Laboratoire
Méthodes
Formelles

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Is there a Fubini-Tonelli-type theorem for continuous valuations on Dcpo?

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Outline

- Continuous valuations on **Top**, and a Fubini-Tonelli theorem
- Continuous valuations on **Dcpo**, and the problem
- Why should we care?
- Positive results: minimal valuations, point-continuous valuations
- Minimal $\subsetneq$ point-continuous $\subsetneq$ continuous

Continuous valuations: a quick introduction

Continuous valuations

- A topological alternative to measures,
  favored in semantics of programming languages since (Jones, Plotkin 1990).
- A **valuation** ν on a topological space X is a map $\nu\colon \mathcal{O}X \to \overline{\mathbb{R}}_+$ satisfying:
  - **strictness**: $\nu(\emptyset) = 0$
  - **monotonicity**: $\nu(U) \leq \nu(V)$ if $U \subseteq V$
  - **modularity**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$
- A **continuous valuation** is not just monotonic, but Scott-continuous.

Continuous valuations and measures

Continuous valuation $\nu : \mathcal{O}X \to \overline{\mathbb{R}}_+$:

- **strict**: $\nu(\emptyset) = 0$
- **Scott-continuous**,
- **modular**: $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$

- **Theorem (Adamski 1977).** Given any **Borel measure**
  on a hereditarily Lindelöf space, (in particular, a second-countable space)
  its restriction t ... **n.**
  ... space is *τ-smooth*.)

Just remember that
continuous valuations~measures,
in most cases

- **Theorem (de Brecht, G** ... **aluation**
  on an LCS-con ... mpact sober space)
  extends to a **B** ...

- A sweet spot: de Brecht's **quasi-Polish** spaces (=2nd countable LCS-complete)
  Those include all ***ω*-continuous dcpos** + all **Polish** spaces

Integration

- For every **lower semicontinuous** map $f\colon X \to \overline{\mathbb{R}}_+$ (i.e., continuous to $\overline{\mathbb{R}}_{+\sigma}$) and every continuous valuation μ on X, there is an **integral** $\int_X f(x)d\mu$ (or $\int f\,d\mu$ for short)
- The easiest way to define it is by the **Choquet formula**:

$$\int_X f(x)d\mu = \int_0^\infty \mu(f^{-1}(]t,\infty]))dt$$

where the term on the right is a Riemann integral (Tix 1995)

Properties of integration

- **Linearity** in f: $\int af\,d\mu = a\int f\,d\mu, \quad \int (f+g)d\mu = \int f\,d\mu + \int g\,d\mu$
- **Scott-continuity** in f: $\int \sup_i^{\uparrow} f_i\,d\mu = \sup_i^{\uparrow} \int f_i\,d\mu$ (μ a continuous valuation)
- **Linearity** in μ: $\int f\,d(a\mu) = a\int f\,d\mu, \quad \int f\,d(\mu+\nu) = \int f\,d\mu + \int f\,d\nu$
- **Scott-continuity** in μ: $\int f\,d\sup_i^{\uparrow}\mu_i = \sup_i^{\uparrow}\int f\,d\mu_i$
- $\int \chi_U\,d\mu = \mu(U)$ $\int f\,d\delta_x = f(x)$, where δ_x is **Dirac valuation** at x ($\delta_x(U) = \chi_U(x)$)

Fubini-Tonelli theorems

Fubini-Tonelli for continuous valuations

- **Theorem (Jones 1989).**
  Given two **continuous dcpos** X and Y,
  a continuous valuation μ on X,
  a continuous valuation ν on Y,
  there is a unique continuous valuation $\mu \otimes \nu$ on $X \times Y$
  such that $(\mu \otimes \nu)(U \times V) = \mu(U)\,.\,\nu(V)$ for all $U \in \mathcal{O}X, V \in \mathcal{O}Y$

- Moreover, the **Fubini-Tonelli** formula holds:

$$\int_{(x,y)} h(x,y)\,d(\mu \otimes \nu) = \int_x \left(\int_y h(x,y)\,d\nu \right) d\mu = \int_y \left(\int_x h(x,y)\,d\mu \right) d\nu$$

  for every Scott-continuous map $h\colon X \times Y \to \overline{\mathbb{R}}_+$

Fubini-Tonelli for continuous valuations

More generally

- **Theorem (folklore).**
  Given two **topological spaces** X and Y,
  a continuous valuation μ on X,
  a continuous valuation ν on Y,
  there is a unique continuous valuation $\mu \otimes \nu$ on $X \times Y$
  such that $(\mu \otimes \nu)(U \times V) = \mu(U) \,.\, \nu(V)$ for all $U \in \mathcal{O}X, V \in \mathcal{O}Y$

- Moreover, the **Fubini-Tonelli** formula holds:

$$\int_{(x,y)} h(x,y) d(\mu \otimes \nu) = \int_x \left(\int_y h(x,y) d\nu \right) d\mu = \int_y \left(\int_x h(x,y) d\mu \right) d\nu$$

for every lower semicontinuous map $h \colon X \times Y \to \overline{\mathbb{R}}_+$

Hence surely… right?

which are certainly topological spaces, in their Scott topology

- **Theorem (?).**
  Given two ~~continuous~~ dcpos X and Y,
  a continuous valuation μ on X,
  a continuous valuation ν on Y,
  there is a unique continuous valuation $\mu \otimes$
  such that $(\mu \otimes \nu)(U \times V) = \mu(U) \,.\, \nu($

That much is a **conjecture**,
and certainly does **not** follow
from the previous theorem.
Do you see why?

- Moreover, the **Fubini-Tonelli** formula holds:

$$\int_{(x,y)} h(x,y)d(\mu \otimes \nu) = \int_x \left(\int_y h(x,y)d\nu \right) d\mu = \int_y \left(\int_x h(x,y)d\mu \right) d\nu$$

for every Scott-continuous map $h \colon X \times Y \to \overline{\mathbb{R}}_+$

Hence surely… right?

- **Theorem (?).**
  Given two ~~continuous~~ dcpos X and Y,
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- Moreover, the **Fubini-Tonelli** formula holds:

$$\int_{(x,y)} h(x,y) d(\mu \otimes \nu) = \int_x \left(\int_y h(x,y) d\nu \right) d$$

for every Scott-continuous map $h \colon X \times Y$ -

which are certainly topological spaces, in their Scott topology

Products in **Top**
≠ products in **Dcpo**
(although they coincide on **continuous** dcpos, in fact even on core-compact dcpos)

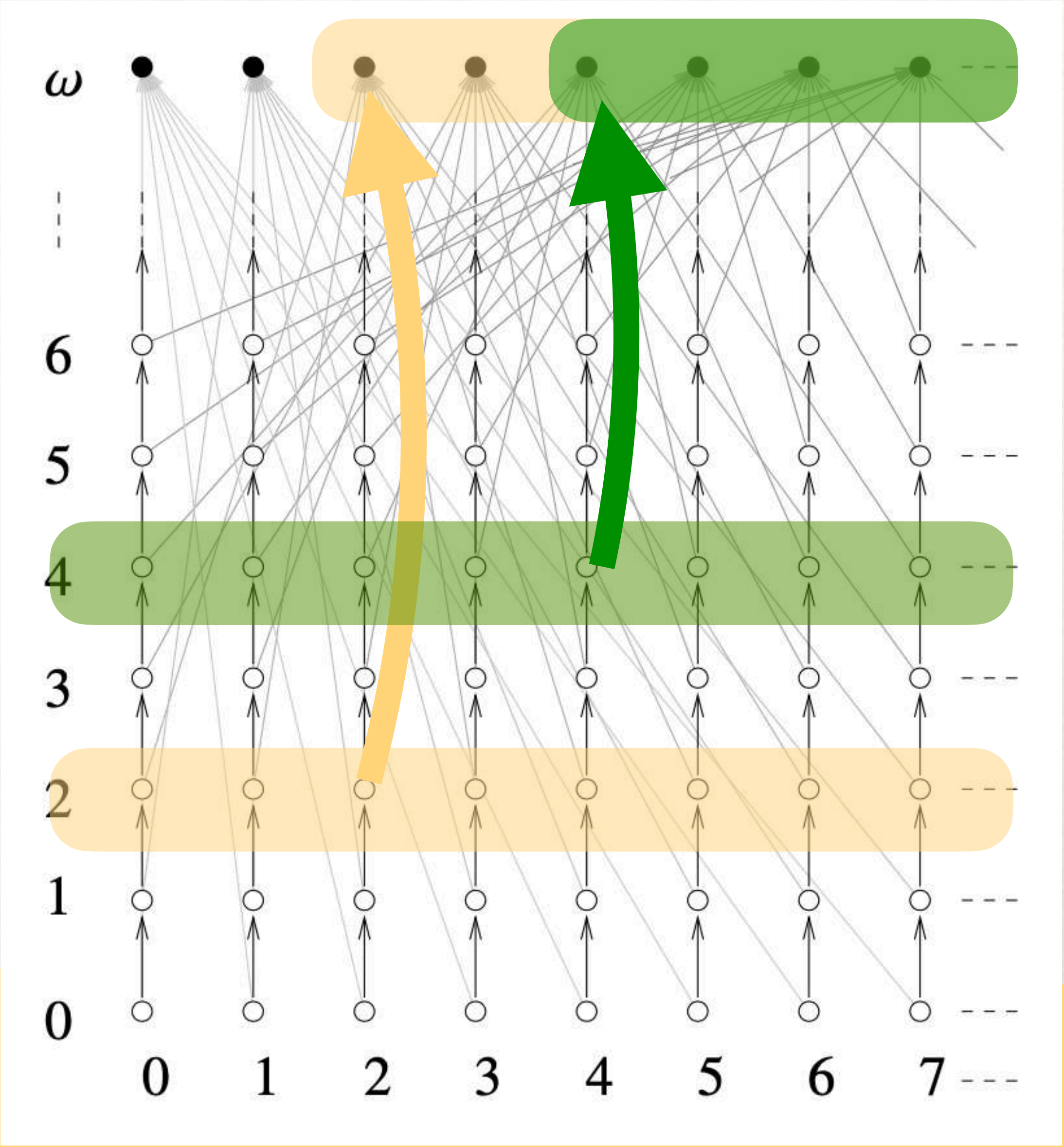
Products in **Top**, products in **Dcpo**

- Given two dcpos X and Y, $X \times Y$ may mean two things:
  - consider X and Y as topological spaces X_σ and Y_σ in their Scott topologies, and build the **topological product** $X_\sigma \times Y_\sigma$ … in the category **Top**
  - build their **poset product** $X \times Y$ … in the category **Dcpo** then equip that with the Scott topology, obtaining $(X \times Y)_\sigma$
- The two may differ (see next slide), although they coincide in many cases:
  — if X_σ or Y_σ is core-compact (Gierz et al. 2003, Theorem II-4.13)
  (… in particular if X or Y is a continuous dcpo)
  — if X_σ and Y_σ are first-countable (de Brecht, 2019)
  (see https://projects.lsv.ens-cachan.fr/topology/?page_id=1852)

Products in **Top** $\neq$ products in **Dcpo**

- Are there dcpos X, Y such that $X_\sigma \times Y_\sigma \neq (X \times Y)_\sigma$? We use:
  Theorem (Exercise 5.2.7, JGL 2013). A topological space Z is core-compact iff $(\in)$, defined as $\{(x, U) \mid x \in U, U \in \mathcal{O}Z\}$, is open in the product topology on $Z \times (\mathcal{O}Z)_\sigma$
- Take $Z = X_\sigma$ where X is a dcpo such that X_σ is not core-compact.
  - Note that $(\in)$ is Scott-open.
  - But $(\in)$ is not open in $X_\sigma \times (\mathcal{O}(X_\sigma))_\sigma$, by the theorem above.
    So $X_\sigma \times Y_\sigma \neq (X \times Y)_\sigma$, where $Y = \mathcal{O}(X_\sigma)$
- Hence it suffices to find a **non-core-compact dcpo**.

The Johnstone dcpo $\mathbf{J}$



- Johnstone's dcpo $\mathbf{J}$ (1981): a well-known non-sober dcpo
  — Points = pairs (m, n) in $\mathbb{N} \times (\mathbb{N} \cup \{\omega\})$
  — $(m, n) \leq (m', n')$ iff
    — $m = m'$ and $n \leq n'$
    — or $n \leq m'$ and $n' = \omega$

- **Fact (Exercise 5.2.15, JGL 2013).**
  In $\mathcal{O}(\mathbf{J}_\sigma)$, U is way-below V if and only if U is empty.
  Hence $\mathbf{J}_\sigma$ is not core-compact.

- **Corollary.** Let $X = \mathbf{J}$, $Y = \mathcal{O}(\mathbf{J}_\sigma)$. We have $X_\sigma \times Y_\sigma \neq (X \times Y)_\sigma$.

So what is the problem with Fubini-Tonelli on **Dcpo**?

- In general, there are **more** open subsets in $(X \times Y)_\sigma$ than in $X_\sigma \times Y_\sigma$
- Hence there are **more** Scott-continuous maps $h : X \times Y \to \overline{\mathbb{R}}_+$
  than lower semi-continuous maps
  (i.e., jointly continuous maps $h : X_\sigma \times Y_\sigma \to \overline{\mathbb{R}}_{+\sigma}$)
- We still have:

$$\int_x \left(\int_y h(x,y) d\nu \right) d\mu = \int_y \left(\int_x h(x,y) d\mu \right) d\nu$$

for every lower semicontinuous map $h : X_\sigma \times Y_\sigma \to \overline{\mathbb{R}}_{+\sigma}$
but does this holds for **Scott-continuous** maps $h : X \times Y \to \overline{\mathbb{R}}_+$?

Why should we care?
A computer scientist's perspective

Higher-order **probabilistic** programs

- Consider a (toy) language like this, called **probabilistic PCF**:

Terms $M, N, \ldots ::= x$		variables
	$\mid MN$	application
	$\mid \lambda x_\sigma . M$	anonymous function
	$\mid$ **rec** M	recursive definition
	$\mid \underline{n}$	natural number (constant)
	$\mid$ **succ** M	add one
	$\mid$ **pred** M	subtract one
	$\mid$ **ifz** $M\ N\ P$	condi
	$\mid M \oplus N$	
	$\mid$ **ret** M	
	$\mid$ **do** $x \leftarrow M; N(x)$	

« choose between M and N with probability 1/2 »

« draw M with probability 1 »

« sample M, put the result in x, then compute $N(x)$ »

Denotational semantics

We work in a category of (pointed) dcpos

- (Leaving out a few details), one defines the **semantics** $[\![M]\!]$ of terms by: […]
- $[\![\mathbf{rec}\ M]\!]$ = least fixed point $\sup_{n \in \mathbb{N}} [\![M]\!]^n(\bot)$

« choose between *M* and *N* with probability 1/2 »

- $[\![M \oplus N]\!] = ½\, [\![M]\!] + ½\, [\![N]\!]$ ($[\![M]\!]$ and $[\![N]\!]$ continuous valuations)

« draw *M* with probability 1 »

- $[\![\mathbf{ret}\ M]\!] = \delta_{[\![M]\!]}$ ($[\![M]\!]$ an ordinary value, $[\![\mathbf{ret}\ M]\!]$ cont. val.)

« sample *M*, put the result in *x*, then compute *N* »

- $[\![\mathbf{do}\ x \leftarrow M;\ N(x)]\!] = \left(U\ open \mapsto \int_x [\![N(x)]\!](U)\ d[\![M]\!] \right)$ (a continuous valuation)

The practitioner's view

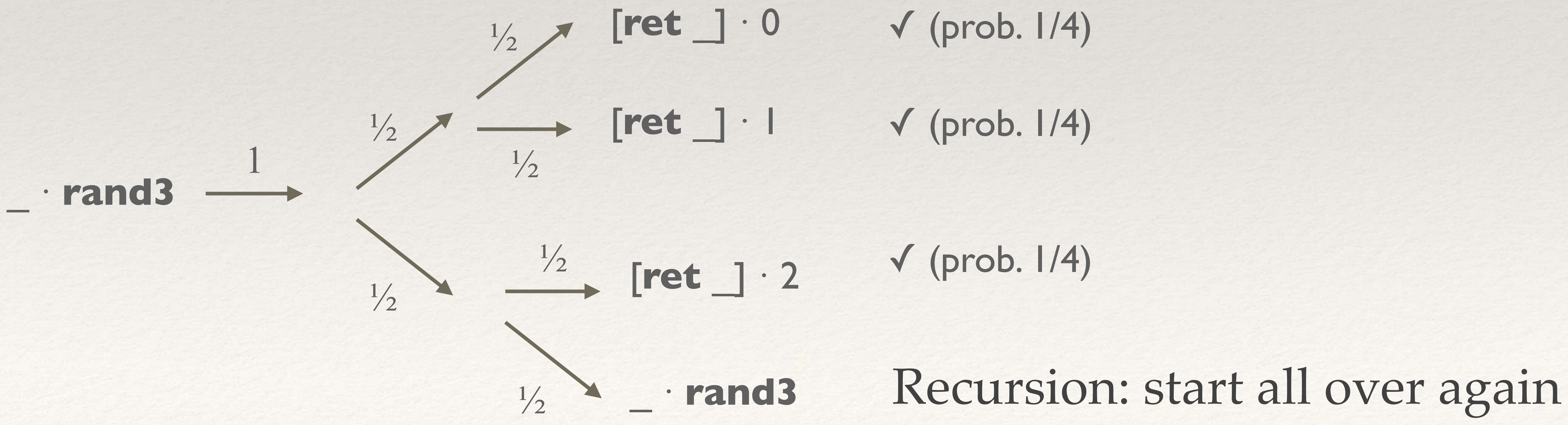
- The denotational semantics is a **theoretical computer scientist**'s view of **what** the program M computes
- The **practitioner**'s view is an implementation: a machine that effects the computation, namely **how** the program M computes
- I will give an idea of what that machine is next

Implementation (abstract machines)

Exploration rules	
$C \cdot E[M] \xrightarrow{1} CE \cdot M \quad (E \text{ elem. context})$	$[\_] \cdot \mathtt{ret}_{\mathtt{int}}\, M \xrightarrow{1} [\mathtt{ret}_{\mathtt{int}}\ \_] \cdot M$
Computation rules	
$C[\_N] \cdot \lambda x_\sigma.M \xrightarrow{1} C \cdot M[x_\sigma := N]$	$C \cdot \mathtt{rec}_\tau\, M \xrightarrow{1} C \cdot M(\mathtt{rec}_\tau\, M)$
$C \cdot M \oplus N \xrightarrow{1/2} C \cdot M$	$C \cdot M \oplus N \xrightarrow{1/2} C \cdot N$
$C[\mathtt{bind}_{\sigma,\tau}\ \_N] \cdot \mathtt{ret}_\sigma\, M \xrightarrow{1} C \cdot NM$	$C[\mathtt{p}\ \_] \cdot n \xrightarrow{1} C \cdot n-1 \quad C[\mathtt{s}\ \_] \cdot n \xrightarrow{1} C \cdot n+1$
$C[\mathtt{if}\ \_ = 0\ \mathtt{then}\ N\ \mathtt{else}\ P] \cdot 0 \xrightarrow{1} C \cdot N$	$C[\mathtt{if}\ \_ = 0\ \mathtt{then}\ N\ \mathtt{else}\ P] \cdot n \xrightarrow{1} C \cdot P \quad (n \neq 0)$

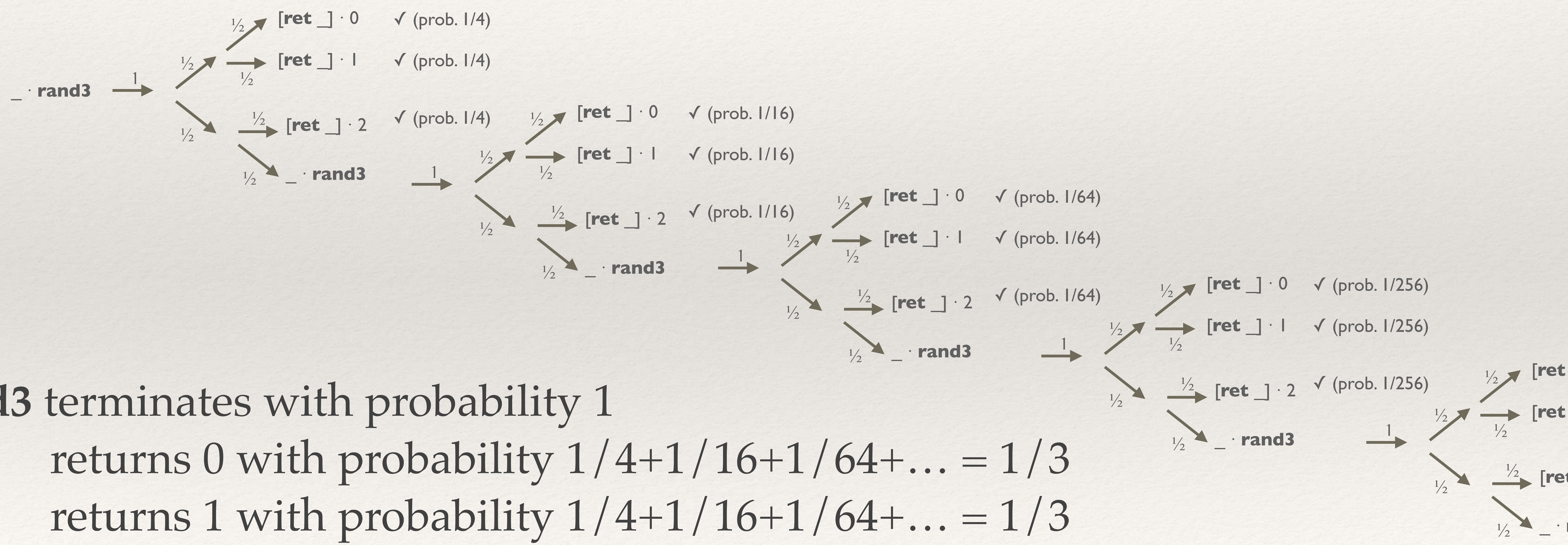
- A probabilistic PCF **machine** works as a transition system working on configurations
- Let us see how the following program runs:

$$\mathbf{rand3} = \mathbf{rec}\ (\lambda p\ .\ (\mathbf{ret}\ 0 \oplus \mathbf{ret}\ 1) \oplus (\mathbf{ret}\ 2 \oplus p))$$



Implementation (abstract machines)

- **rand3** = **rec** (λp . (**ret** 0 ⊕ **ret** 1) ⊕ (**ret** 2 ⊕ p))



- **rand3** terminates with probability 1
  returns 0 with probability 1/4+1/16+1/64+… = 1/3
  returns 1 with probability 1/4+1/16+1/64+… = 1/3
  returns 2 with probability 1/4+1/16+1/64+… = 1/3

Back to denotational semantics

- **rand3** = **rec** $(\lambda p\ .\ (\textbf{ret}\ 0 \oplus \textbf{ret}\ 1) \oplus (\textbf{ret}\ 2 \oplus p))$
- **rand3** terminates with probability 1
  returns 0 with probability 1/4+1/16+1/64+… = 1/3
  returns 1 with probability 1/4+1/16+1/64+… = 1/3
  returns 2 with probability 1/4+1/16+1/64+… = 1/3
- Compare this with the denotational semantics:
  $$[\![\textbf{rand3}]\!] = \text{least fixed point of } \nu \mapsto \frac{1}{4}\delta_0 + \frac{1}{4}\delta_1 + \frac{1}{4}\delta_2 + \frac{1}{4}\nu$$
- There is just one fixed point: $\frac{1}{3}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{3}\delta_2$ (no need to sum a series!)

Bugs, and verification

- A central problem in computer science is **bugs**
- How do you make sure that a program M:
  — computes what you want?
  — computes something that satisfies a given property P?
  — computes the same thing as another program N?
- E.g., do the following two programs compute the same thing?

  do x←**rand3**; (**do** y←**ret** 0 ⊕ **ret** 1; **ret** (x–y))

  do y←**ret** 0 ⊕ **ret** 1; (**do** x←**rand3**; **ret** (x–y))

Equivalence of programs

- **Theorem.** If $[\![M]\!]=[\![N]\!]$ then M and N **compute the same thing**
  — formally, if $[\![M]\!]=[\![N]\!]$ then M and N are **observationally equivalent**
  i.e., for every context C (of basic type), I
- Let $M \stackrel{\text{def}}{=}$ **do** $x\leftarrow$**rand3**; (**do** $y\leftarrow$**ret** $0 \oplus$ **ret** 1; **re**
  $N \stackrel{\text{def}}{=}$ **do** $y\leftarrow$**ret** $0 \oplus$ **ret** 1; (**do** $x\leftarrow$**rand3**; **ret**
- Are M and N observationally equivalent?
- $[\![M]\!]=[\![N]\!]$ is **Fubini-Tonelli:** $\displaystyle\int_x \left(\int_y \delta_{x-y}(U)d\nu\right) d\mu = \int_y \left(\int_x \delta_{x-y}(U)d\mu\right) d\nu$

  for every open set U; where $\mu = [\![\textbf{rand3}]\!]$, $\nu = [\![\textbf{ret}\ 0 \oplus \textbf{ret}\ 1]\!]$

… on **Dcpo**

Whence our initial question:
is there a Fubini-Tonelli theorem
for continuous valuations
on **Dcpo**?

A question by Jung and Tix

- Let $\mathbf{V}X$ be the dcpo of continuous valuations on X, ordered by $\leq$ (pointwise)
- Is there a Cartesian-closed category (in order to give meaning to MN, $\lambda x_\sigma . M$)
  closed under **V** (… and to $M \oplus N$, **ret** M, **do** $x \leftarrow M; N(x)$)
  of **continuous** dcpos (then Fubini-Tonelli will holds, see Jones 1990)
  ?
- See Jung and Tix (1998)
  Bc-domains, Scott domains, algebraic domains: not closed under **V**.
  RB-domains, FS-domains: closed under **V**? (unknown)
- New direction: do not insist on having **continuous** dcpos,
  rather look for Fubini-Tonelli on larger Cartesian-closed categories of dcpos

Positive results:
minimal and point-continuous valuations

Simple valuations

- A **simple valuation** is a finite sum $\sum_{i=1}^{n} a_i \delta_{x_i}$, with $a_i < \infty$
- Fubini-Tonelli on **Dcpo** holds if one of the valuations is **simple**, say $\mu = \sum_{i=1}^{n} a_i \delta_{x_i}$:

$$\int_X \left(\int_Y f(x,y) d\nu \right) d\mu = \sum_{i=1}^{n} a_i \int_X f(x_i, y) d\nu$$

$$= \int_Y \sum_{i=1}^{n} a_i f(x_i, y) d\nu$$

$$= \int_Y \left(\int_X f(x,y) d\mu \right) d\nu$$

Minimal valuations

- Let $\mathbf{V}_{\mathrm{fin}}X$ be the subset of $\mathbf{V}X$ consisting of simple valuations
- The smallest subdcpo $\mathbf{M}X$ of $\mathbf{V}X$ containing $\mathbf{V}_{\mathrm{fin}}X$ is the dcpo of **minimal valuations**
- Explicitly, a minimal valuation is a directed supremum of directed suprema of … of simple valuations (iterated transfinitely)
- **Prop.** Fubini-Tonelli holds on **Dcpo** if one of the valuations is minimal.
- *Proof sketch*: integration commutes with directed suprema.

Point-continuous valuations

- The **pointwise topology** on $[X \to Y]$ is the coarsest that makes
  $f \mapsto f(x)$ continuous, for each $x \in X$ **Notation**: $[X \to Y]_p$
- Let $\mathbb{S}$ be Sierpiński space $\{0 < 1\}$
- By equating $\mathcal{O}X$ with $[X \to \mathbb{S}]$
  through $U \cong \chi_U$, yields **pointwise topology** on $\mathcal{O}X$
  (**coarser** than Scott on $(\mathcal{O}X)_\sigma$) **Notation**: $(\mathcal{O}X)_p$
- **Definition** (Heckmann 1996). A valuation ν is **point-continuous**
  iff it is continuous from $(\mathcal{O}X)_p$ (not $(\mathcal{O}X)_\sigma$)
  to $\overline{\mathbb{R}}_{+\sigma}$

Properties of point-continuous valuations

- **Prop.** Simple ⇒ Minimal ⇒ Point-continuous ⇒ Continuous.
- **Prop.** $g \longmapsto \int_x g(x) d\mu$ continuous:

  — from $[X \to \overline{\mathbb{R}}_{+\sigma}]_\sigma$ to $\overline{\mathbb{R}}_{+\sigma}$ for every continuous valuation μ

  — from $[X \to \overline{\mathbb{R}}_{+\sigma}]_p$ to $\overline{\mathbb{R}}_{+\sigma}$ for every **point-continuous** valuation μ
- Many other properties: see Heckmann 1996.

The weak topology, and the Schröder-Simpson theorem

- The **weak topology** on $\mathbf{V}X$ is the coarsest that makes

  $\nu \mapsto \int_x f(x) d\nu$ continuous from $\mathbf{V}X$ to $\overline{\mathbb{R}}_{+\sigma}$,

  for every continuous map $f \colon X \to \overline{\mathbb{R}}_{+\sigma}$

- Let $\mathbf{V}_w X$ be $\mathbf{V}X$ with the weak topology (**coarser** than Scott on $\mathbf{V}X$)
  We will use:

- **Theorem** (**Schröder and Simpson** 2005; JGL 2015).
  Every linear continuous map $F \colon \mathbf{V}_w X \to \overline{\mathbb{R}}_{+\sigma}$

  is equal to $\mu \mapsto \int_x h_F(x) d\mu$ for some unique $h_F \in [X \to \overline{\mathbb{R}}_{+\sigma}]$

  (explicitly, $h_F(x) = F(\delta_x)$)

Fubini-Tonelli on **Dcpo** for point-continuous valuations

- Why do we care about point-continuous valuations?

  Prop (Jia, Lindenhovius, Mislove, Zamdzhiev 2021). Fubini-Tonelli holds on **Dcpo** if one of the valuations is point-continuous.

- *Proof.* With f, ν fixed, $G\colon \mu \mapsto \int_y \left(\int_x f(x,y)d\mu \right) d\nu$ is the composition of:

$$\underbrace{\begin{array}{c} \mathbf{V}_w X \longrightarrow [Y \to \overline{\mathbb{R}}_+]_p \\ \mu \longmapsto \left(y \mapsto \int_x f(x,y)d\mu \right) \end{array}}_{\text{continuous by definition of the topologies involved}} \quad \text{and} \quad \underbrace{\begin{array}{c} [Y \to \overline{\mathbb{R}}_+]_p \longrightarrow \overline{\mathbb{R}}_+ \\ g \longmapsto \int_y g(y)d\nu \end{array}}_{\text{continuous if } \nu \text{ point-continuous}}$$

Fubini-Tonelli on **Dcpo** for point-continuous valuations

- Why do we care about point-continuous valuations?

Prop (Jia, Lindenhovius, Mislove, Zamdzhiev 2021). Fubini-Tonelli holds on **Dcpo** if one of the valuations is point-continuous.

- *Proof.* Fix f and a **point-continuous** valuation ν, and let

$$F(\mu) = \int_x \left(\int_y f(x,y) d\nu \right) d\mu \qquad G(\mu) = \int_y \left(\int_x f(x,y) d\mu \right) d\nu$$

- F and G are **linear** and **continuous** from $\mathbf{V}_w X$ to $\overline{\mathbb{R}}_{+\sigma}$

(We have just proved continuity for G; F is continuous by definition of the weak topology)

- Hence…

Fubini-Tonelli on **Dcpo** for point-continuous valuations

- Why do we care about point-continuous valuations?

Prop (Jia, Lindenhovius, Mislove, Zamdzhiev 2021). Fubini-Tonelli holds on **Dcpo** if one of the valuations is point-continuous.

- *Proof.* Fix f and a **point-continuous** valuation ν
  F and G are linear and continuous from $\mathbf{V}_w Y$ to $\overline{\mathbb{R}}_{+\sigma}$

$$F(\mu) = \int_x \left(\int_y f(x,y) d\nu \right) d\mu$$

$$G(\mu) = \int_y \left(\int_x f(x,y) d\mu \right) d\nu$$

- We use the Schröder-Simpson theorem:

$$h_F(x) = \int_y f(x,y) d\nu = h_G(x)$$

for every $x \in X$

Theorem (**Schröder and Simpson** 2005; JGL 2015).
Every linear continuous map $F\colon \mathbf{V}_w X \to \overline{\mathbb{R}}_+$
is equal to $\mu \mapsto \int_x h_F(x) d\mu$ for some unique $h_F \in [X \to \overline{\mathbb{R}}_{+\sigma}]$
(explicitly, $h_F(x) = F(\delta_x)$)

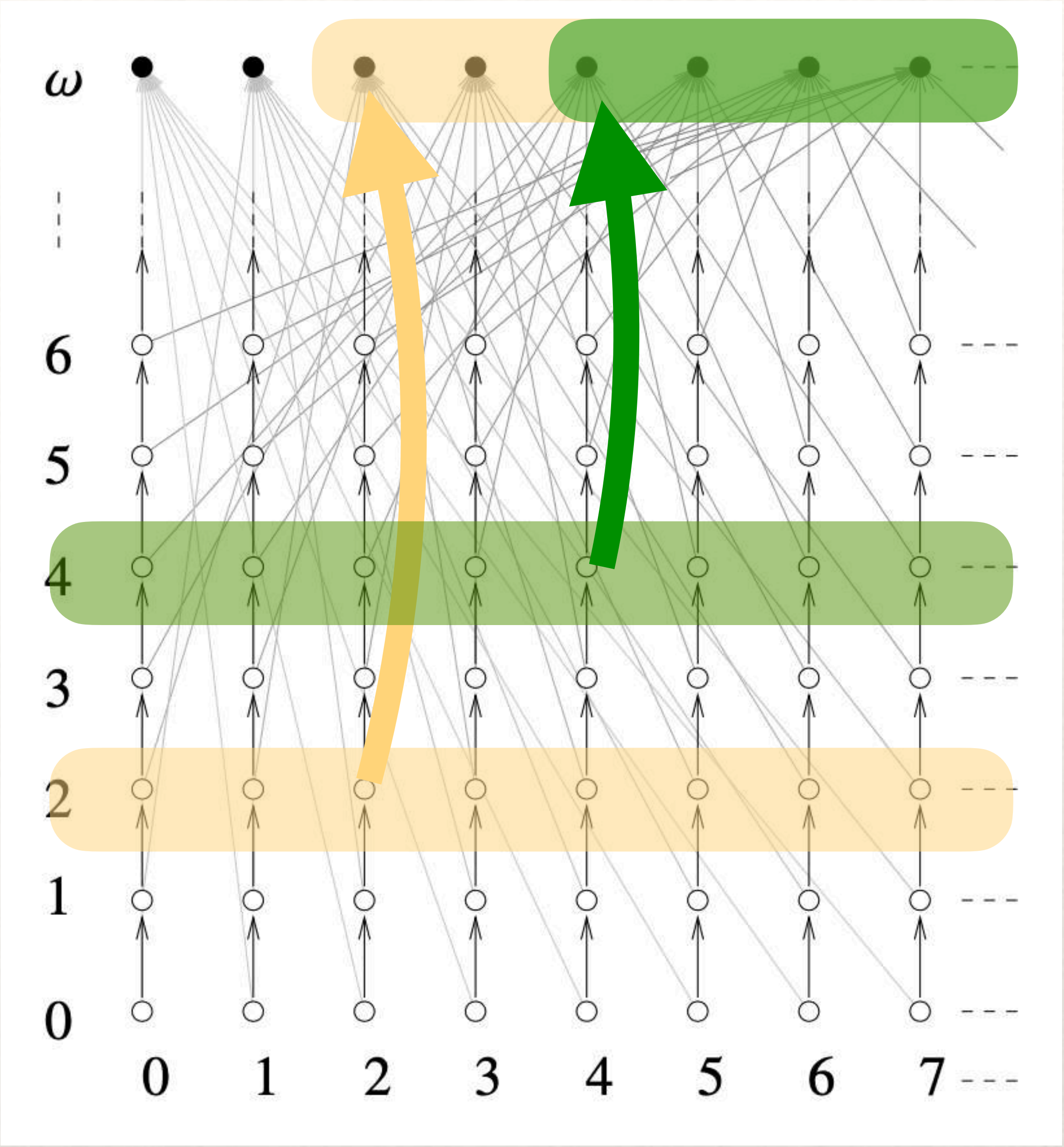
- so $F = G$. $\square$

So where are we now?

- **Prop.** Simple ⇒ Minimal ⇒ Point-continuous ⇒ Continuous.
- **Prop.** Fubini-Tonelli holds on **Dcpo**
  if one of the valuations is point-continuous.
- **Conjecture.** Every continuous valuation on a dcpo is point-continuous.
- That would imply Fubini-Tonelli for all continuous valuations on **Dcpo**.
- We will see (briefly) that this conjecture is **wrong**.
  (JGL, X. Jia. *Separating minimal valuations, points-continuous valuations, and continuous valuations.* Math. Struct. Computer Science 31(6), 2021, pages 614–632 https://doi.org/10.1017/S0960129521000384)
- In fact, all the implications above are **strict**.

Separating minimal from point-continuous valuations

A funny valuation on J



- On Johnstone's dcpo **J**, there is a continuous valuation μ defined by:

  $\mu(U) = 1$ for every non-empty Scott-open set U
  $\mu(\emptyset) = 0$

- Modularity $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$ comes from the fact that $\mathbf{J}_\sigma$ is **hyperconnected**:

  any two non-empty open sets intersect.

  (Check it! Observe that every non-empty open set contains all points (m, ω) for m large enough.)

- We will show that μ is not minimal.

Discrete and good valuations

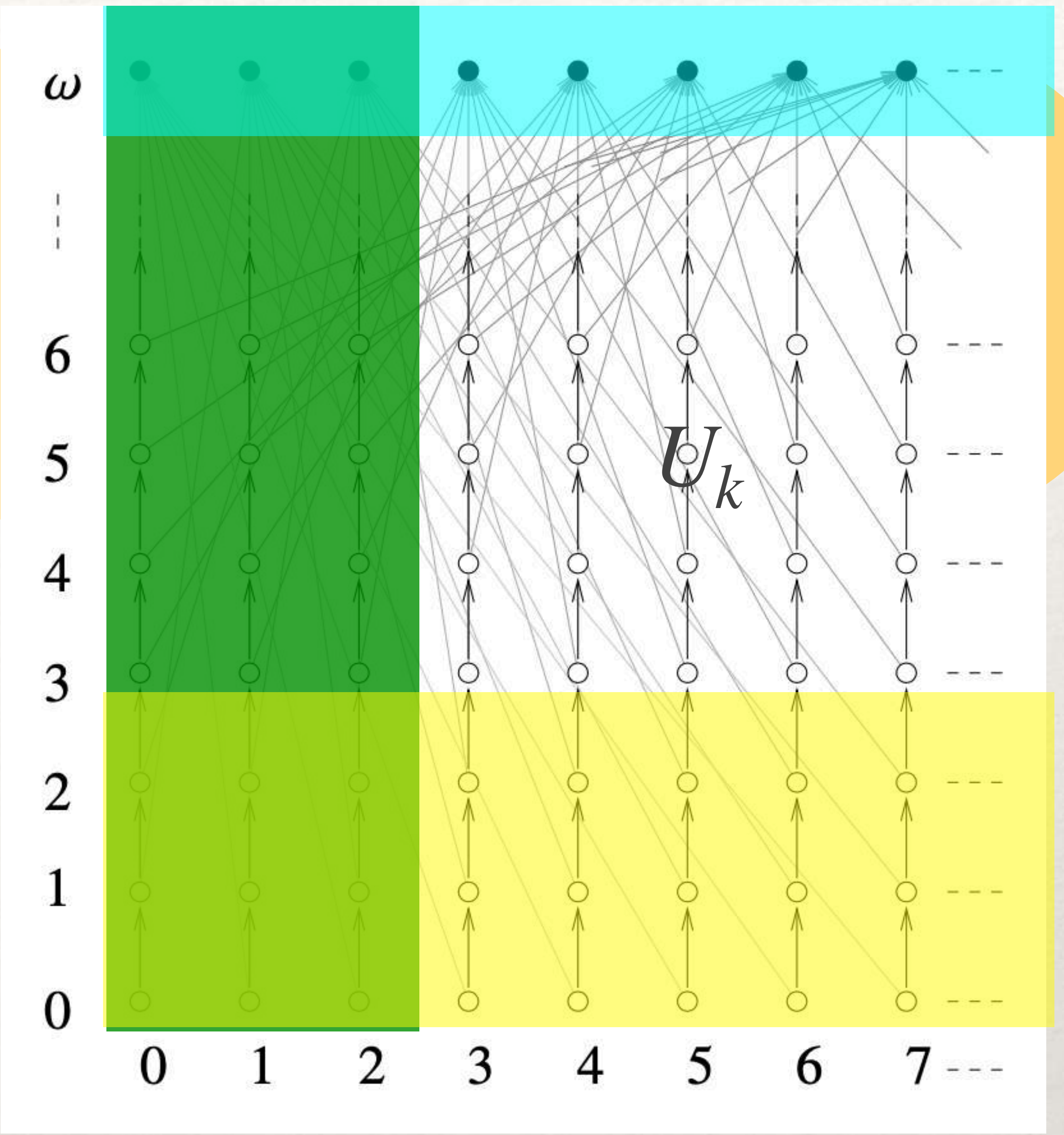
- A bounded valuation ν on X is **good** iff it extends to a Borel measure
- Every bounded **discrete** valuation $\sum_x a_x \delta_x$ (infinite sum, in general)
  is **good** (Alvarez-Manilla, Edalat, Saheb-Djahromi 2000)
- Every subset of $\mathbf{J}_\sigma$ is Borel.
- One can show that every bounded continuous valuation ν on $\mathbf{J}_\sigma$
  is of the form $\theta + r \,.\, \mu$, where
  — θ is discrete (hence good)
  — $r \geq 0$

Namely, $\theta = \sum_{x \in \mathbf{J}} \nu(\{x\}) \,.\, \delta_x$

$r = \nu(\mathbf{J}) - \sum_{x \in \mathbf{J}} \nu(\{x\})$

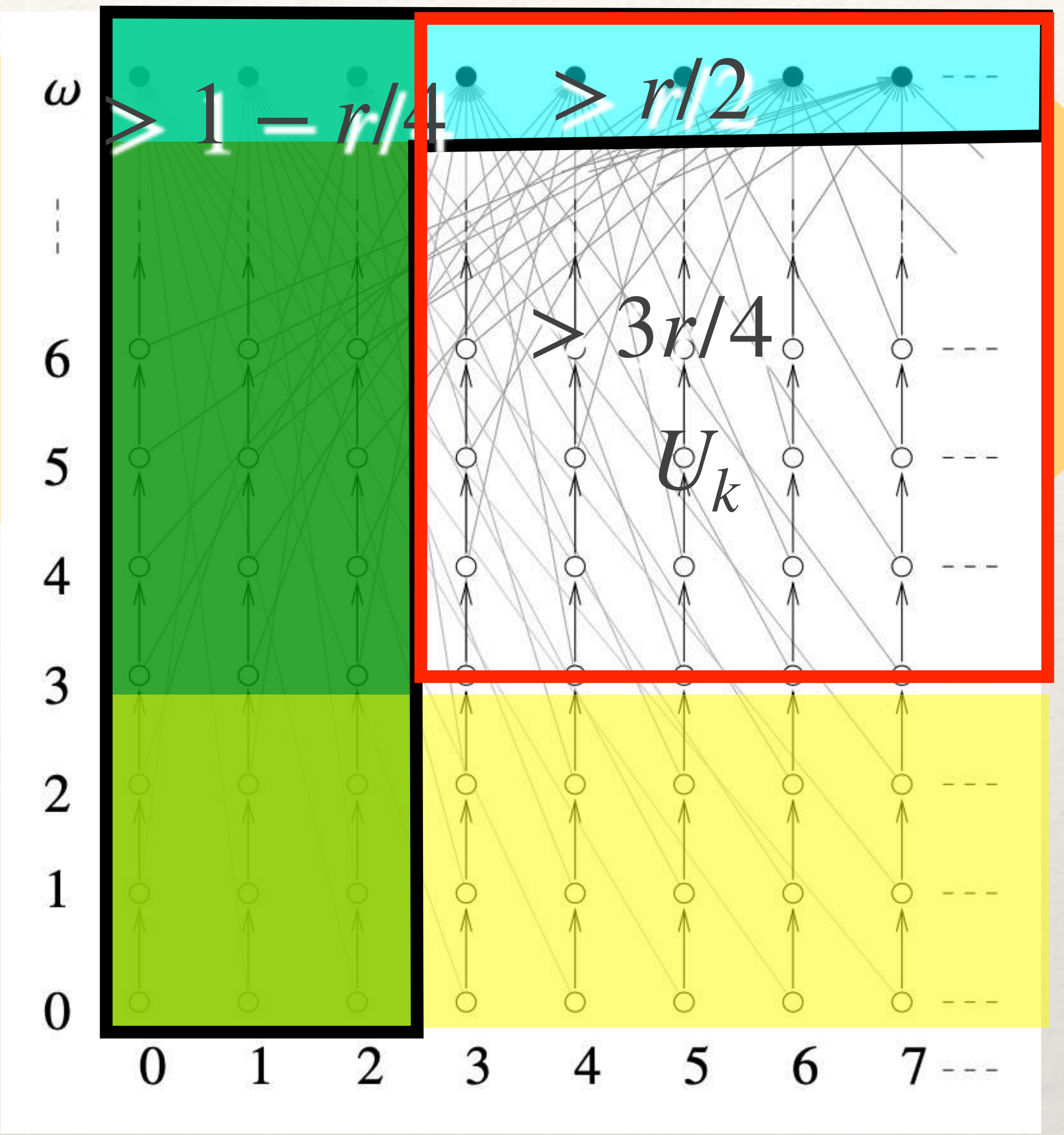
Good valuations on $\mathbf{J}$ are Scott-closed

- **Lemma.** If $\theta + r\,.\,\mu$ is bounded and a directed supremum of good valuations θ_i on $\mathbf{J}_{\sigma'}$, then $r = 0$.
- *Proof.* We assume $r \neq 0$.
  Wlog., we also assume total mass $(\theta + r\,.\,\mu)(\mathbf{J}) = 1$
- Let D_k={k leftmost columns} (in green)
- $\downarrow D_k$ is closed (green+yellow); let U_k be its complement
- $\uparrow D_k$: green+blue



Good valuations on $\mathbf{J}$ are Scott-closed

- **Lemma.** If $\theta + r\,.\,\mu$ is bounded and a directed supremum of good valuations θ_i on $\mathbf{J}_\sigma$, then $r = 0$.

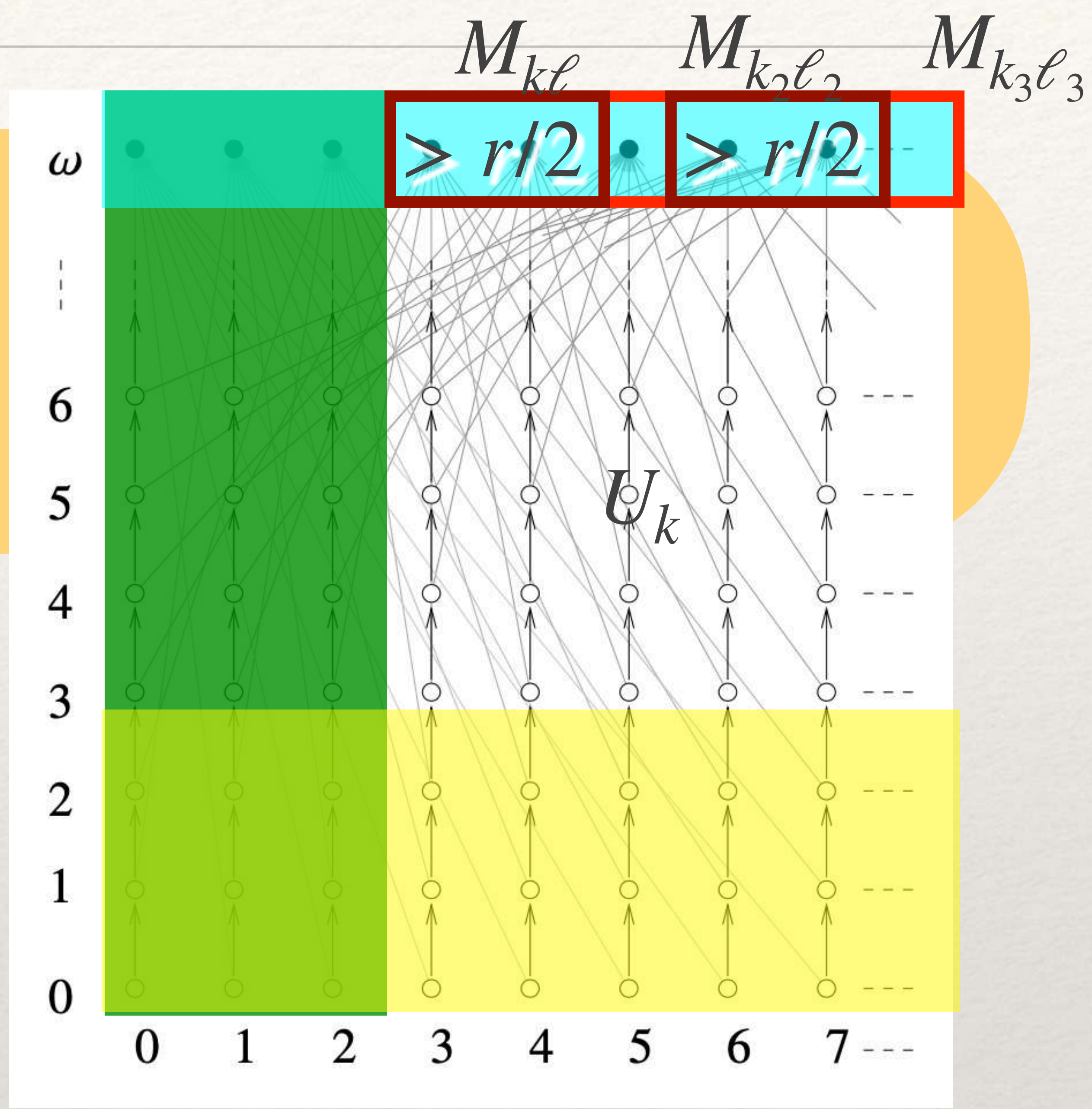


D_k

- We have $(\theta + r\,.\,\mu)(\mathbf{J}) = 1 = \sup_i \theta_i(\mathbf{J}) > 1 - r/4$ so $\theta_i(\mathbf{J}) > 1 - r/4$ for i large enough;
- $\mathbf{J} = \bigcup^{\uparrow}_{k \in \mathbb{N}} \uparrow D_k$, union of a countable chain of Borel sets
  Since θ_i is (extends to) a measure, for k large enough,
  $$\theta_i(\uparrow D_k) > 1 - r/4$$
- We have $(\theta + r\,.\,\mu)(U_k) \geq r > 3r/4$
  For j large enough, $\theta_j(U_k) > 3r/4$ (and $\theta_j \geq \theta_i$, so $\theta_j(\uparrow D_k) > 1 - r/4$)
- Total mass of $\theta_j \leq 1$, so $\theta_j(\uparrow D_k \cap U_k) > (1 - r/4) + 3r/4 - 1 = r/2$

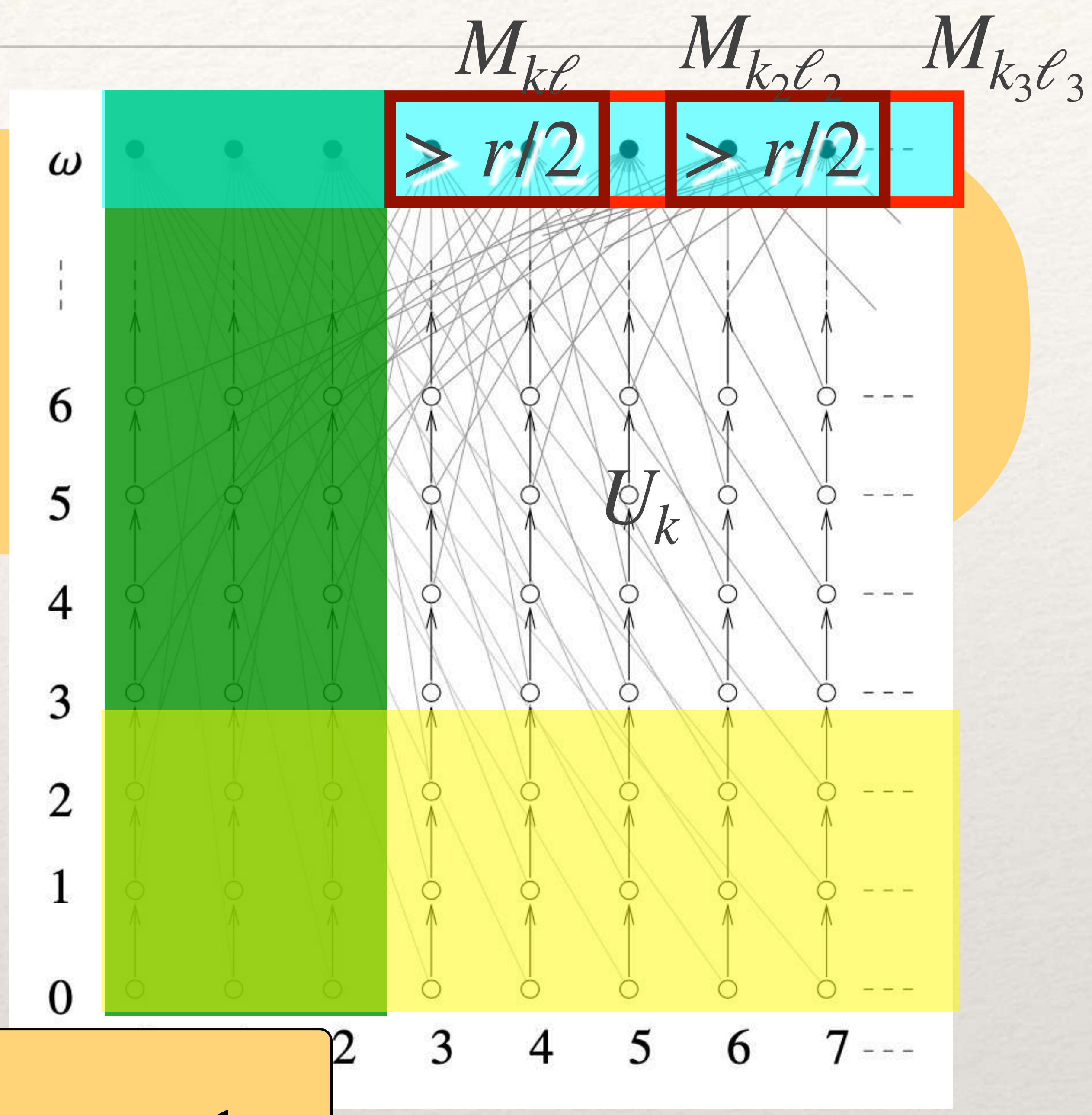
Good valuations on **J** are Scott-closed

- **Lemma.** If $\theta + r \,.\, \mu$ is bounded and a directed supremum of good valuations θ_i on $\mathbf{J}_\sigma$, then $r = 0$.



- $\theta_j(\uparrow D_k \cap U_k) > r/2$
- Let $M_{k\ell} = \{(k,\omega), (k+1,\omega), \cdots, (\ell-1,\omega)\}$
- Since θ_j is a measure, $\theta_j(M_{k\ell}) > r/2$ for some $\ell > k$
- Similarly, for some j_2 such that $\theta_{j_2} \geq \theta_j$, there are $\ell_2 > k_2 \geq \ell$ such that $\theta_{j_2}(M_{k_2\ell_2}) > r/2$
- Then, for some j_3 such that $\theta_{j_3} \geq \theta_{j_2}$, there are $\ell_3 > k_3 \geq \ell_2$ such that $\theta_{j_3}(M_{k_3\ell_3}) > r/2$
- Etc. Eventually, $\theta_{j_N}(M_{kl} \uplus \cdots \uplus M_{k_N\ell_N})$ $> Nr/2$ > 1 : contradiction. □

μ is not minimal



- **Lemma.** If $\theta + r\,.\,\mu$ is bounded and a directed supremum of good valuations θ_i on $\mathbf{J}_{\sigma'}$, then $r = 0$.
- Hence directed suprema of good valuations θ_i of total mass ≤ 1 are good valuations θ
- By transfinite induction, every minimal valuation on $\mathbf{J}$ of total mass ≤ 1 is good.
- But μ itself is **not** good: otherwise $\mu\left(\bigcap^{\downarrow}_{k\in\mathbb{N}} U_k\right) = \inf^{\downarrow}_{k\in\mathbb{N}} \mu(U_k)$

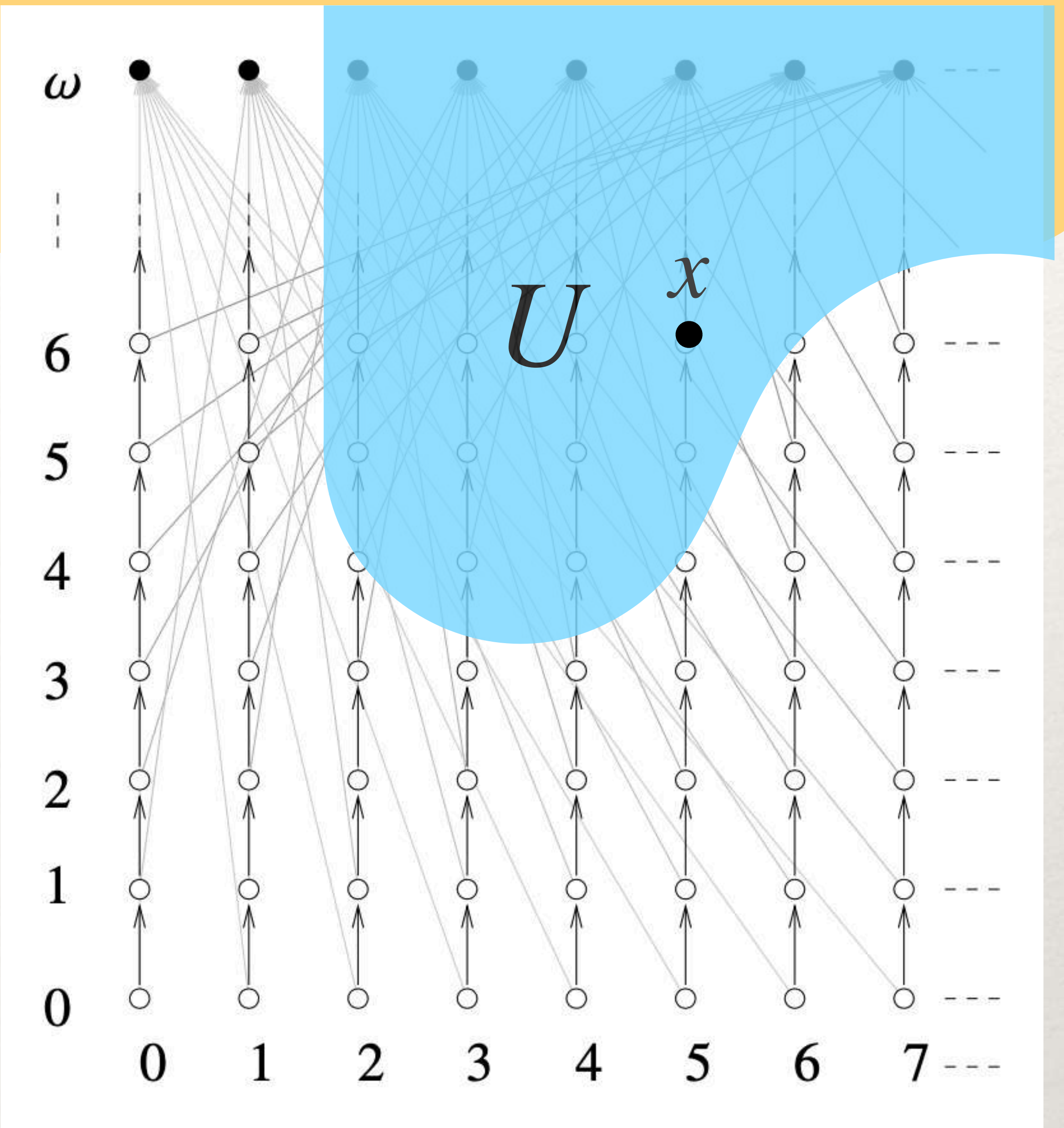
$= \mu(\emptyset) = 0$

$= 1$

- **Theorem.** μ is not minimal on $\mathbf{J}_\sigma$.

μ is not minimal

- **Theorem.** μ is not minimal on $\mathbf{J}_\sigma$.
- However,
  μ is point-continuous:
  It suffices to show that it is continuous from $(\mathcal{O}X)_p$ to $\overline{\mathbb{R}}_{+\sigma}$
- We assume $\mu(U) > r$, so U is non-empty, and $r < 1$.
- Pick $x \in U$. Then $[x \in\ ] = \{V \in \mathcal{O}\mathbf{J}_\sigma \mid x \in V\}$ is open in $(\mathcal{O}X)_p$ and for every $V \in [x \in\ ]$, $\mu(V) > r$ (since V is non-empty). □
- In fact:
- **Theorem.** Every continuous valuation on $\mathbf{J}_\sigma$ is point-continuous.



Separating point-continuous from continuous valuations

Lebesgue is not point-continuous

- An easy to find an example of a non-point-continuous, continuous valuation
- Consider Lebesgue measure λ on $\mathbb{R}$
  (with its metric topology)

Reminder

- **Theorem (Adamski 1977).** Given any **Borel measure** on a hereditarily Lindelöf space, (in particular, on a second-countable space) its restriction to open sets is a **continuous valuation**.

- Heckmann showed that a valuation ν is point-continuous iff
  for every open set V, if $r < \nu(V)$ then
  there are finitely many points $x_1, \ldots, x_n$ in V such t
  for every open neighborhood U of $x_1, \ldots, x_n$,
  $r < \nu(U)$

Can we find a similar counter-example **on a dcpo**?

- This is wrong for λ: $\lambda\left(\bigcup_{i=1}^{n}]x_i - \frac{\epsilon}{2n}, x_i + \frac{\epsilon}{2n}[\right) \leq \epsilon$ arbitrarily small

The Sorgenfrey line $\mathbb{R}_\ell$

- A famous counterexample in topology:
  Sorgenfrey topology on $\mathbb{R}$ generated by basic open sets $[a, b[$
  (topology of convergence from the right)

Still not a dcpo,
so we will consider its
Smyth powerdomain

Nice	N...
paracompact, T_4	
zero-dimensional	
Choquet-complete, henc...	
first-countable, with countable dense subset ...	
hereditarily Lindelöf	
completely quasi-metrizable	

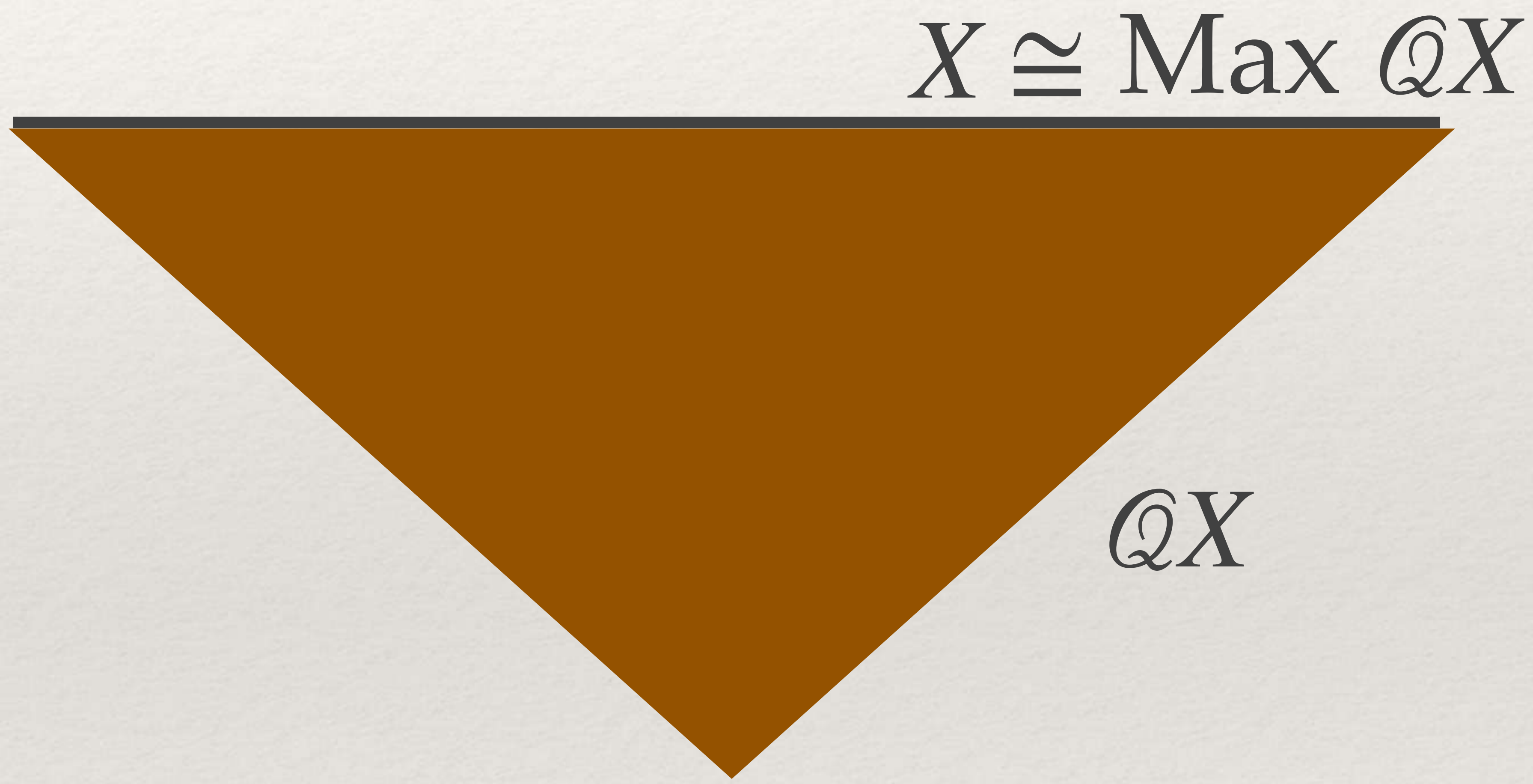
Reminder

- **Theorem (Adamski 1977).** Given any **Borel measure** on a hereditarily Lindelöf space, its restriction to open sets is a **continuous valuation**.

Additionally, $\mathbb{R}$ and $\mathbb{R}_\ell$ have the same Borel σ-algebra
So there is a **Lebesgue** continuous valuation λ on $\mathbb{R}_\ell$

The Smyth powerdomain of $\mathbb{R}_\ell$

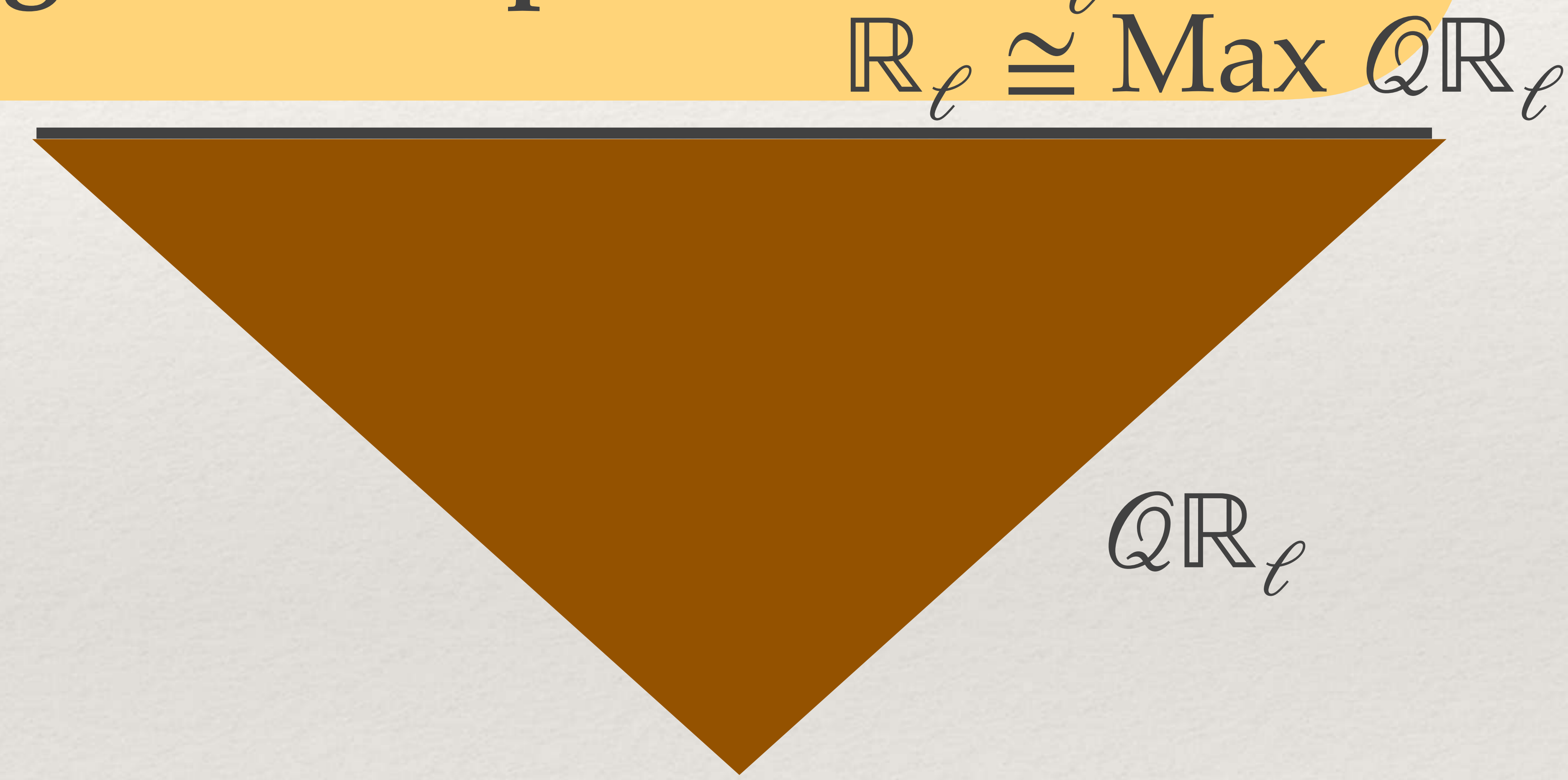
- For every topological space X,
  the **Smyth powerdomain** $\mathcal{Q}X$ = {compact saturated subsets of X},
  ordered by $\supseteq$



- $\mathcal{Q}X$ is a **dcpo** if X is well-filtered
  … and $\mathbb{R}_\ell$ is well-filtered, since Hausdorff
- $x \mapsto \{x\}$ is a **topological embedding** of X into $\mathcal{Q}X$
  ~~if X is locally compact T_1 … but $\mathbb{R}_\ell$ is not locally compact~~
- or if X is T_1, well-filtered and first-countable (He, Li, Xi, Zhao 2019)
  … and $\mathbb{R}_\ell$ **is** first-countable

The Smyth powerdomain of $\mathbb{R}_\ell$

- **Proposition.** $\mathcal{Q}\mathbb{R}_\ell$ is a dcpo model of $\mathbb{R}_\ell$:
  through $x \mapsto \{x\}$, $\mathbb{R}_\ell$ embeds as a **topological subspace** of $\mathcal{Q}\mathbb{R}_\ell$.



- Explicitly, the open subsets of $\mathbb{R}_\ell$
  are the intersections $\mathscr{U} \cap \mathbb{R}_\ell$
  where $\mathscr{U}$ is Scott-open in $\mathcal{Q}\mathbb{R}_\ell$

- The Lebesgue valuation λ on $\mathbb{R}_\ell$ induces
  a continuous valuation $\overline{\lambda}$ on $\mathcal{Q}\mathbb{R}_\ell$ by

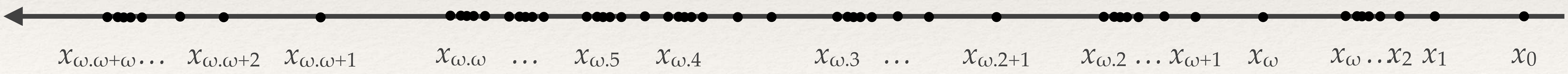
$$\overline{\lambda}(\mathscr{U}) = \lambda(\mathscr{U} \cap \mathbb{R}_\ell)$$

(the image continuous valuation through the embedding $\mathbb{R}_\ell \to \mathcal{Q}\mathbb{R}_\ell$)

The elements of $\mathcal{Q}\mathbb{R}_\ell$

- We will show that $\bar{\lambda}$ is not point-continuous.
  To this end, we need:

- **Proposition.** The compact (saturated) subsets of $\mathbb{R}_\ell$ are exactly
  the **well-founded subdcpos** of $(\mathbb{R}, \geq)$.
  They are all **countable**.



$\overline{\lambda}$ is not point-continuous

- **Proposition.** $\overline{\lambda}$, defined by $\overline{\lambda}(\mathcal{U}) = \lambda(\mathcal{U} \cap \mathbb{R}_\ell)$, is not point-continuous.
- It suffices to show that given any finite set $\{Q_1, \cdots, Q_n\}$ of points of $\mathcal{Q}\mathbb{R}_\ell$, we can find Scott-open neighborhoods $\mathcal{U}$ of those points, such that $\overline{\lambda}(\mathcal{U})$ is arbitrarily small
- Enumerate the **countably many** points of $Q_1 \cup \cdots \cup Q_n$ as $x_k, k \in \mathbb{N}$
- Let $U = \bigcup_{k \in \mathbb{N}}]x_k - \frac{\epsilon}{2^k}, x_k + \frac{\epsilon}{2^k}[$, and $\mathcal{U} = \{Q \in \mathcal{Q}\mathbb{R}_\ell \mid Q \subseteq U\}$
- Then $\overline{\lambda}(\mathcal{U}) = \lambda(U) \leq 4\epsilon$. $\square$

Reminder

Heckmann showed that a valuation ν is point-continuous iff for every open set $\mathcal{V}$, if $r < \nu(\mathcal{V})$ then there are finitely many points $Q_1, \ldots, Q_n$ in $\mathcal{V}$ such th for every open neighborhood $\mathcal{U}$ of $x_1, \ldots, x_n$, $r < \nu(\mathcal{U})$

Conclusion

Conclusion, open problems, and a comment

- Fubini-Tonelli for continuous valuations: holds on **Top**

  Open: Does Fubini-Tonelli for all continuous valuations on Dcpo?

- Holds for minimal and even point-continuous valuations on **Dcpo**

- Simple ⇒ Minimal ⇒ Point-continuous ⇒ Continuous,

  all strict implications

- Fubini-Tonelli also holds (essentially by definition) for the even larger class of **central valuations** (Jia, Mislove, Zamdzhiev 2021)

  Open: Are there any non-central continuous valuations on a dcpo?

- The greater picture: questions from computer science

  (reasoning about computer programs).