *)
    | _, _ -> failwith « stumped »
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… which really runs

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goubault@macbook-pro-de-jean Topics2023 % gimml
GimML comes with ABSOLUTELY NO WARRANTY (See file COPYRIGHT).
GimML for Darwin 20.5.0, by Jean Goubault-Larrecq (c) 2021.
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- **Note:** only works at the **observable** type **nat**
  (although some variants can be made to work at function types, too [BloomRiecke 88])

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- Let $M \cong N$ if and only if
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- **Fact.** If $[\![M]\!]=[\![N]\!]$ then $M \cong N$.

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- *Proof.* If $C, M \to^* \_, n$ then $[\![C[M]]\!] = n$ (soundness)
  so $[\![C[N]]\!] = n$ (denotational semantics is **compositional**)
  so $C, N \to^* \_, n$ (adequacy) $\square$

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- **Full abstraction** is when this is an equivalence…
  and it fails in PCF [Sazonov76, Plotkin 77]:
  some murky issue with **parallel or**
  (a function that exists **semantically** but that not in the **syntax**)
  … which I will ignore here

Probabilistic PCF

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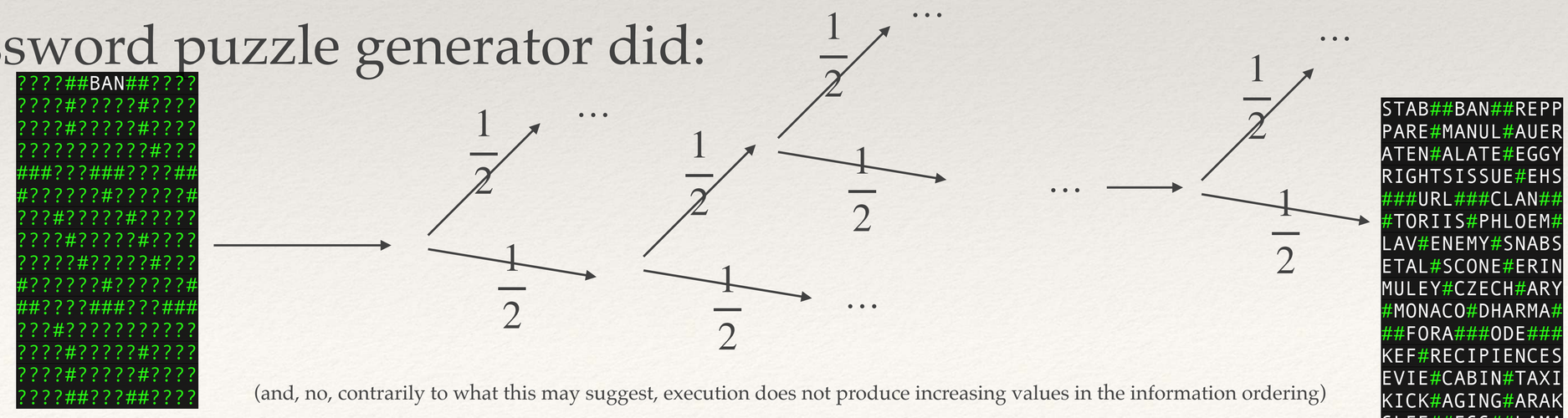
- Denotational semantics is now (as a first approach) a (sub)**probability distribution** over possible outcomes
- What our crossword puzzle generator did:

(more or less)



(and, no, contrarily to what this may suggest, execution does not produce increasing values in the information ordering)

Probabilistic PCF

$M, N, P, \ldots ::= \ldots$	(as in PCF)
$\mid M \oplus N$	probabilistic choice
$\mid \mathbf{ret}\ M$	monad unit
$\mid \mathbf{do}\ x_\sigma = M; N$	sequential composition

Probabilistic PCF

- $M, N, P, \ldots ::= \ldots$ (as in PCF)
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  - $\mid \mathbf{ret}\ M$ monad unit
  - $\mid \mathbf{do}\ x_\sigma = M; N$ sequential composition
- Types: $\sigma, \tau, \ldots ::= \mathbf{nat} \mid \mathbf{unit} \mid \sigma \to \tau \mid \mathbf{T}\tau$

  $\mathbf{T}\tau$: monadic types [Moggi 91]

$$\frac{M : \mathbf{T}\tau \qquad N : \mathbf{T}\tau}{M \oplus N : \mathbf{T}\tau} \qquad \frac{}{* : \mathbf{unit}}$$

$$\frac{M : \tau}{\mathbf{ret}\ M : \mathbf{T}\tau} \qquad \frac{M : \mathbf{T}\sigma \qquad N : \mathbf{T}\tau}{\mathbf{do}\ x_\sigma = M; N : \mathbf{T}\tau}$$

$\mathbf{T}\tau$ = type of **(first-class)** distributions

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  $\mathbf{T}\tau$: monadic types [Moggi 91]

- New operational rules:

Exploration rules

$$C, \mathbf{do}\ x = M; N \to C[\mathbf{do}\ x = \_; N], M$$
$$\_, \mathbf{ret}\ M \to \mathbf{ret}\ \_, M$$

Computation rules

$$C[\mathbf{do}\ x = \_; N], \mathbf{ret}\ M \to C, N[x := M]$$
$$C, M \oplus N \to^{1/2} M$$
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type term = V of string                    (* variables *)
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    | Do(x,m,n) ->  exec (Dc(c,x,n),m)
    | Ret(m) when c==Ec
                  -> exec (Rc,m)
      (* computation *)
    | _ ->            compute c t
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and compute (c:ctx) (t:term) =
    match c, t with
      ...
    | Dc(c',x,n), Ret(m)   ->     exec c' (subst n x m)
    | _, Flip(m,n)         ->
      if Random.bool() then      exec c m
                       else      exec c n
    | Rc, result           ->    result (* finished! *)
    | Ec, result           ->    result (* finished! *)
    | _, _ -> failwith « stumped »
```

$$C[\mathbf{do}\ x = \_; N], \mathbf{ret}\ M \to C, N[x := M]$$
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Let’s run it, on another example

```
goubault@macbook-pro-de-jean Topics2023 % gimml
GimML comes with ABSOLUTELY NO WARRANTY (See file COPYRIGHT).
GimML for Darwin 20.5.0, by Jean Goubault-Larrecq (c) 2021.
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_ · *Rand3*

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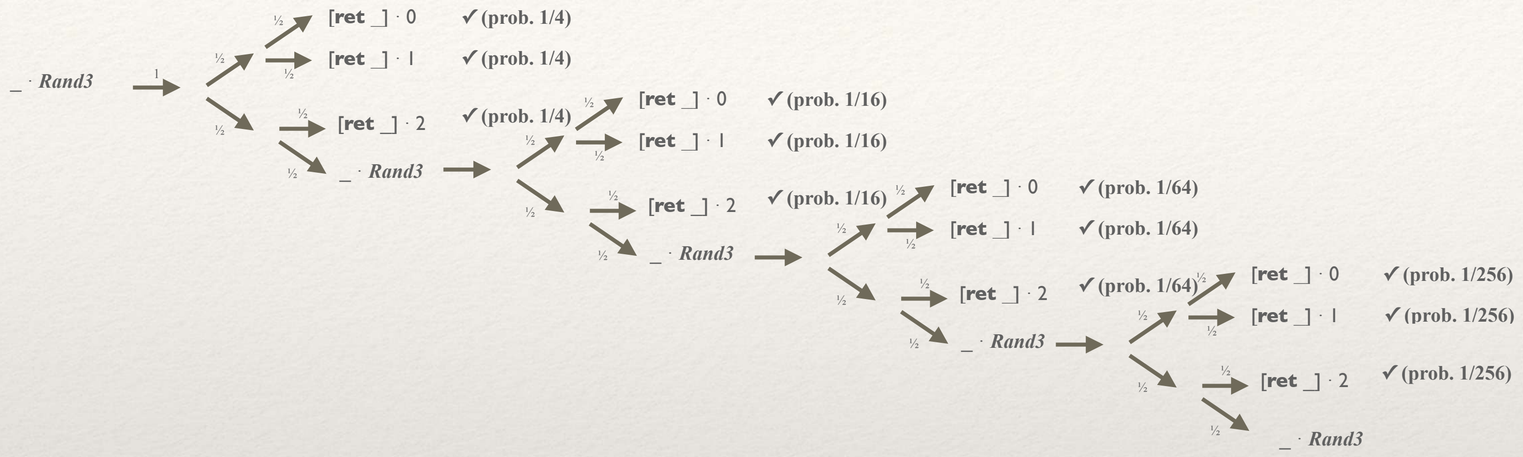

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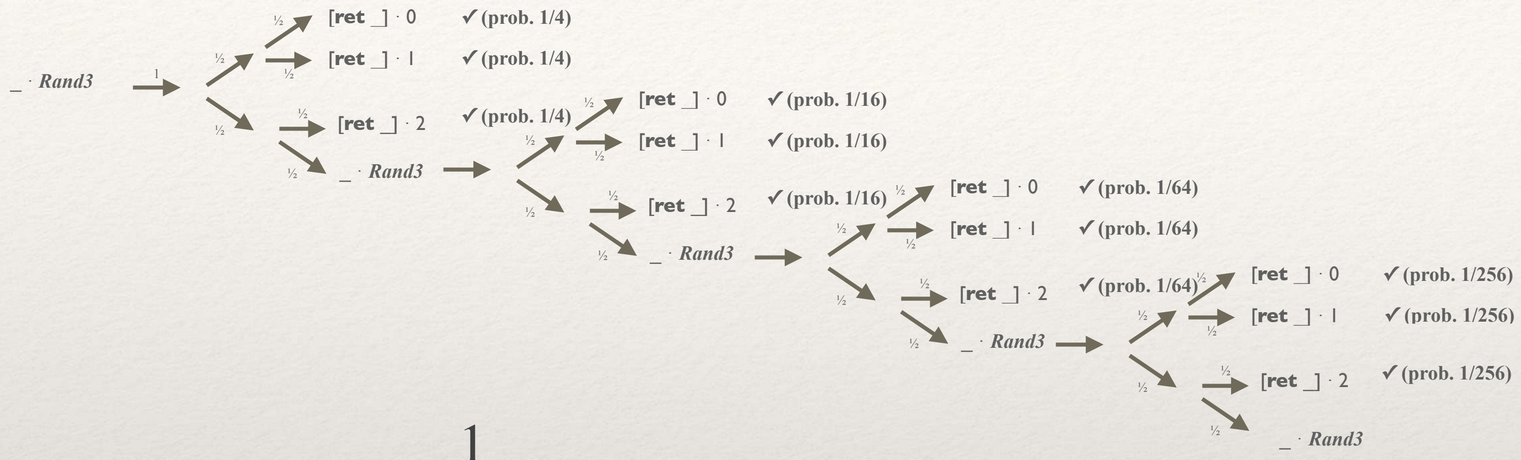

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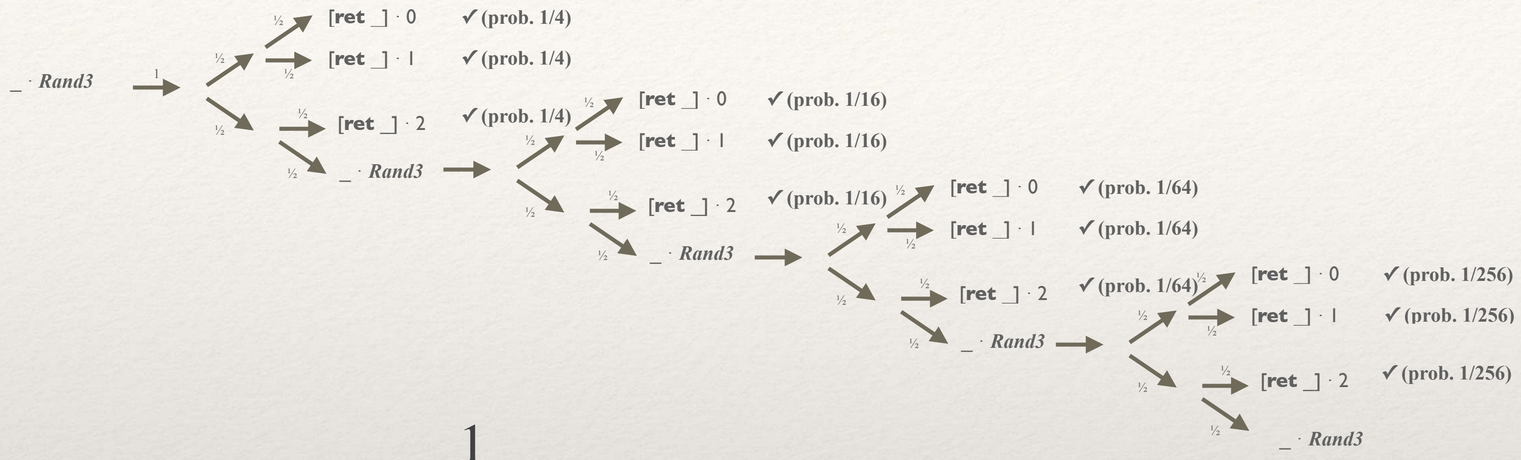

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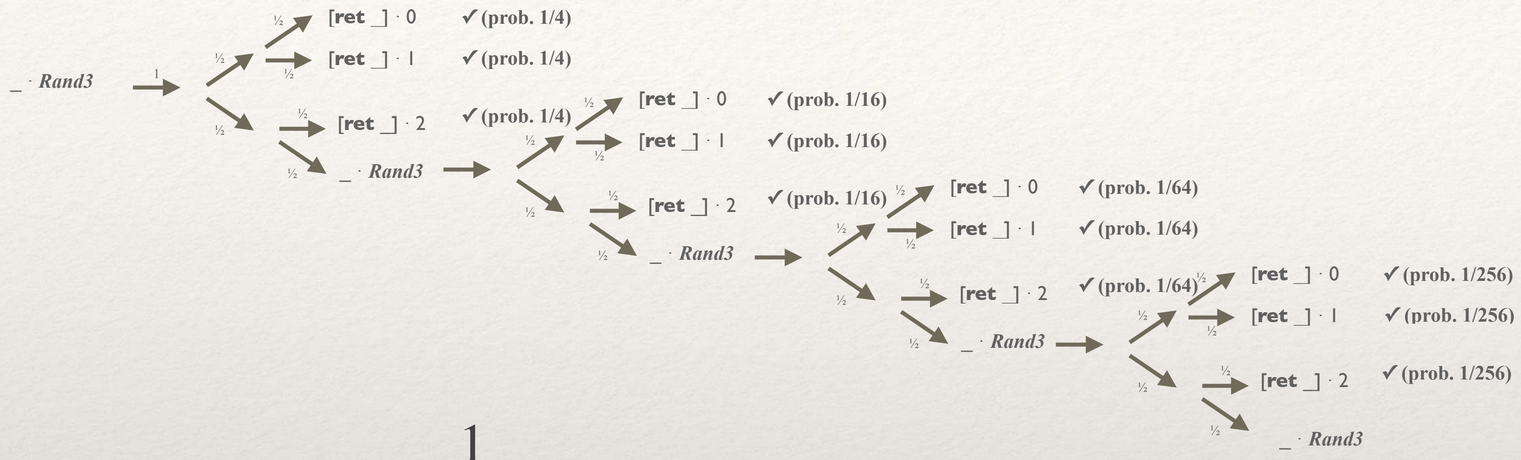

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- **Operationally**: $\dfrac{1}{4} + \dfrac{1}{16} + \dfrac{1}{64} + \dfrac{1}{256} + \cdots = \dfrac{1}{3}$
- **Denotationally**: $[\![Rand3]\!]\rho(\{n\})$ (denotational measure of getting n) $= x$

  where $\dfrac{1}{4} + \dfrac{1}{4}\,.\,x = x$ (**rec** defines a **fixed point**)

Denotational semantics for probabilistic PCF

- Introduced in Claire Jones' PhD thesis [Jones 90]
- $[\![\mathbf{T}\tau]\!] \triangleq \mathbf{V}_{\leq 1}([\![\tau]\!])$ dcpo of **subprobability valuations** on $[\![\tau]\!]$
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Let us define all that first!

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Continuous valuations

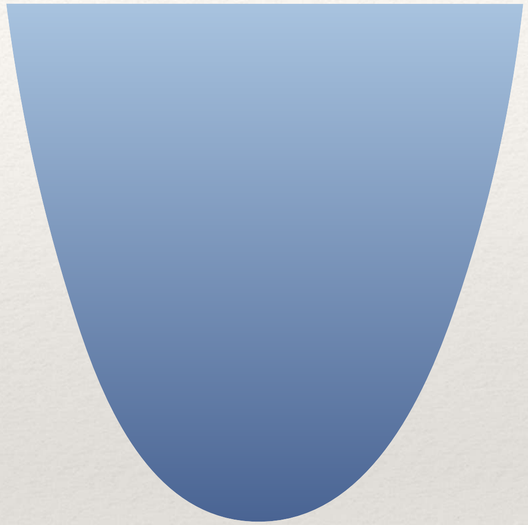
… We come to some topology at last

- Measures give mass to **measurable** subsets
- Following a remark by [Smyth 83] that testable property=open, Jones remarks that it is more natural to measure **open** subsets
- Hence what are the **open** subsets of a dcpo?

  Let me introduce the **Scott topology.**

The Scott topology

- A subset U of a dcpo X is **Scott-open** iff:



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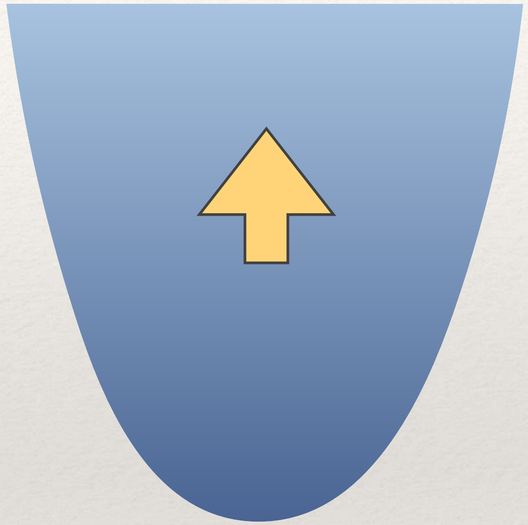

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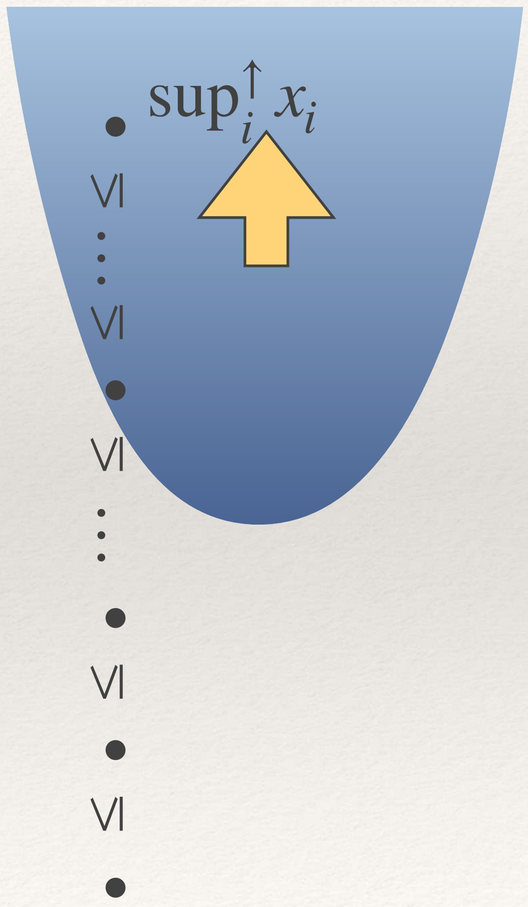

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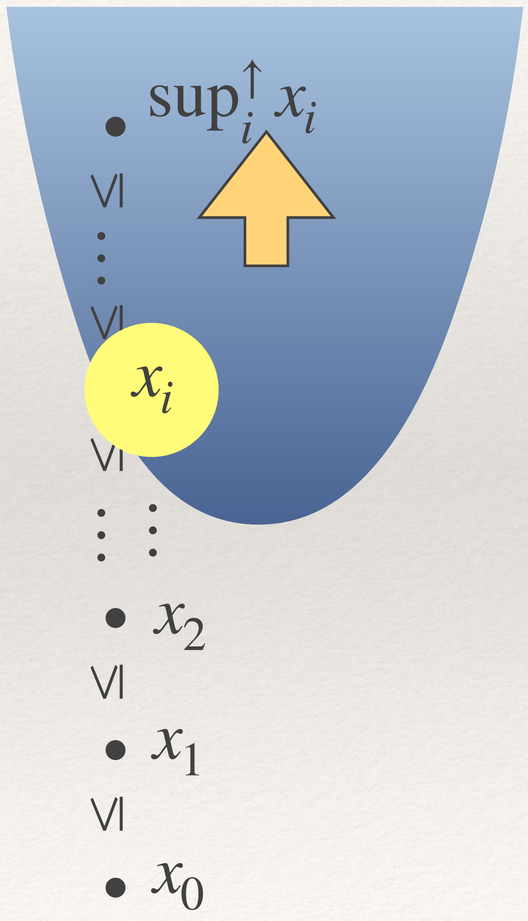

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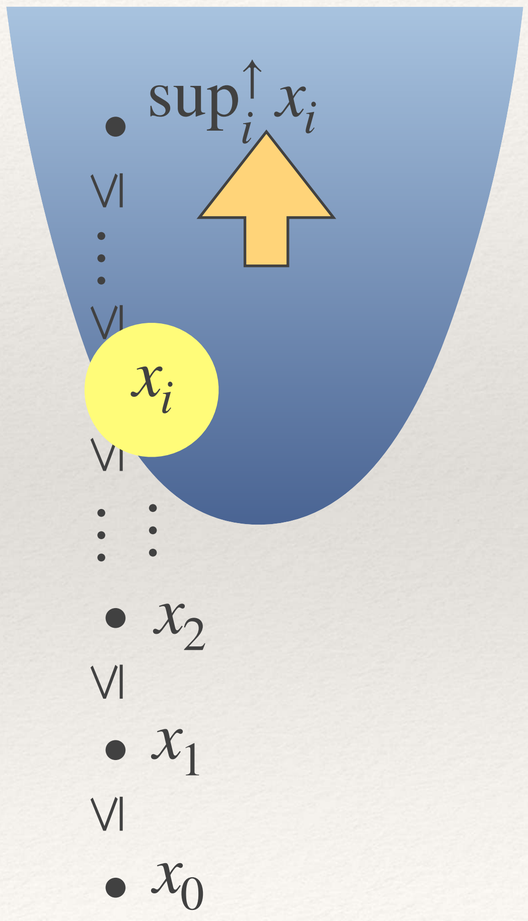

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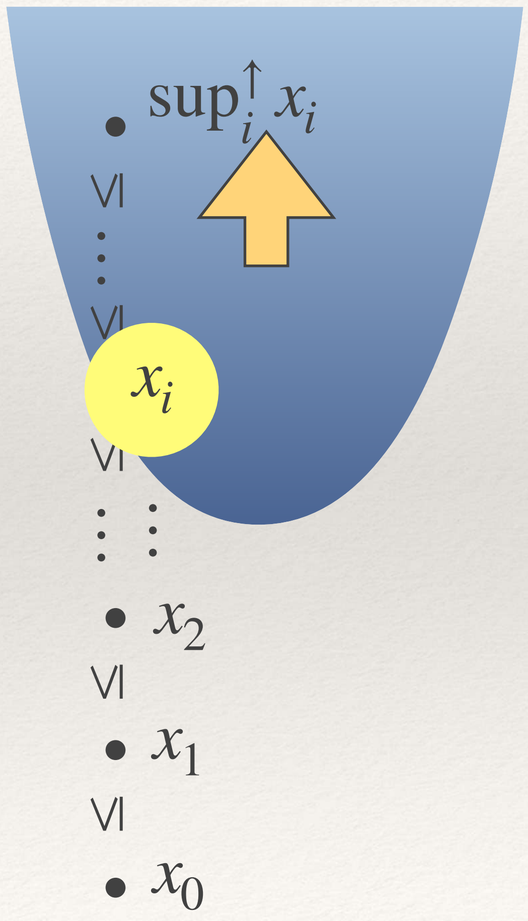

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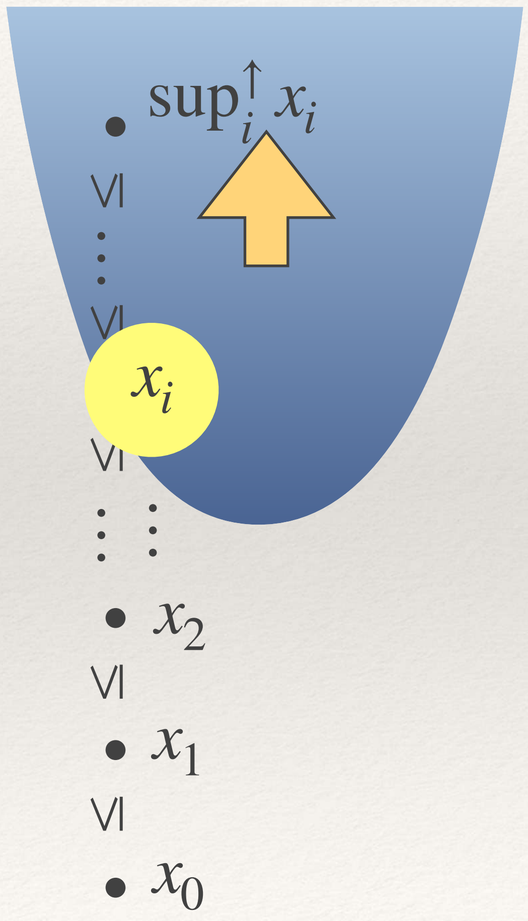
- The **Scott-continuous** maps are exactly the **continuous** maps for the Scott topologies



Continuous valuations

- First studied by [SahebDjahromi 80]: gives mass to **Scott-open** subsets
- Makes sense on **every topological space**, not just dcpos with the Scott topology
- Let $\mathcal{O}X$ denote the lattice of open subsets of a space X
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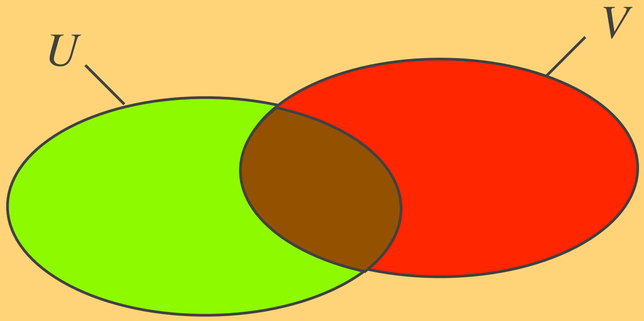

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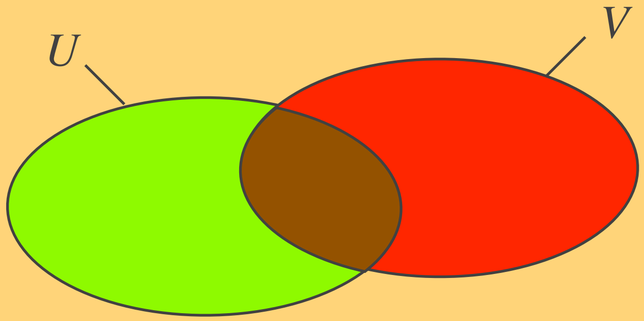

- A **continuous valuation** is not just monotonic, but Scott-continuous: $\nu(\sup_i^{\uparrow} U_i) = \sup_i^{\uparrow} \nu(U_i)$.

Simple valuations

- **Definition.** The **Dirac valuation** δ_x:

$$\delta_x(U) \triangleq \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

  is a continuous valuation.

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- always Scott-continuous

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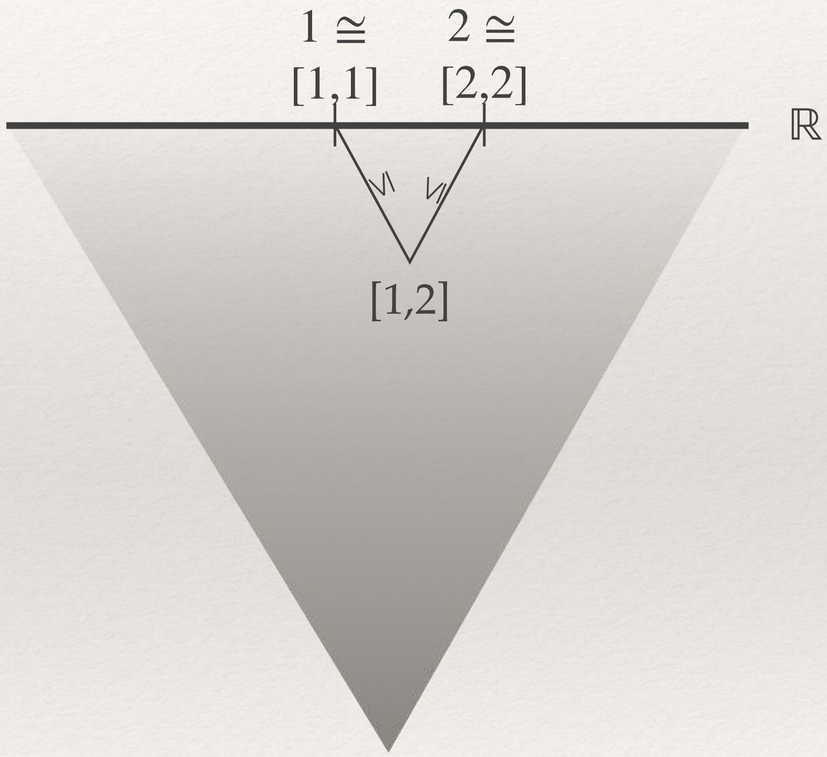
- Here is an example on a **dcpo**…

The dcpo $\mathbb{IR}$

- Elements of $\mathbb{IR}$: intervals $[a, b]$ with $a \leq b \in \mathbb{R}$
  We equate $[a, a]$ with $a \in \mathbb{R}$, so $\mathbb{R} \subseteq \mathbb{IR}$
- Ordered by **reverse** inclusion:
  $[a, b] \leq [c, d]$ iff $[a, b] \supseteq [c, d]$



The dcpo $\mathbf{I}\mathbb{R}$

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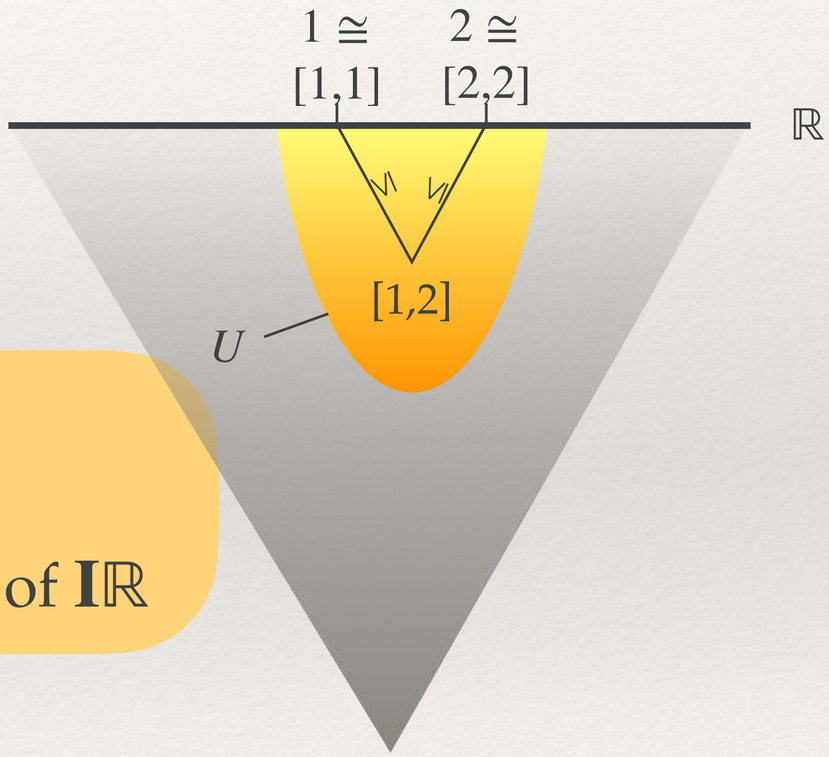

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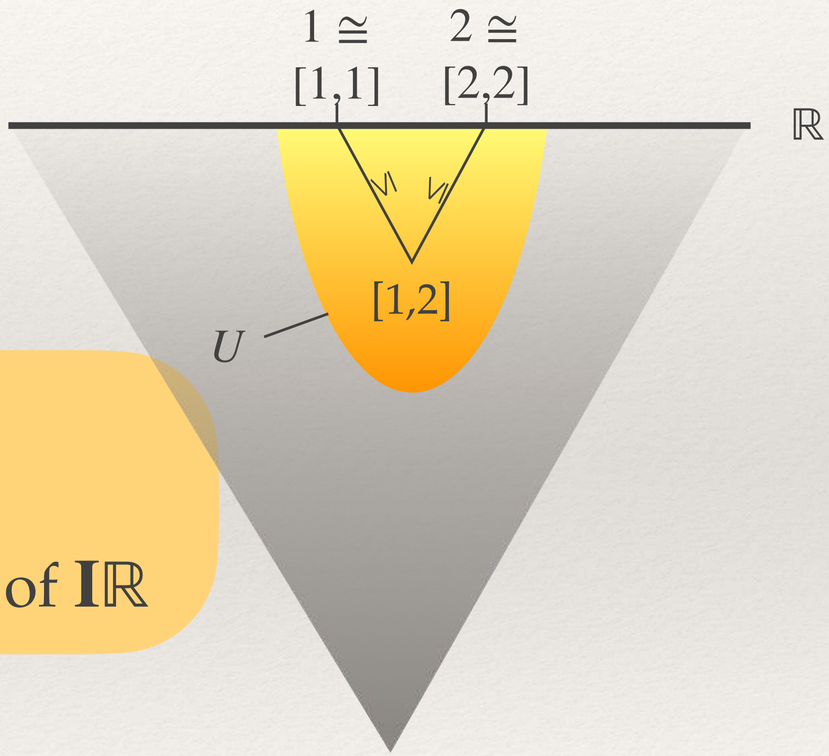
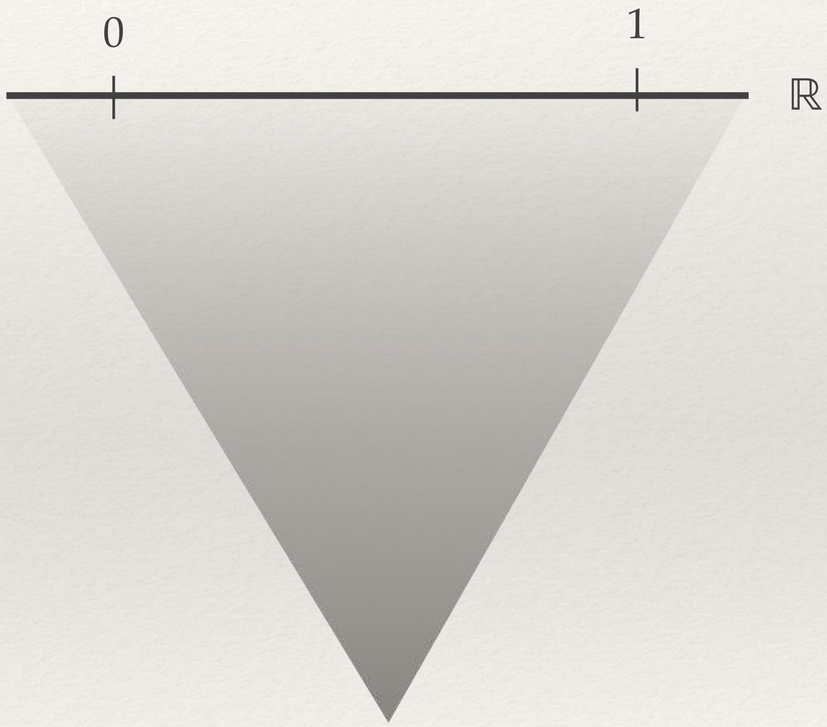


- Now let $\bar{\lambda}(U) \triangleq \lambda(U \cap \mathbb{R})$, for every $U \in \mathcal{O}(\mathbf{I}\mathbb{R})$:
  $\bar{\lambda}$ is a continuous valuation on the dcpo $\mathbf{I}\mathbb{R}$
  that is **not a simple valuation** (Note: $\bar{\lambda}$ is supported on $\mathbb{R}$)

Beyond simple valuations

- Maybe we only need **simple** valuations in computer science?
- No: simple valuations do not form a **dcpo**



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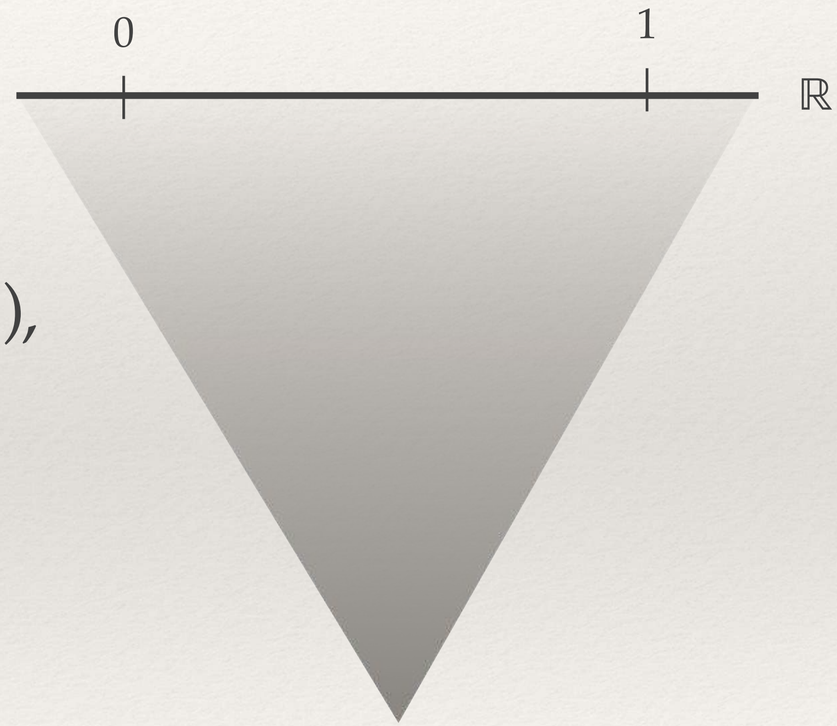

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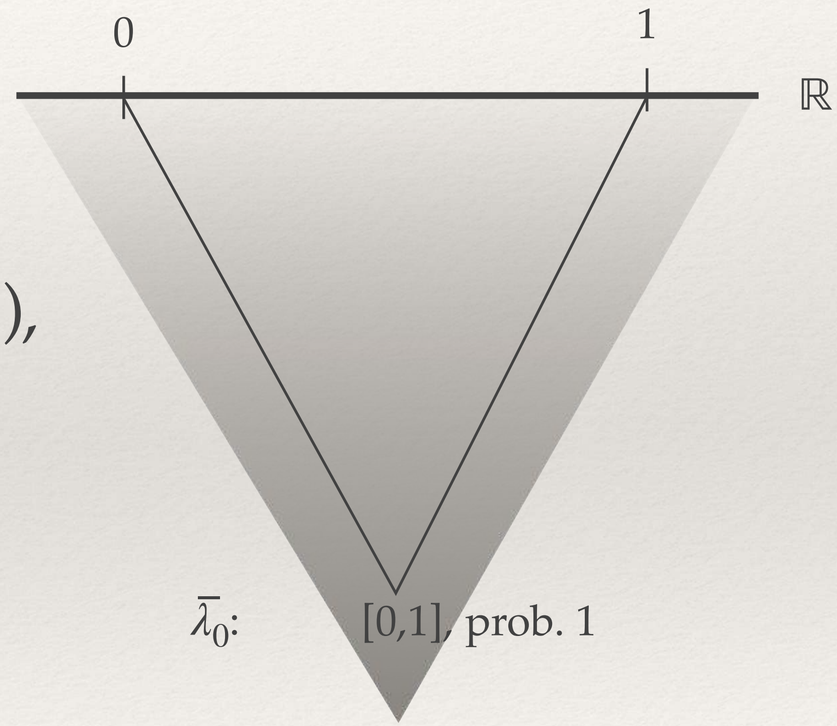

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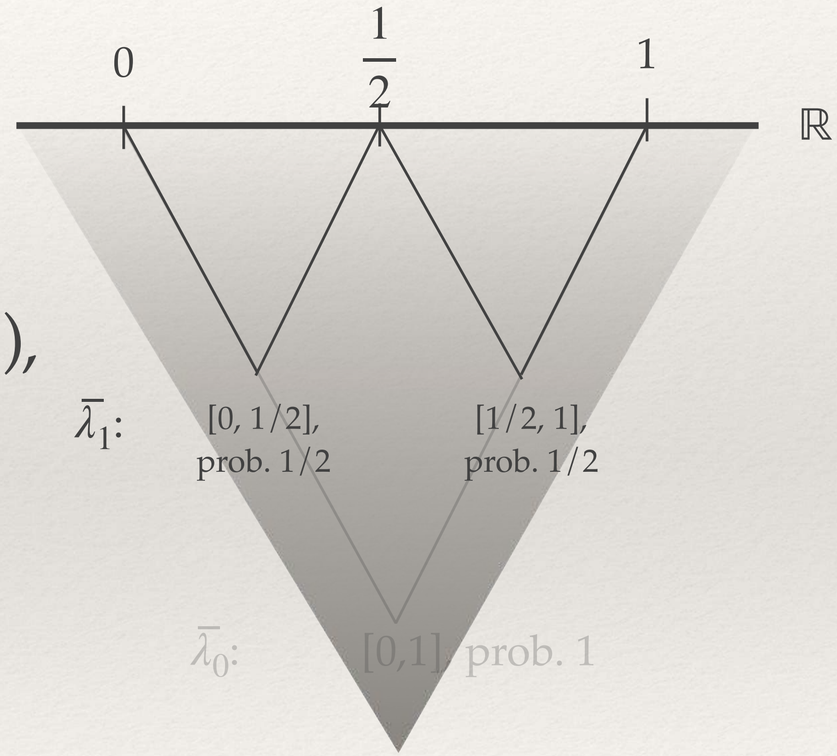

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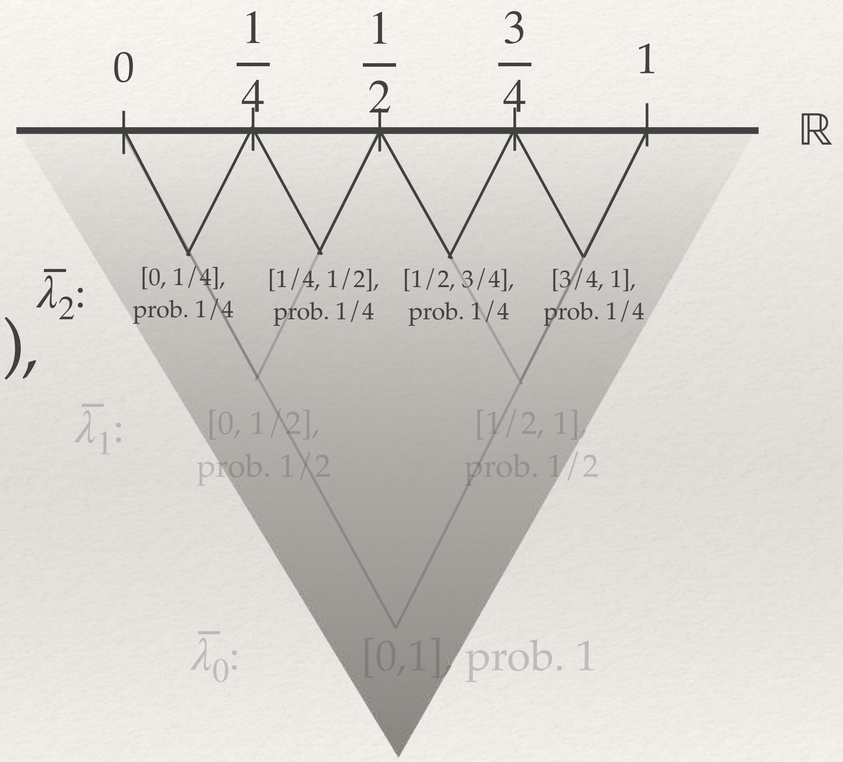

Continuous valuations and measures

Continuous valuation $\nu : \mathcal{O}X \to \overline{\mathbb{R}}_+$:
- **strict**: $\nu(\emptyset) = 0$
- **Scott-continuous,**
- **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$

- **Theorem [Adamski 77].** Given any **Borel measure** on a hereditarily Lindelöf space, (in particular, a second-countable space) its restriction to open sets is a **continuous valuation**.
  (i.e., every Borel measure on such a space is *τ-smooth.*)

- **Theorem [de Brecht, JGL, Jia, Lyu 2019].** Every **continuous valuation** on an LCS-complete space (a G_δ subspace of a locally compact sober space) extends to a **Borel measure**.

- A sweet spot: de Brecht's **quasi-Polish** spaces (=2nd countable LCS-complete)
  Those include all **ω-continuous dcpos** + all **Polish** spaces

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$\overline{\mathbb{R}}_+$:

$) = \nu(U) + \nu(V)$

Just remember that
continuous valuations~measures,
in most cases (not all, though)

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Integration

- For every **lower semicontinuous** map $f\colon X \to \overline{\mathbb{R}}_+$ and every continuous valuation μ on X,

  there is an **integral** $\displaystyle\int_x f(x)d\mu$ (or $\displaystyle\int f\, d\mu$ for short)

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If X is a dcpo, this just means **Scott-continuous**, but the definition works for any topological space X

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  Really defines a **strong monad** on **Dcpo**

Soundness, adequacy

- Operationally, let $\Pr[C, M \downarrow n] \mathrel{\hat{=}}$ the (sub)probability that $C, M \to^* \mathbf{ret}\ \_, n$
  — sum over all executions $C, M \to^* \mathbf{ret}\ \_, n$
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- **Theorem (adequacy).** $\Pr[C, M \downarrow n] = [\![C[M]]\!]\rho(\{n\})$ (at type **nat**)
- Soundness is a bit harder than before: requires $[\![C]\!]$ **linear**
- Adequacy: by a suitable form of logical relations

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- Fact. If $[\![M]\!]=[\![N]\!]$ then $M \cong N$. (as in PCF, consequence of adequacy)
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- There is a second obstacle to full abstraction here:
  the **statistical testers** $[V > r] \in [\![\mathbf{T}\ \tau \to \mathbf{T}\ \mathbf{unit}]\!]$
  mapping $\nu \in [\![\mathbf{T}\ \tau]\!]$ to δ_* if $\nu(V) > r$
  to 0 otherwise
  are **not definable** in probabilistic PCF
  … leading to another cause of failure of full abstraction

No full abstraction (2/2)

- Fact. If $[\![M]\!]=[\![N]\!]$ then $M \cong N$.
- Even if we added primitive parallel or and statistical testers, we do not have the **technology** to prove the converse.
- The problem is that we need all dcpos $[\![\tau]\!]$ to be at least

  continuous dcpos (a.k.a. **domains**)

  to hope to obtain a proof

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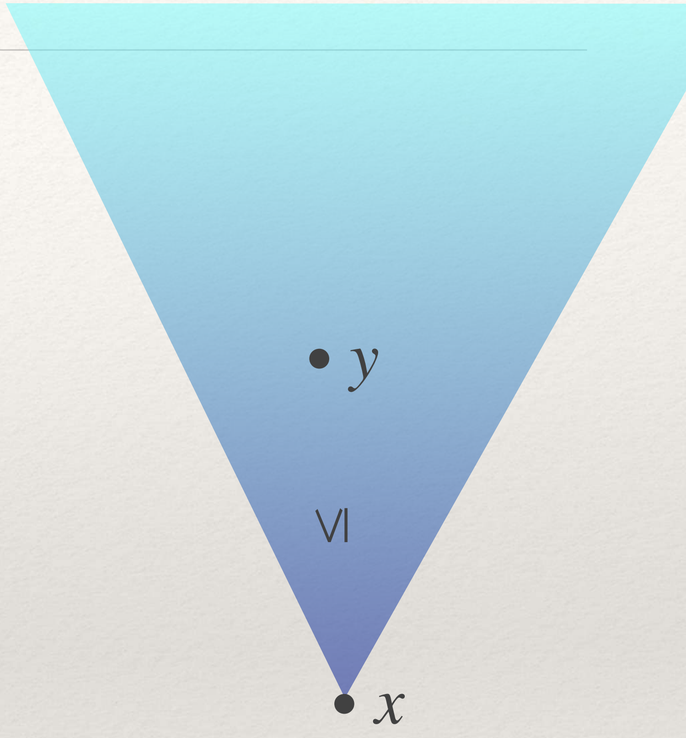

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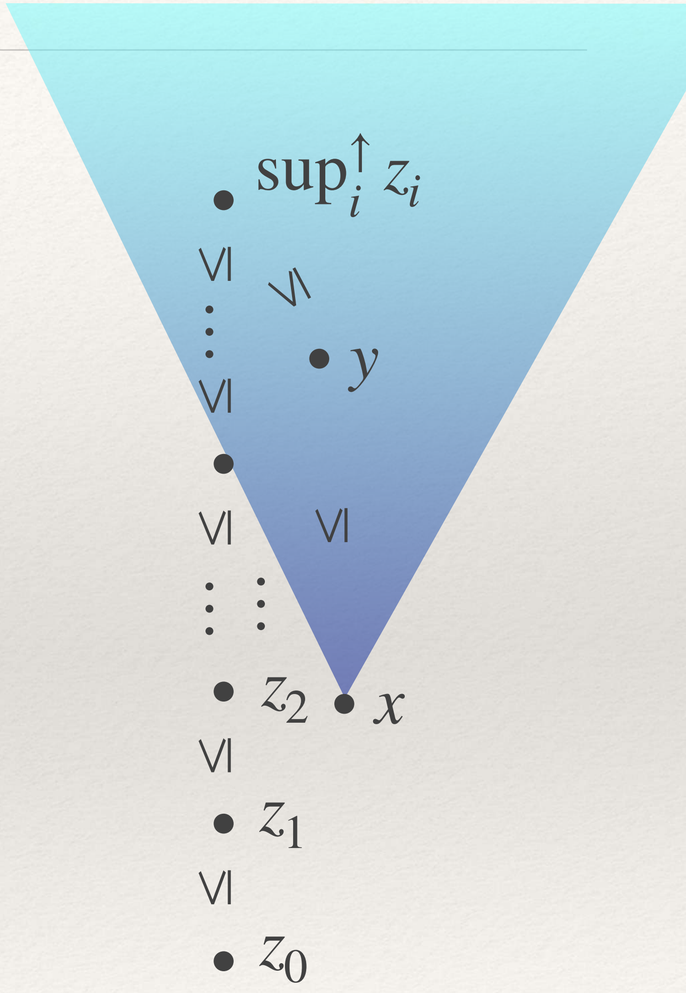

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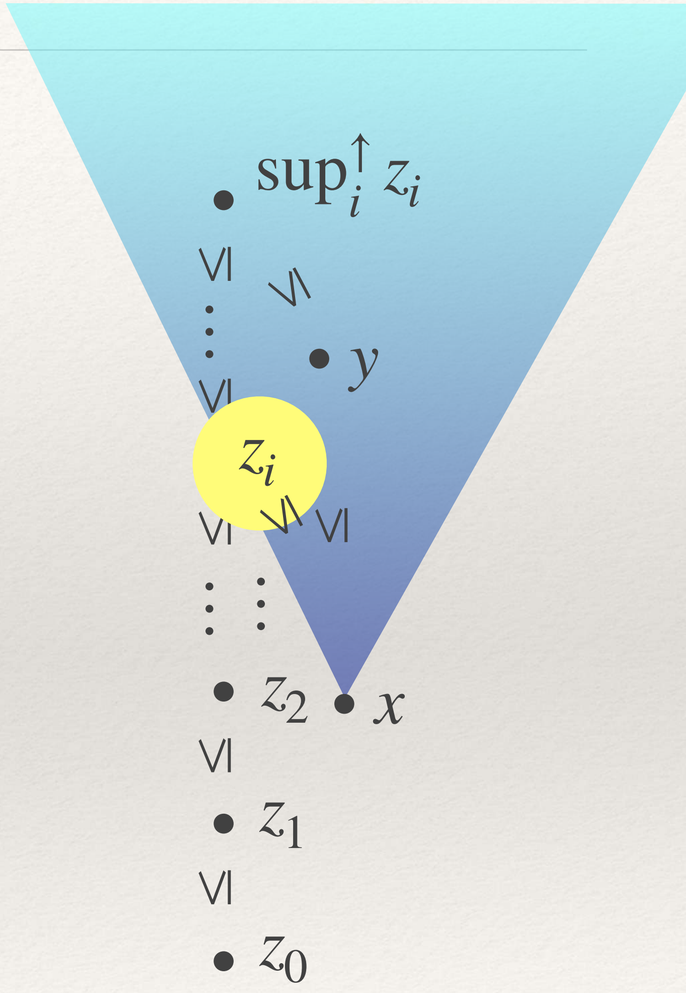

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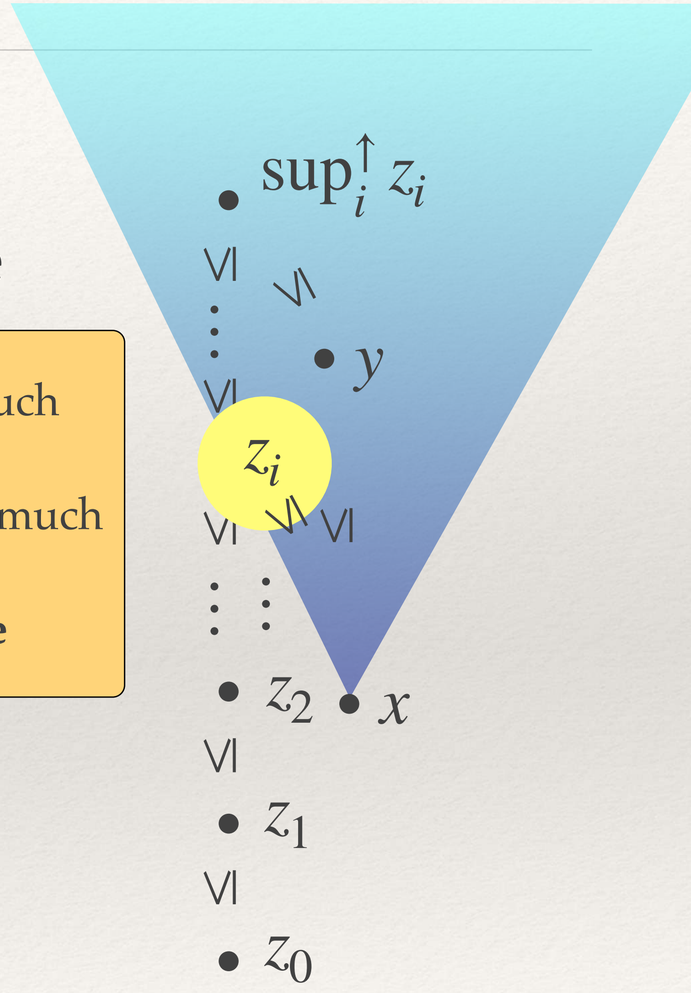

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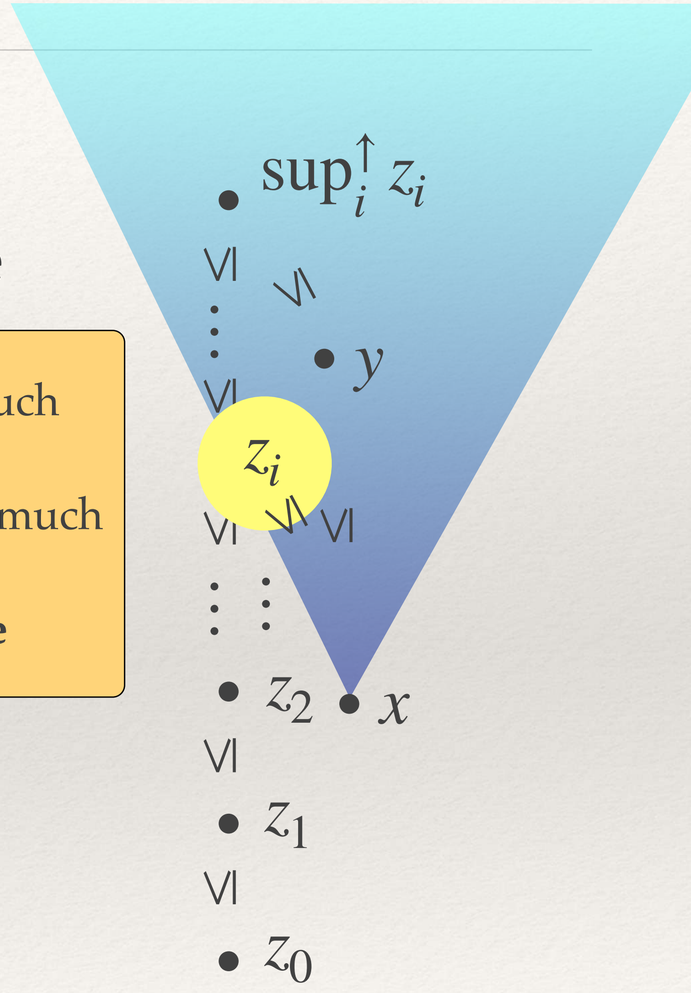
- **Definition.** A dcpo is **continuous** iff
  every point x is the supremum
  of some directed family of points **way-below** x.



Examples, counterexamples

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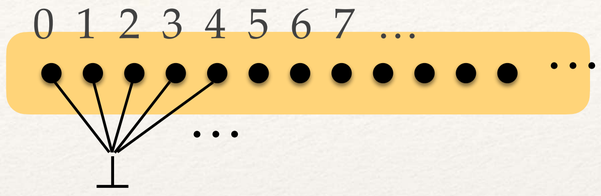

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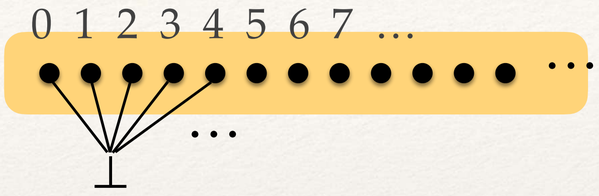

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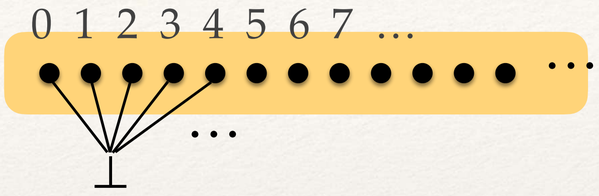

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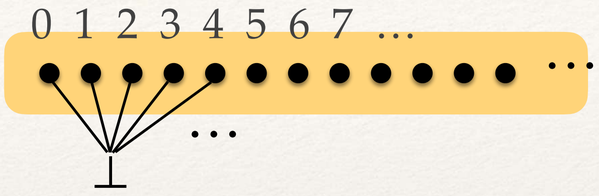

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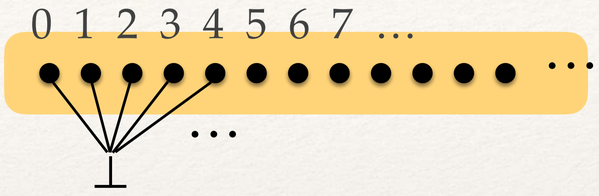

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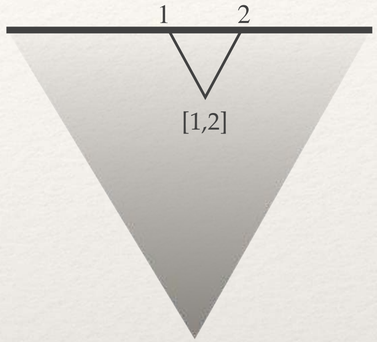

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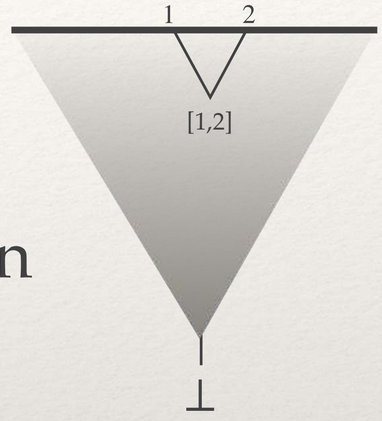

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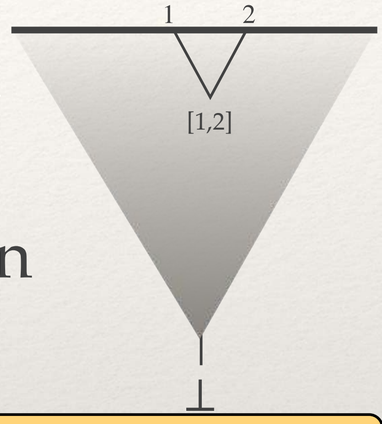

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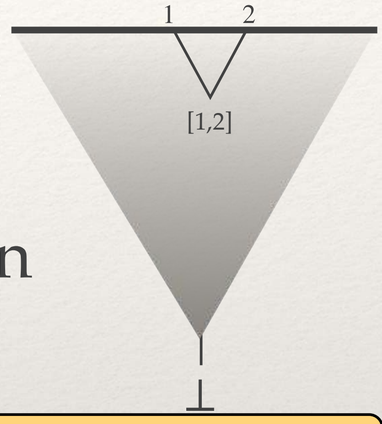

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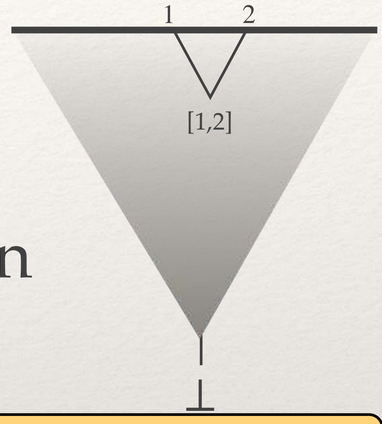

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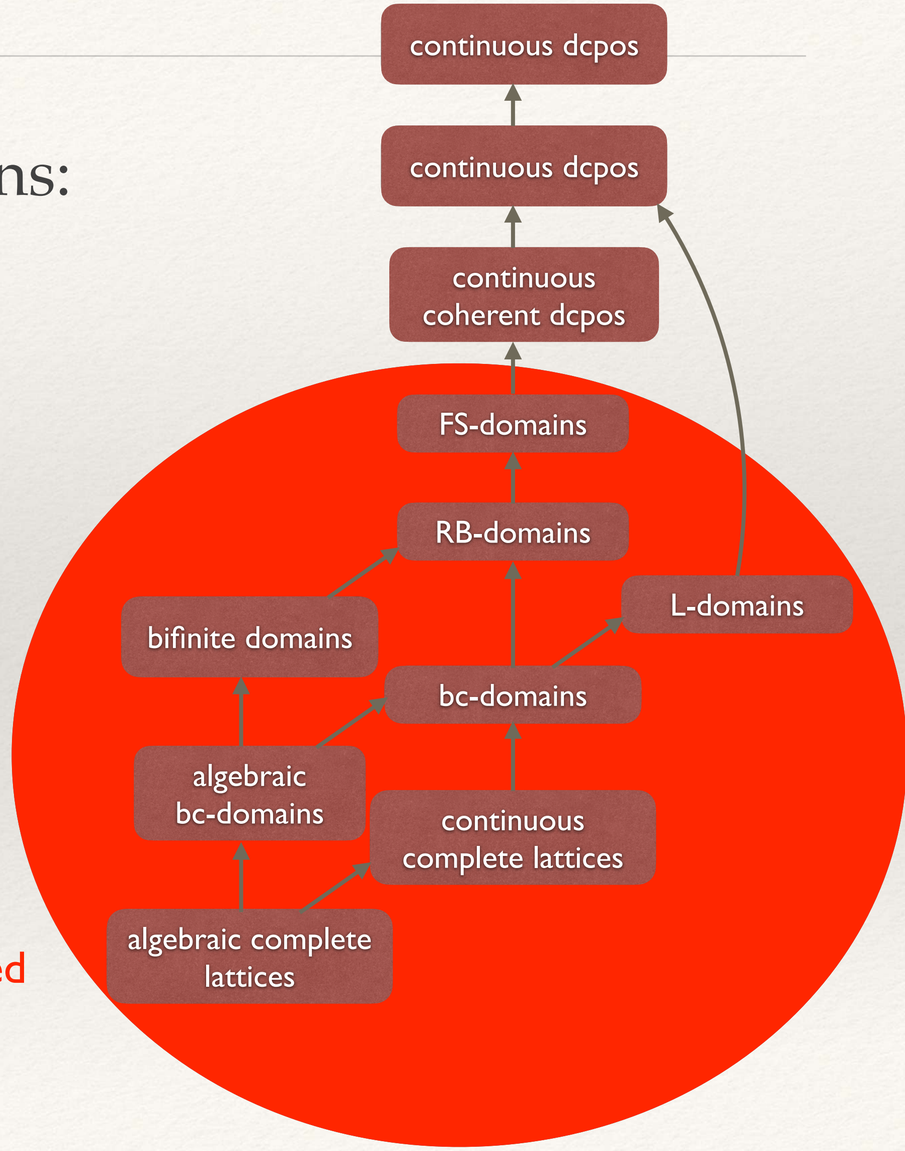

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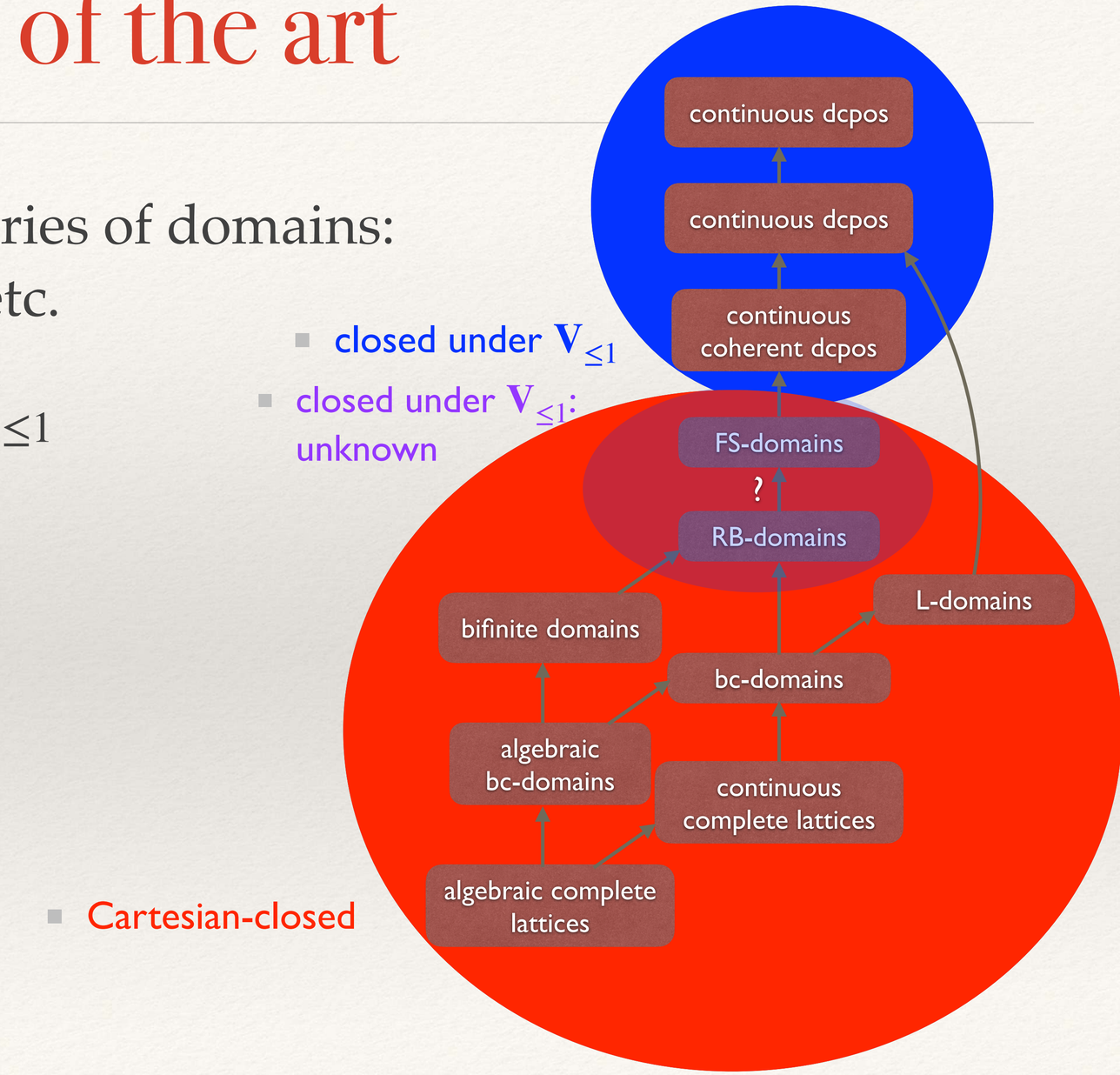

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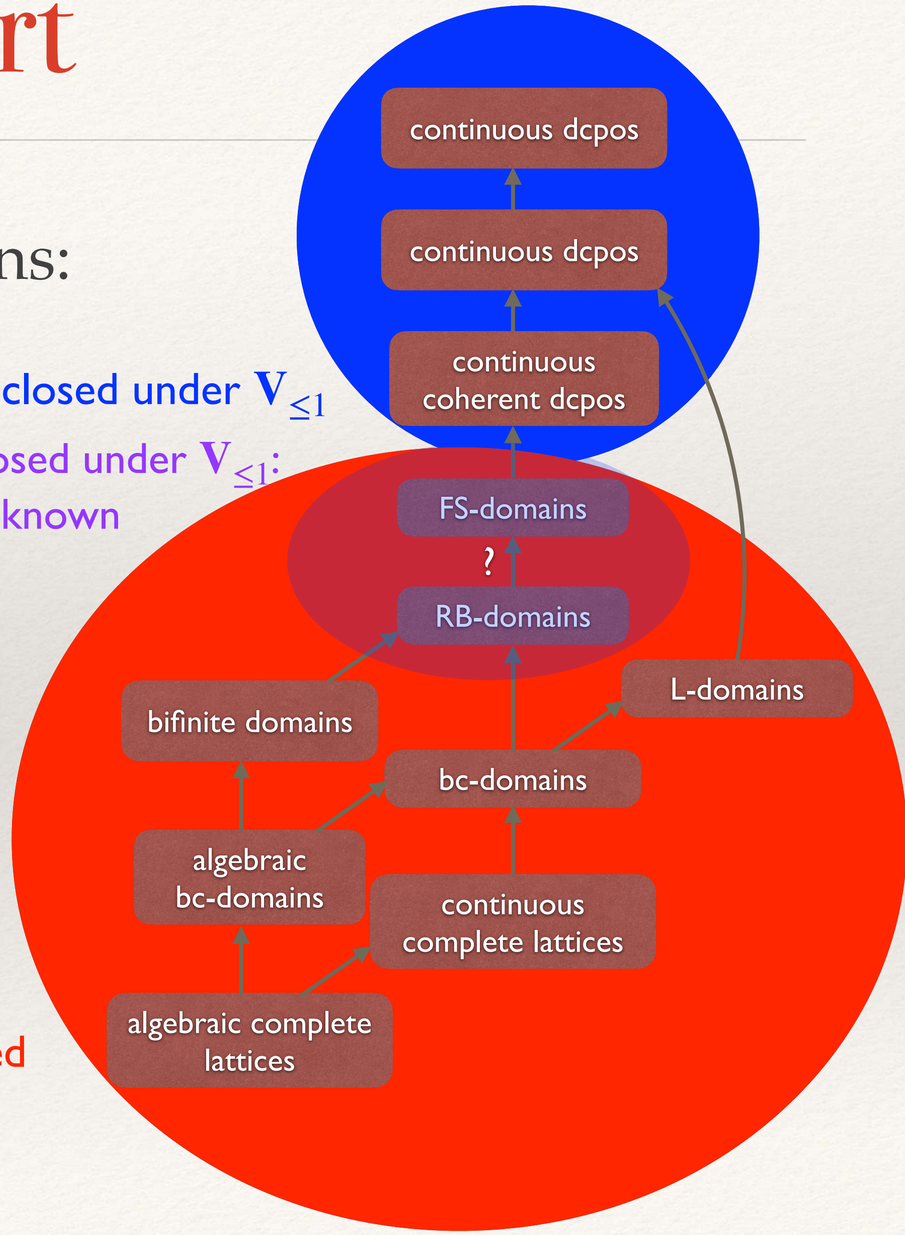
- As of 2023, the best results are still those of [Jung,Tix 98]

apart from [JGL 12] ($\mathbf{V}_{\leq 1}$(QRB-domain) is a QRB-domain)
or [Mislove 20] ($\mathbf{V}_{\leq 1}$(chain) is a continuous lattice)
or [JGL 22] ($\mathbf{V}_{\leq 1}$(quasi-cont. dcpo) is quasi-continuous)



Statistical programming languages

Statistical programming languages

- Meanwhile, statisticians were inventing their own languages in order to describe **probabilistic processes:**
  - — Church [Goodman,Mansinghka,Roy,Bonawitz,Tenenbaum 08]
  - — Anglican [Wood,van de Meent,Mansinghka 14]
  - — WebPPL [Wood,Stuhlmüller 14]
  - — Venture [Mansinghka,Selsam,Perov 14]
  - — Hakaru [Shan, Ramsey 17]
  - — PPCF [Ehrhard, Pagani, Tasson 18]
  - — PCFSS [Dal Lago, Hoshino 19]
  - — etc.

```
(define (DP alpha proc)
  (let ((sticks (mem (lambda x (beta 1.0 alpha))))
        (atoms  (mem (lambda x (proc)))))
    (lambda () (atoms (pick-a-stick sticks 1)))))

(define (pick-a-stick sticks J)
  (if (< (random) (sticks J))
      J
      (pick-a-stick sticks (+ J 1))))

(define (DPmem alpha proc)
  (let ((dps (mem (lambda args
                    (DP alpha
                        (lambda () (apply proc args
                        )))))
    (lambda argsin ((apply dps argsin))) ))
```

Dirichlet processes in Church

```
[assume initial-state-dist (list (/ 1 3) (/ 1 3) (/ 1 3))]
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Hidden Markov models in Venture

Statistical programming languages

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Dirichlet processes in Church

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distributions over the reals

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Dirichlet processes in Church

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reals

conditioning

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Dirichlet processes in Church

Usually implemented as **floating-point**…
but we can do better
(from a formal
point of view)

reals

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Hidden Markov models in Venture

Semantics

- The difficulty of handling **continuous** distributions and **real numbers** spurred the invention of new semantical domains:
  - quasi-Borel spaces/predomains [Heunen,Kammar,Staton,Yang 17; Vákár, Kammar, Staton 21]
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- … especially since [Jung, Tix 98] was **misinterpreted** as a failure of domain theory — we will give a dcpo semantics

Exact real arithmetic

- **Exact real arithmetic** can be implemented in many ways
- E.g., [Boehm 87] encodes a real number x by a **function** $n \in \mathbb{N} \mapsto x_n \in \mathbb{Z}$

  such that $\left| x - \frac{x_n}{4^n} \right| \leq \frac{1}{4^n}$

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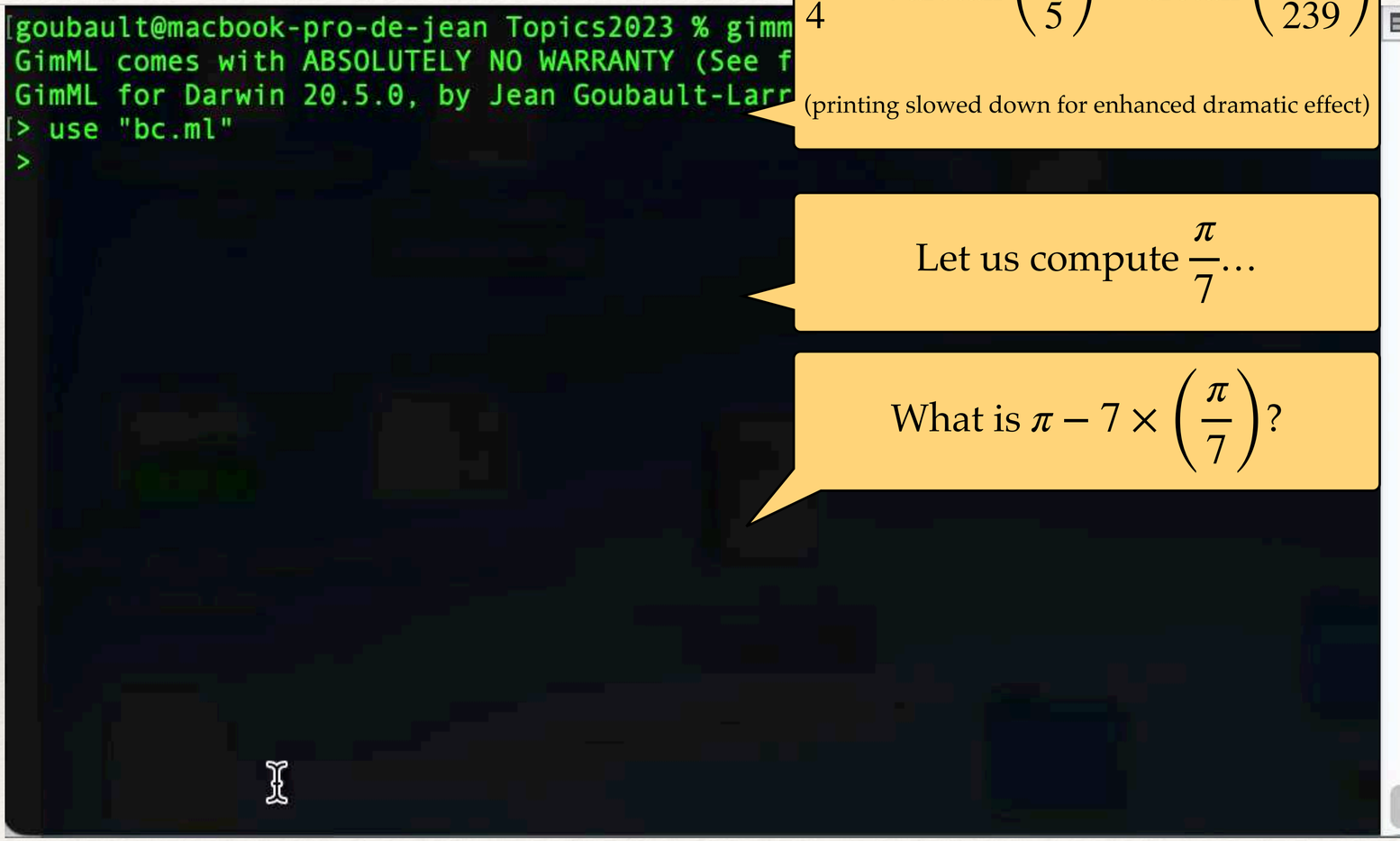
- x is really $\sup^{\uparrow}_{n \in \mathbb{N}} \left[\frac{x_n}{4^n}, \frac{x_n + 1}{4^n} \right]$ in $\mathbb{IR}_{\perp}$, as we now demonstrate…



Eg., compute π by Machin's formula

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

(printing slowed down for enhanced dramatic effect)

Let us compute $\frac{\pi}{7}$…

What is $\pi - 7 \times \left(\frac{\pi}{7}\right)$?

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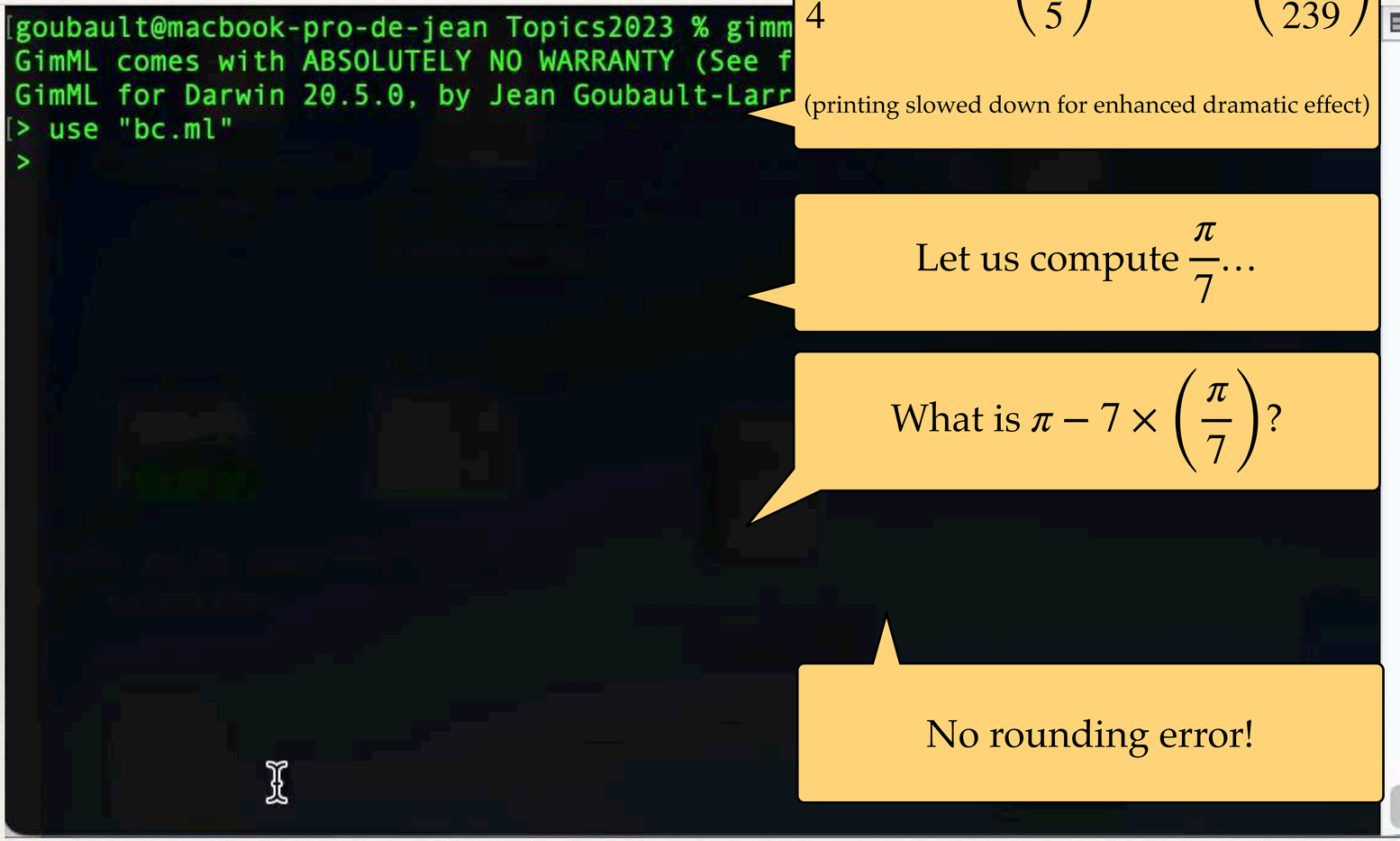

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No rounding error!

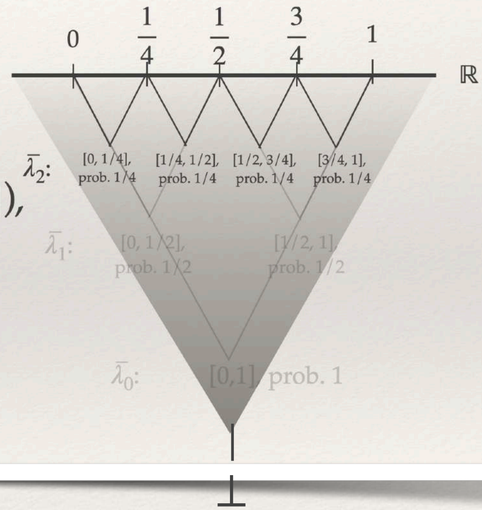
Lebesgue is implementable

- Remember $\bar{\lambda}_{|[0,1]}$?
- This is also implementable, as a **program** that draws **exact real numbers** at random in [0, 1]…

Beyond simple valuations

- Maybe we only need **simple** valuations in computer science?
- No: simple valuations do not form a **dcpo**
- E.g., $\bar{\lambda}_{|[0,1]} = \sup^{\uparrow}_{n \in \mathbb{N}} \bar{\lambda}_n$ (uniform measure on [0,1]), where $\bar{\lambda}_n \mathrel{\hat{=}} \sum_{i=1}^{2^n} \frac{1}{2^n} \delta_{[\frac{i-1}{2^n}, \frac{i}{2^n}]}$

  (there is a similar formula for $\bar{\lambda}$ itself, but I wish to show you a **probability** valuation)



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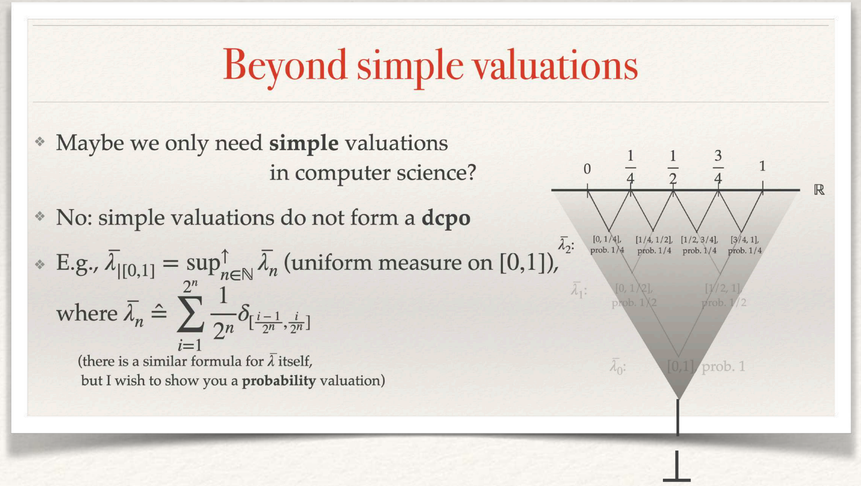

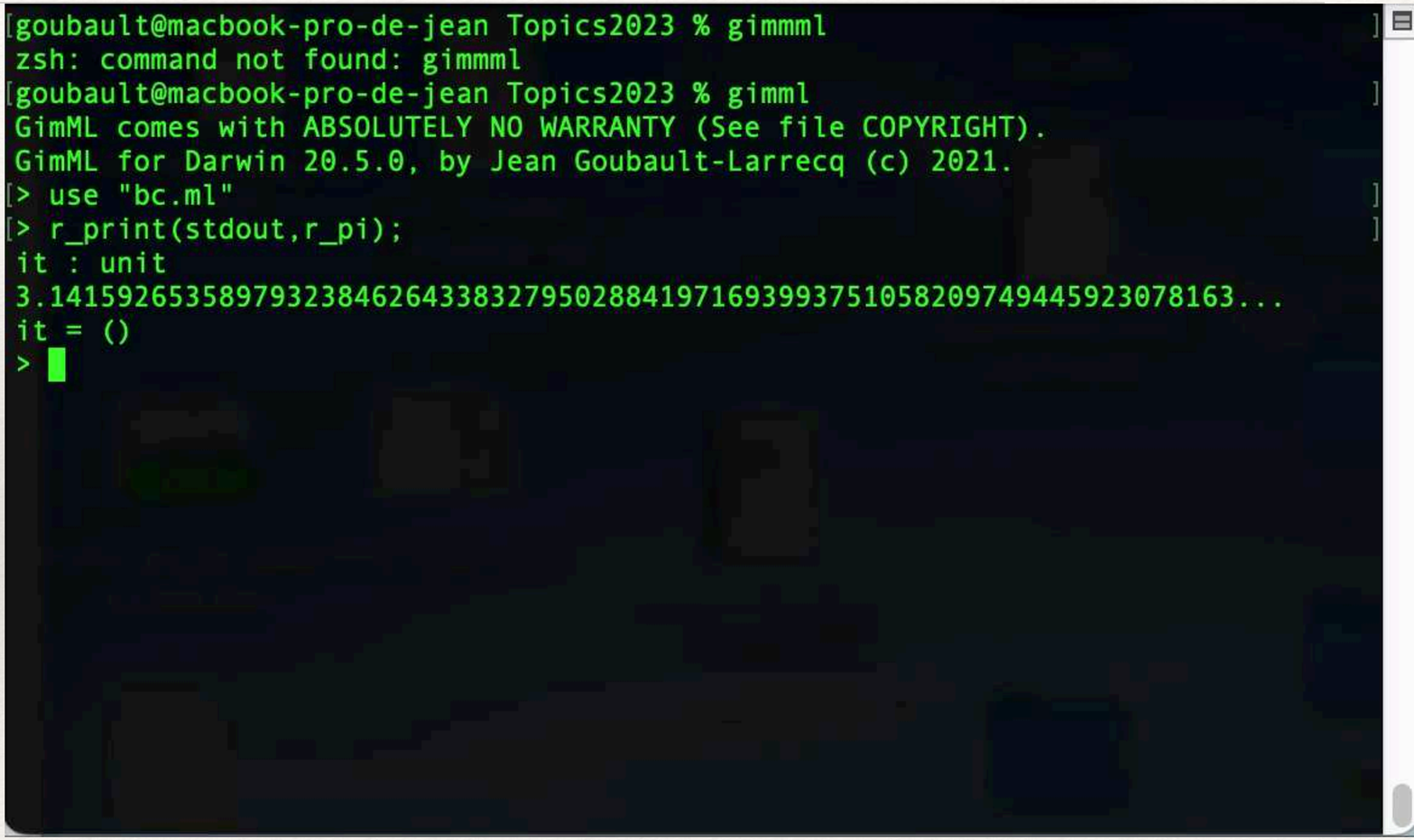

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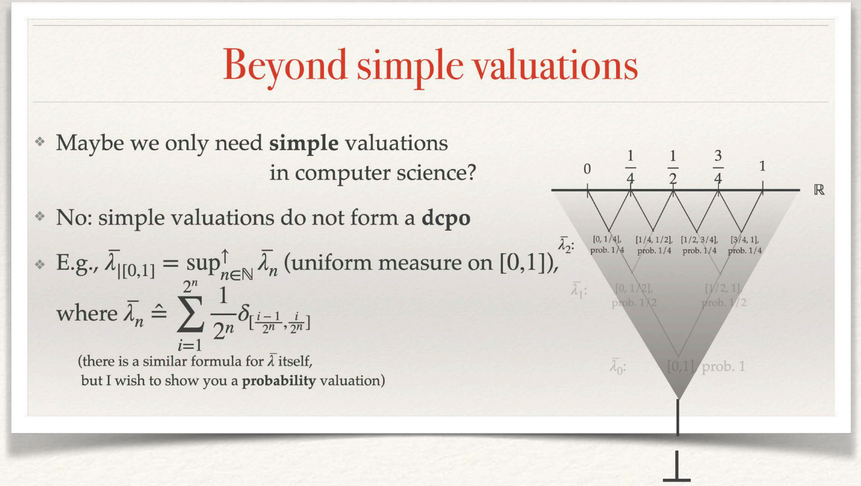

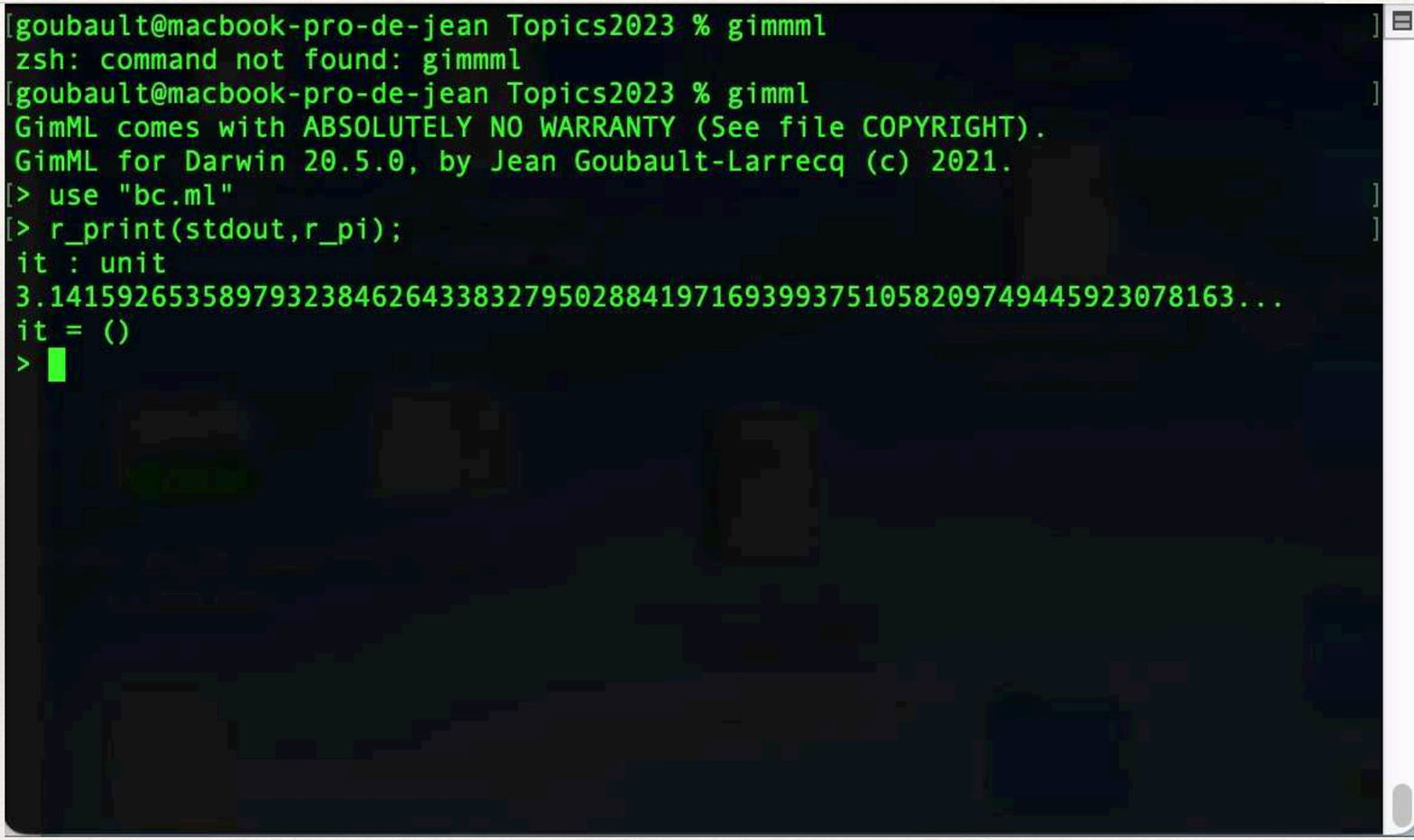

Interval Statistical PCF (ISPCF), v1

- $$
\begin{aligned}
M, N, P, \ldots ::= \; & \ldots && \text{(as in PCF)} \\
| \; & \mathbf{ret}\ M && \text{monad unit} \\
| \; & \mathbf{do}\ x_\sigma = M; N && \text{sequential composition} \\
| \; & \cancel{M \oplus N} && \cancel{\text{probabilistic choice}} \\
| \; & \mathbf{sample}[0,1] && (\bar{\lambda}_{|[0,1]}) \\
| \; & \underline{r} && (\text{real constants, } r \in \mathbb{R}) \\
| \; & \underline{f}(M_1, \cdots, M_n) && (f \in \{ +, -, >, \cdots \})
\end{aligned}
$$

(subsumed by **sample**[0,1])

- Types: $\sigma, \tau, \ldots ::= \mathbf{nat} \mid \mathbf{unit} \mid \mathbf{real} \mid \sigma \to \tau \mid \mathbf{T}\tau$

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The denotational semantics of ISPCF v1

- $[\![\mathbf{nat}]\!] \triangleq \mathbb{N}_\perp$, $[\![\mathbf{unit}]\!] \triangleq \{\perp, *\}$, $[\![\sigma \to \tau]\!] \triangleq [[\![\sigma]\!] \to [\![\tau]\!]]$,
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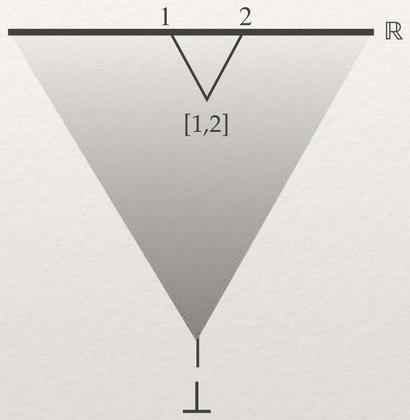

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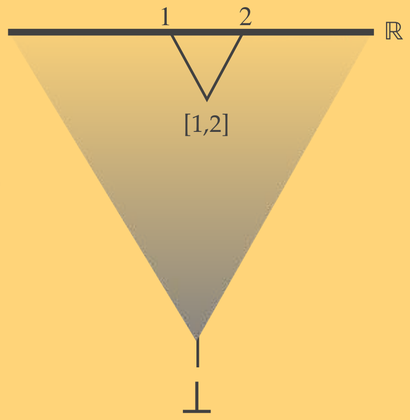

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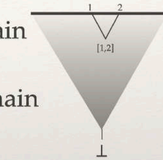
The extension theorem for bc-domains

- A **bc-domain** is a continuous dcpo that:
  — is pointed (has a $\perp$)
  — any two elements x, y with an upper bound have a **least** upper bound $x \vee y$
- $\mathbb{N}_\perp$ is a bc-domain
- not $\mathbb{IR}$ (no $\perp$)
- $\mathbb{IR}_\perp$ is a bc-domain



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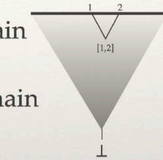

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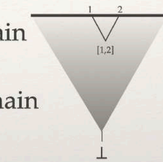

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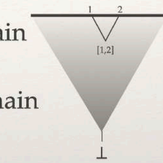

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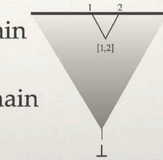

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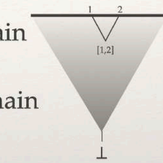

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The operational semantics of ISPCF v1

Exploration rules

$$C, \mathbf{do}\ x = M; N \to C[\mathbf{do}\ x = \_; N], M$$
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  $\Pr[C, M \downarrow n] = [\![C[M]]\!]\rho$
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with **actual** real numbers (in $\mathbb{R}$)
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What does this compute?

```
goubault@macbook-pro-de-jean Topics2023 % gimml
GimML comes with ABSOLUTELY NO WARRANTY (See file COPYRIGHT).
GimML for Darwin 20.5.0, by Jean Goubault-Larrecq (c) 2021.
> use "ispcf1.ml"
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```

$$
\begin{aligned}
\mathit{longest\_decreasing\_run} \mathrel{\hat{=}} \mathbf{rec}(\lambda f_{\mathbf{real}\to\mathbf{int}\to\mathbf{T}\ \mathbf{int}} \cdot \lambda x_{\mathbf{real}} \cdot \lambda n_{\mathbf{int}} \cdot\ & \\
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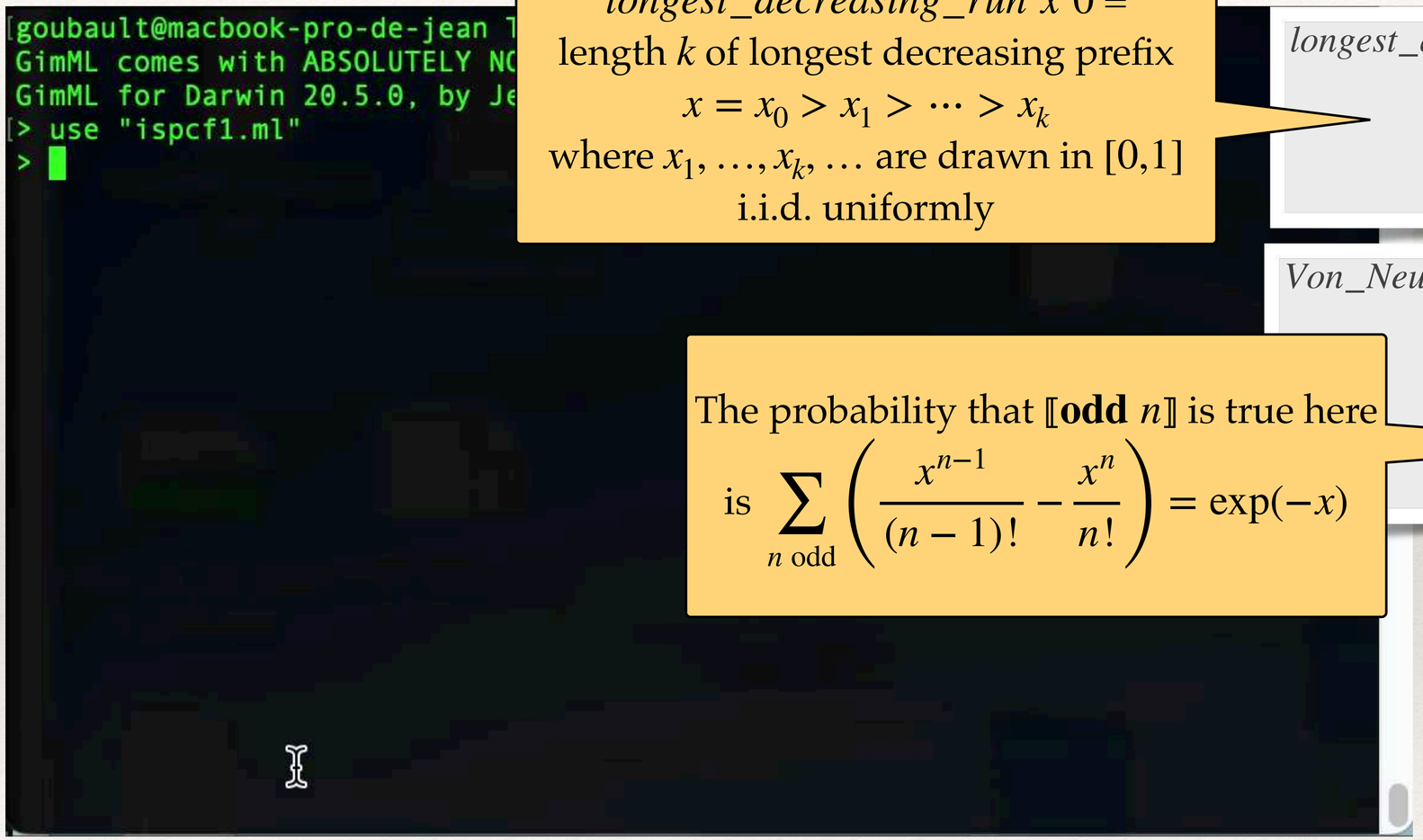

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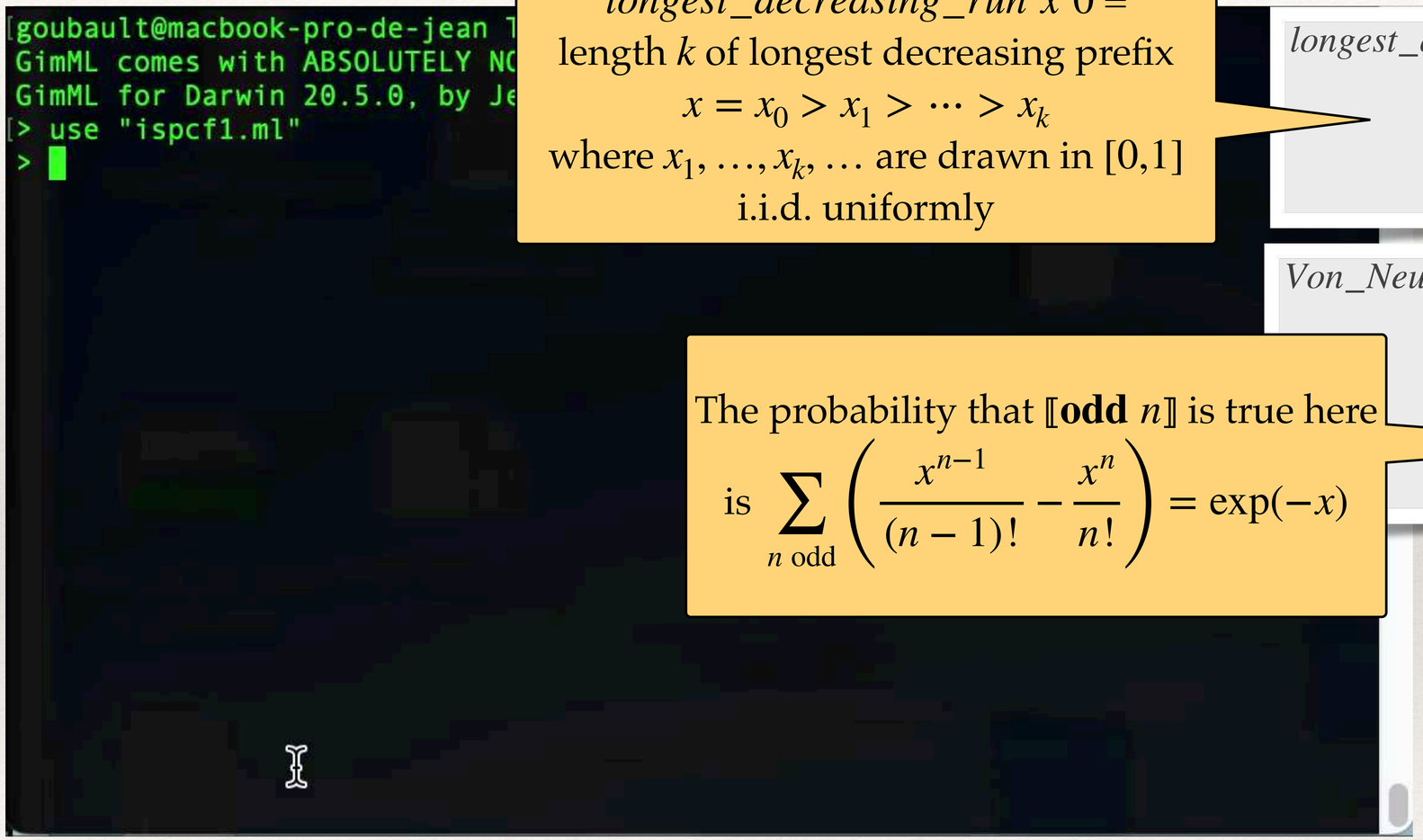

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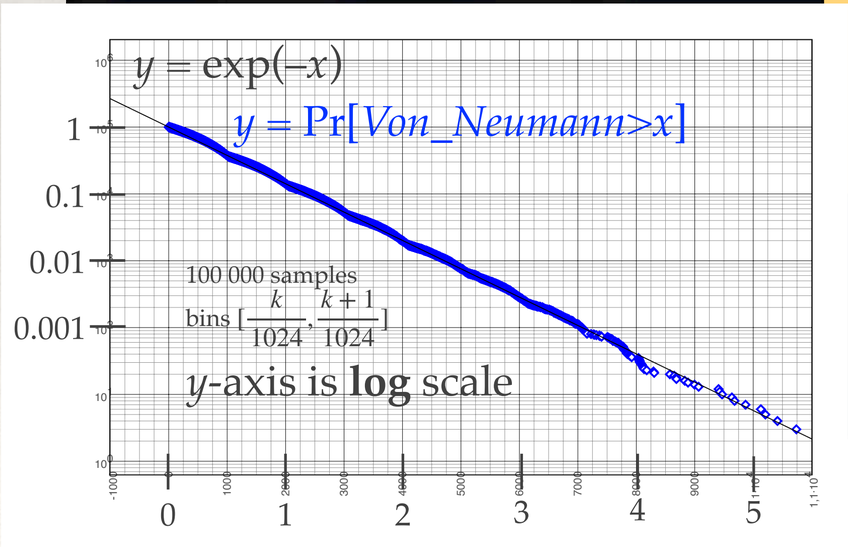

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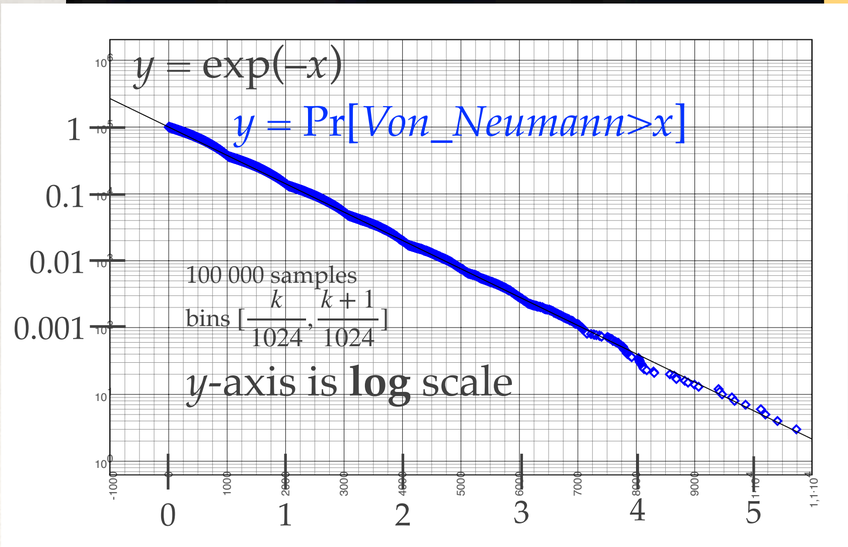

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- **Soft** conditioning [Vákár, Kammar, Staton 21]: argument to **score** is a **density**
  (here, some Gaussian noise around the [formally incompatible] conditions $a = 1.1$, $2a = 1.9$, $3a = 2.7$)

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let a = *normal-rng* $(0, 2)$ in
score(*normal-pdf*$(1.1 \mid a * 1, 0.25)$);
score(*normal-pdf*$(1.9 \mid a * 2, 0.25)$);
score(*normal-pdf*$(2.7 \mid a * 3, 0.25)$);
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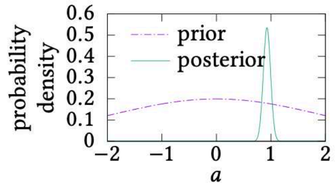
(a)

Prior: $a \sim N(0, 2)$
Observations:
$1.1 \sim N(1a, \frac{1}{4})$
$1.9 \sim N(2a, \frac{1}{4})$
$2.7 \sim N(3a, \frac{1}{4})$

(b)



(c)

Fig. 1. Bayesian linear regression as (a) a first-order probabilistic program, (b) an informal specification, and (c) a plot of the prior and posterior distributions. Here $N(\mu, \sigma)$ is the normal (Gaussian) distribution with density *normal-pdf*$(x \mid \mu, \sigma) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$ and random number generator *normal-rng*(μ, σ).

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- $[\![\mathbf{score}\ M]\!] \mathrel{\hat{=}} |[\![M]\!]| \,.\, \delta_*$,
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- We must also redefine $[\![\mathbf{T}\ \tau]\!]$ as
  $\mathbf{V}([\![\tau]\!])$ = dcpo of **all** continuous valuations on $[\![\tau]\!]$
  (not just the subprobability valuations)

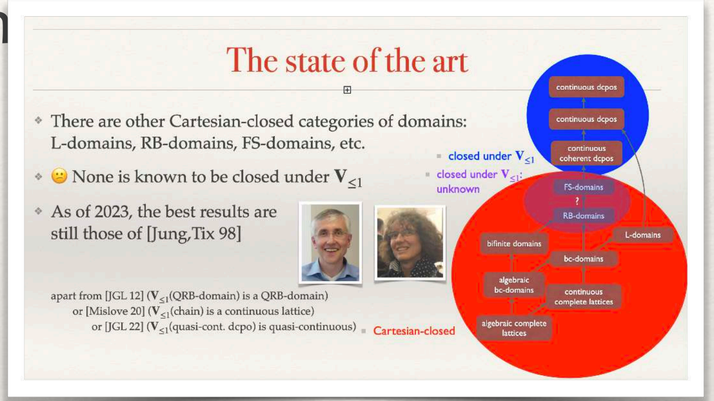
The semantics of **score**

- **score** : **real** $\to$ **T unit**
- $[\![\mathbf{score}\ M]\!] \mathrel{\hat{=}} |[\![M]\!]| \,.\, \delta_*$,
  where $|\_|$ is largest continuous map : $\mathbf{I}\mathbb{R} \to \overline{\mathbb{R}}_+$
  below absolute value : $\mathbb{R} \to \overline{\mathbb{R}}_+$ on $\mathbb{R}$
  $|[a,b]| \mathrel{\hat{=}} a$ if $a > 0$, $-b$ if $b < 0$, 0 if $a \leq 0 \leq b$
- We must also redefine $[\![\mathbf{T}\ \tau]\!]$ as
  $\mathbf{V}([\![\tau]\!])$ = dcpo of **all** continuous valuations on $[\![\tau]\!]$
  (not just the subprobability valuations)
- Soundness, adequacy… as usual [JGL, Jia 23]
  … but operational semantics uses **unbounded** distributions

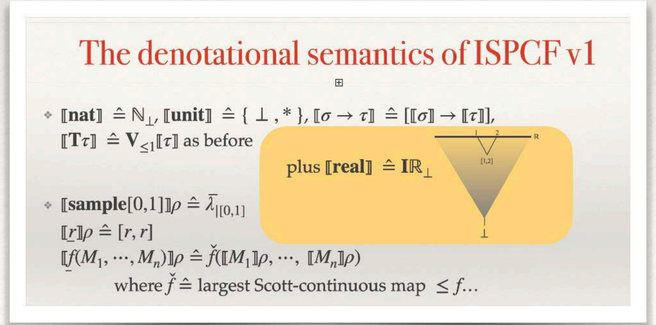
Commutativity

The catch

- I once told you that we needed at least **continuous** dcpos if we aim to prove a form of full abstraction
- … but that leads to difficult problems



- Then I showed you a semantics of ISPCF in (**non**-continuous) dcpos that had all sorts of nice properties
- The catch is that…



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- The (rather silly) catch is that we cannot prove that

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There is a very subtle issue here … as Fubini-Tonelli holds on the **larger** category **Top** 😖

Fubini-Tonelli for continuous valuations

- **Theorem [folklore; JGL, Xia 23].** Let X, Y be arbitrary topological spaces.

$$\int_x \left(\int_y h(x,y) d\nu \right) d\mu = \int_y \left(\int_x h(x,y) d\mu \right) d\nu$$

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More generally:
— if X_σ or Y_σ is core-compact [Gierz,Hofmann,Keimel,Lawson,Mislove 03]
— if X_σ and Y_σ are first-countable [de Brecht, priv. comm., 19]

Minimal valuations

- Let $\mathbf{V}_{\mathrm{fin}}X$ be the subset of $\mathbf{V}X$ consisting of simple valuations
- The smallest subdcpo $\mathbf{M}X$ of $\mathbf{V}X$ containing $\mathbf{V}_{\mathrm{fin}}X$ is the dcpo of **minimal valuations**

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- **Prop [Jia,Lindenhovius,Mislove,Zamdzhiev 21; JGL, Jia 23].**
  Fubini-Tonelli holds on **Dcpo** if one of the valuations is minimal.
- *Proof sketch*: Integration commutes with directed suprema.
  This reduces the question to the case of simple valuations,
  where commutation is easy.

ISPCF, v2

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  Instead of $\mathbf{V}[\![\tau]\!]$
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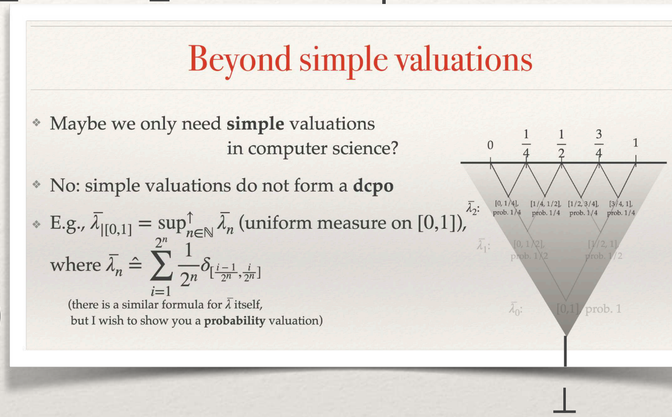

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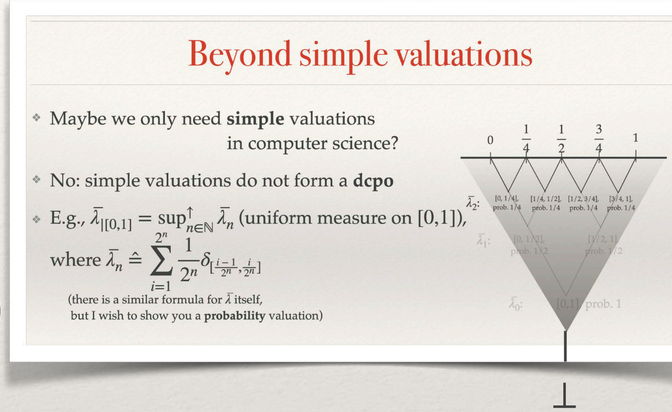

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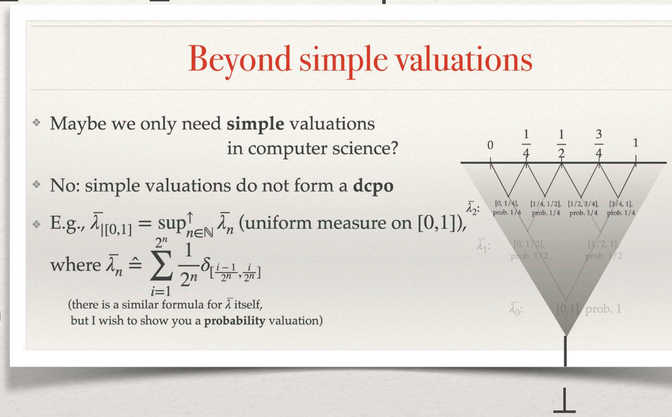

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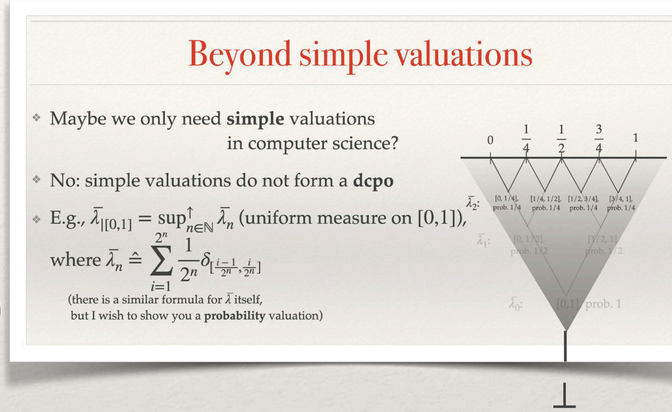

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Problem **solved**.
In particular, $\mathbf{do}\ x = M; \mathbf{do}\ y = N; P$
$\cong$ $\mathbf{do}\ y = N; \mathbf{do}\ x = M; P$

Hierarchies of continuous valuations

Simple valuations

$$\sum_{i=1}^{n} a_i \delta_{x_i}$$

$\subseteq$ Minimal valuations

dcpo-closure of simple valuations

$\subseteq$ Point-continuous valuations

sobrification of simple valuations [Heckmann 97]

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center of the **V** monad on **Dcpo** [Jia, Mislove, Zamdzhiev 21]

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[SahebDjahromi 80, Jones 90]

All those satisfy Fubini-Tonelli on **Dcpo**

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- Is any of those inclusions **strict**?
- If every continuous valuation on a dcpo is minimal/point-continuous/central, then we would obtain Fubini-Tonelli on **Dcpo**.

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$\subsetneq$

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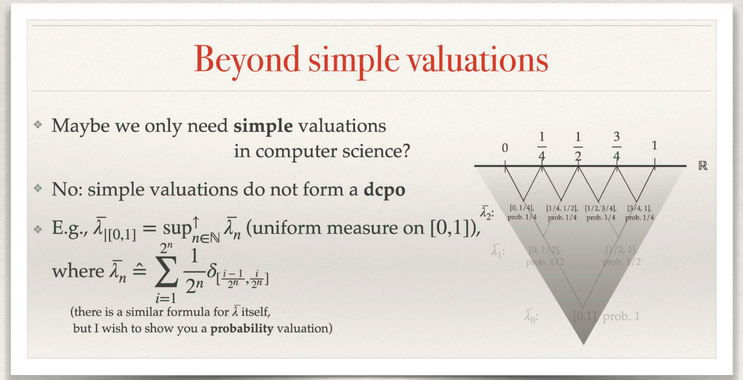


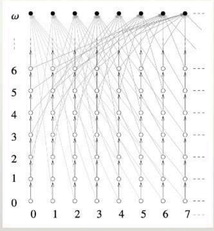
Hierarchies of continuous valuations

Simple valuations $\sum_{i=1}^{n} a_i \delta_{x_i}$ $\subsetneq$ Minimal valuations (dcpo-closure of simple valuations) $\subsetneq$ Point-continuous valuations (sobrification of simple valuations [Heckmann 97]) $\subseteq$ Central valuations (center of the **V** monad) $\subseteq$ Continuous valuations [SahebDjahromi 80]

- **Theorem [JGL, Jia 21].** On Johnstone's dcpo **J**,
  — every continuous valuation is point-continuous
  — there is a continuous valuation μ that is **not minimal**.
- *Proof:* pretty elaborate, uses that every bounded discrete valuation extends to a measure [Alvarez-Manilla, Edalat, Saheb-Djahromi 00], plus a detailed analysis of valuations of the form $\theta + r \,.\, \mu$ with θ bounded discrete. □

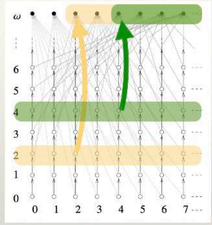
The Johnstone dcpo J

- Johnstone's dcpo **J** (1981): a well-known non-sober dcpo
  — Points = pairs (m, n) in $\mathbb{N} \times (\mathbb{N} \cup \{\omega\})$
  — $(m, n) \leq (m', n')$ iff
    — $m = m'$ and $n \leq n'$
    — or $n \leq m'$ and $n' = \omega$



A funny valuation on J

- On Johnstone's dcpo **J**, there is a continuous valuation μ defined by:
  $\mu(U) = 1$ for every non-empty Scott-open set U
  $\mu(\emptyset) = 0$
- Modularity $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$ comes from the fact that $\mathbf{J}_\sigma$ is **hyperconnected**:
  any two non-empty open sets intersect.
  (Check it! Observe that every non-empty open set contains all points (m, ω) for m large enough.)
- We will show that μ is not minimal.



Hierarchies of continuous valuations

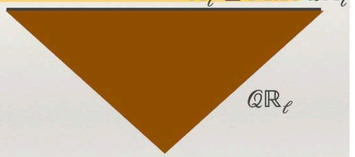
The Sorgenfrey line $\mathbb{R}_\ell$

- A famous counterexample in topology:
  Sorgenfrey topology on $\mathbb{R}$ generated by basic open sets $[a, b[$
  (topology of convergence from the right)

Nice	Not so nice
paracompact, T_4	product with itself **not** T_4
zero-dimensional	**not** locally compact
Choquet-complete, hence Baire	**not** consonant
first-countable, with countable dense subset ($\mathbb{Q}$)	
hereditarily Lindelöf	**not** second-countable
completely quasi-metrizable	**not** metrizable

The Smyth powerdomain of $\mathbb{R}_\ell$

- **Proposition.** $\mathcal{Q}\mathbb{R}_\ell$ is a dcpo model of $\mathbb{R}_\ell$:
  through $x \mapsto \{x\}$, $\mathbb{R}_\ell$ embeds as a **topological subspace** of $\mathcal{Q}\mathbb{R}_\ell$.
- Explicitly, the open subsets of $\mathbb{R}_\ell$
  are the intersections $\mathcal{U} \cap \mathbb{R}_\ell$
  where $\mathcal{U}$ is Scott-open in $\mathcal{Q}\mathbb{R}_\ell$
- The Lebesgue valuation λ on $\mathbb{R}_\ell$ induces
  a continuous valuation $\bar{\lambda}$ on $\mathcal{Q}\mathbb{R}_\ell$ by
  $\bar{\lambda}(\mathcal{U}) = \lambda(\mathcal{U} \cap \mathbb{R}_\ell)$
  (the image continuous valuation through the embedding $\mathbb{R}_\ell \to \mathcal{Q}\mathbb{R}_\ell$)



Point-continuous valuations
sobrification
of simple valuations
[Heckmann 97]

$\subseteq$

Central valuations
center of the **V** monad
on **Dcpo**
[Jia, Mislove, Zamdzhiev 21]

$\subsetneq$

Continuous valuations
[SahebDjahromi 80, Jones 90]

Theorem [JGL, Jia 21]. On the dcpo $\mathcal{Q}(\mathbb{R}_\ell)$ $\bar{\lambda}$ is **not point-continuous**.

Proof: pretty involved, uses Adamski's theorem, results by [He, Li, Xi, Zhao 19] on the Smyth powerdomain of first-countable spaces, and a complete characterization of the compact subsets of the Sorgenfrey line $\mathbb{R}_\ell$. $\square$

Concluding remarks

Conclusion, past, and future

- A short tour of **semantics** of probabilistic programming languages
- This is just the tip of the iceberg…
  in fact I have barely touched the subject of **topology**
  — and only through domain theory
- Let me quickly tell you some of the other things…

Full abstraction

- Probabilistic PCF + (angelic) **non-determinism**
  + statistical termination testers $[V > r]$
  is **fully abstract** [JGL 15]

- Denotationally, redefine $[\![\mathbf{T}\tau]\!]$ as $\mathbf{P}_{\mathrm{A}}[\![\tau]\!]$,
  domain of **Hoare previsions** on $[\![\tau]\!]$ [JGL 07]

- The key to full abstraction is theorems on
  coincidence of topologies
  [JGL 15]

$M \precsim^{\mathrm{may}} N$ *iff* $[\![M]\!]_S \leq [\![N]\!]_S$, *at all types*.

Definition

A *continuous upper prevision* F is a functional from $\langle X \to \mathbb{R}^+\rangle$ to $\mathbb{R}^+$ such that:

- F is (Sco[illegible]$_{Hoare}$ tinuous: $F(\sup^{\uparrow}_{i\in I} h_i) = \sup^{\uparrow}_{i\in I} F(h_i)$;
- F is positively homogeneous: $F(ah) = aF(h)$ $(a \in \mathbb{R}^+)$;
- F is convex: $F(h + h') \leq F(h) + F(h')$.

This models probabilistic + angelic non-determinism

(Exercise: Check that sup is indeed upper.)

Fortunately:

Lemma

For every type τ, $[\![\tau]\!]_S$ is a bc-domain.

(One of the nice CCCs of continuous domains.)

Proposition (Key result — *coincidence of topologies*)

If X and Y are bc-domains, then:

- *Scott topology on $[X \to Y]$ = pointwise convergence*
  Subbasis: $[a \in V] = \{f \mid f(a) \in V\}$, $a \in X$, V open in Y
- *Scott topology on previsions on X = weak topology*
  Subbasis: $[h > r] = \{F \mid F(h) > r\}$, $h \in [X \to I]$, $r \in \mathbb{Q}$

Previsions

- For a continuous valuation ν, $h \mapsto \int_X h d\nu$ is continuous and **linear**

  This is in fact an isomorphism $\mathbf{V}X \cong [[X \to \overline{\mathbb{R}}_+] \to_{\mathrm{lin}} \overline{\mathbb{R}}_+]$

- **Previsions** relax the linearity requirement

  e.g., Hoare previsions are **sublinear**: $F(h_1 + h_2) \leq F(h_1) + F(h_2)$ (and $F(r \,.\, h) = r \,.\, F(h)$)

- **Isomorphisms** between spaces of previsions on X

  and convex hyperspaces over $\mathbf{V}X$,

  (as used by [Mislove 00], [McIver, Morgan 01], [Tix 99; Tix, Keimel, Plotkin 09])

  [JGL 08]: on continuous dcpos

  [JGL 17]: on large families of **topological spaces**

A few open questions

- Full abstraction for ISPCF+angelic non-determinism+statistical testers? [Yes]
  — any connection between statistical testers and conditioning?
- Fubini-Tonelli on **Dcpo**?
  — equivalently, is every continuous valuation central on **Dcpo**?
- Can the Jung-Tix dream be realized?
  — a Cartesian-closed category of continuous dcpos, closed under **V**?
- Can we trust **pseudo-random** number generators
  instead of true random number generators
  in the implementation of **sample**[0,1]?