# A few things on Noetherian spaces

Jean Goubault-Larrecq, ENS Cachan, France

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# Noetherian spaces

- **Defn.** A space is **Noetherian** iff every open is compact.
- Here compact does not entail any kind of separation.
- **Fact.** The following are equivalent:
  (1) $X$ is Noetherian
  (2) Every subspace of $X$ is compact
  (3) Ascending sequences $U_1 \subseteq U_2 \subseteq \ldots \subseteq U_n \subseteq \ldots$ of opens stabilize
  (4) Descending sequences $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ of closed sets stabilize
- We shall see other characterizations later.

# Outline

- Characterizations of Noetherian spaces (half of them well-known)
- Transfering results from wqo theory to topology
- Applications in software verification
- Representations
- Conclusion

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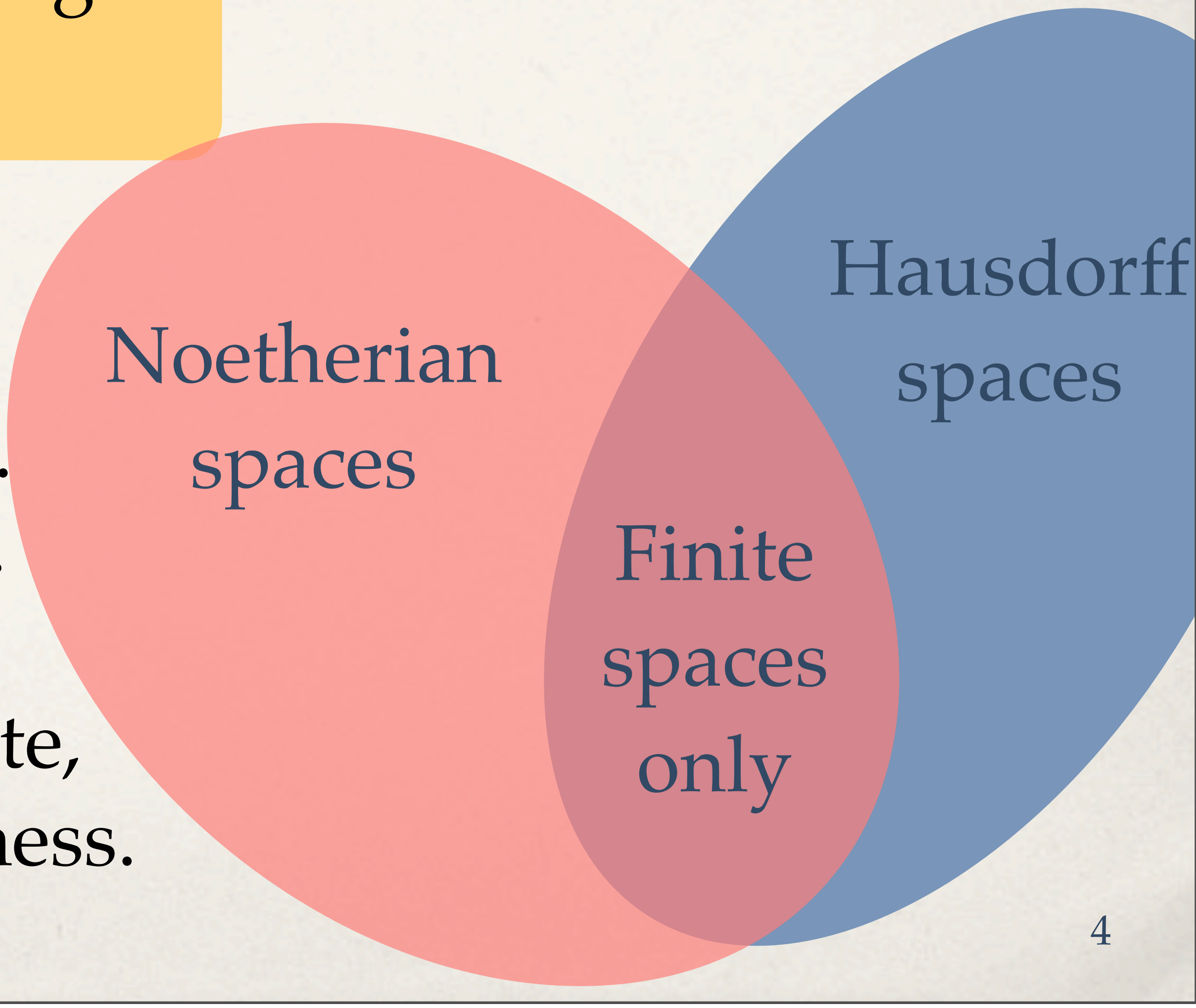
# Noetherian spaces, classically

- **Defn.** A space is **Noetherian** iff every open is compact.
- **Prop.** The spectrum of a Noetherian ring is a Noetherian space.
- E.g., the spectrum of a polynomial ring over $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.
  **Not** my first source of inspiration here.
  We shall see many (simpler) examples.
- **Note.** Noetherian + Hausdorff ⇔ finite, so we shall definitely **drop** Hausdorffness.



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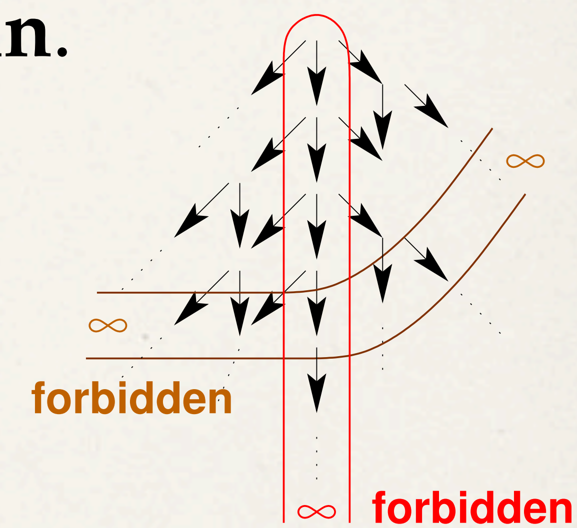
# Well-quasi-orders

- **Fact.** The following are equivalent for a quasi-ordering $\leq$:
(1) Every sequence $(x_n)_{n\in\mathbb{N}}$ is **good**: $x_m \leq x_n$ for some $m<n$
(2) Every sequence $(x_n)_{n\in\mathbb{N}}$ is **perfect**: has a monotone subsequence
(3) $\leq$ is **well-founded** and has **no infinite antichain**.

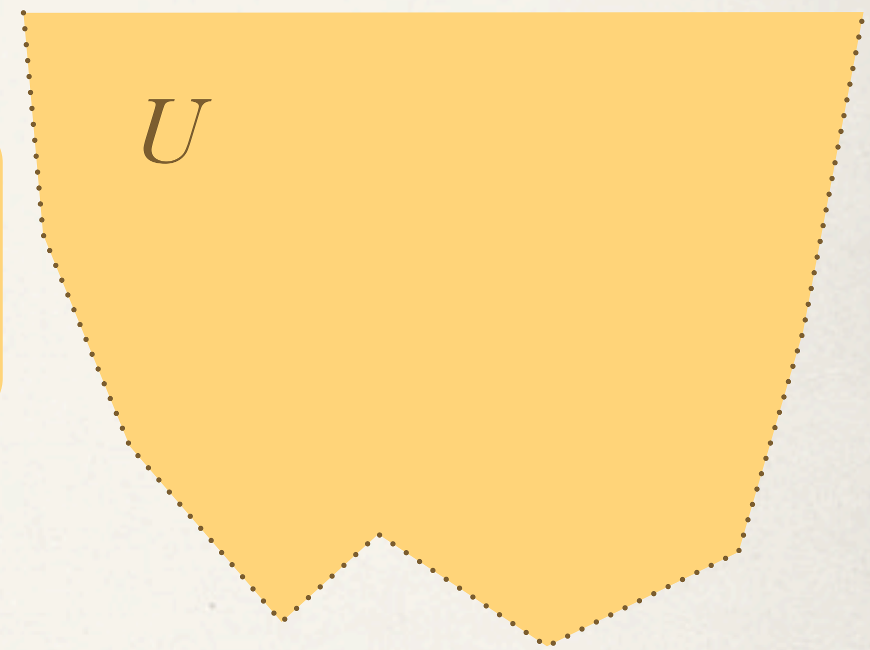
- **Defn.** Such a quasi-ordering $\leq$ is called a **well-quasi-order (wqo).**




- Applications:
classification of graphs (Kuratowski, Robertson-Seymour)
verification (computer science)
model theory (logic: Fraïssé, Jullien, Pouzet)

# The starting observation

- Given a qo $(X, \leq)$, its **Alexandroff topology** has as opens $U$ all upwards-closed subsets of $X$.

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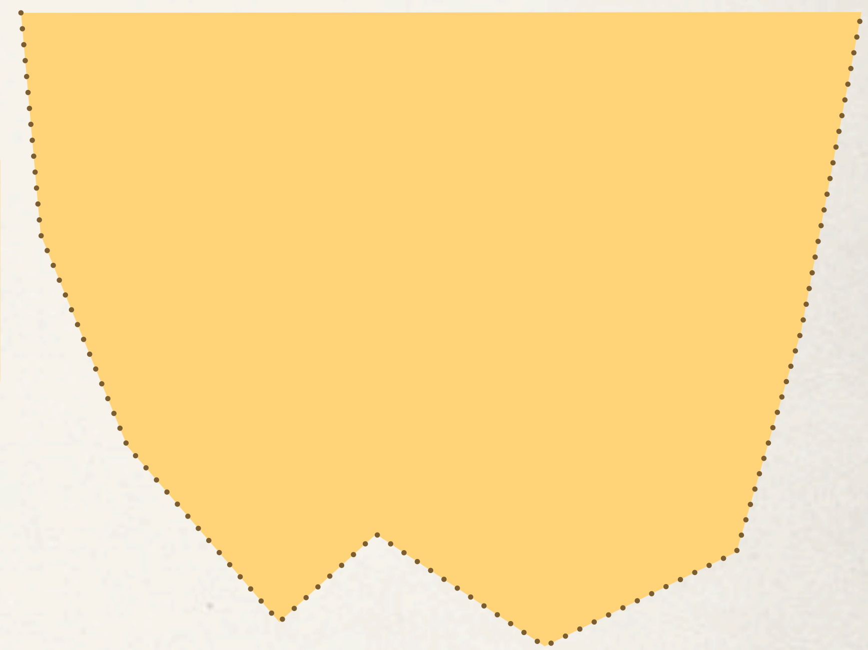

- **Prop.** Let $(X, \leq)$ be wqo.
  With its Alexandroff topology, $X$ is Noetherian.

- *Proof*. Consider an infinite ascending sequence
  $U_1 \subsetneq U_2 \subsetneq \ldots \subsetneq U_n \subsetneq \ldots$ of opens.
  Pick $x_n$ in $U_n$, not in any previous $U_m$.
  By wqo, $x_m \leq x_n$ for some $m<n$.
  Since $x_m \in U_m$ upwards-closed, $x_n \in U_m$:
  contradiction. $\square$

- Plenty of wqos ⇨ plenty of Noetherian spaces.

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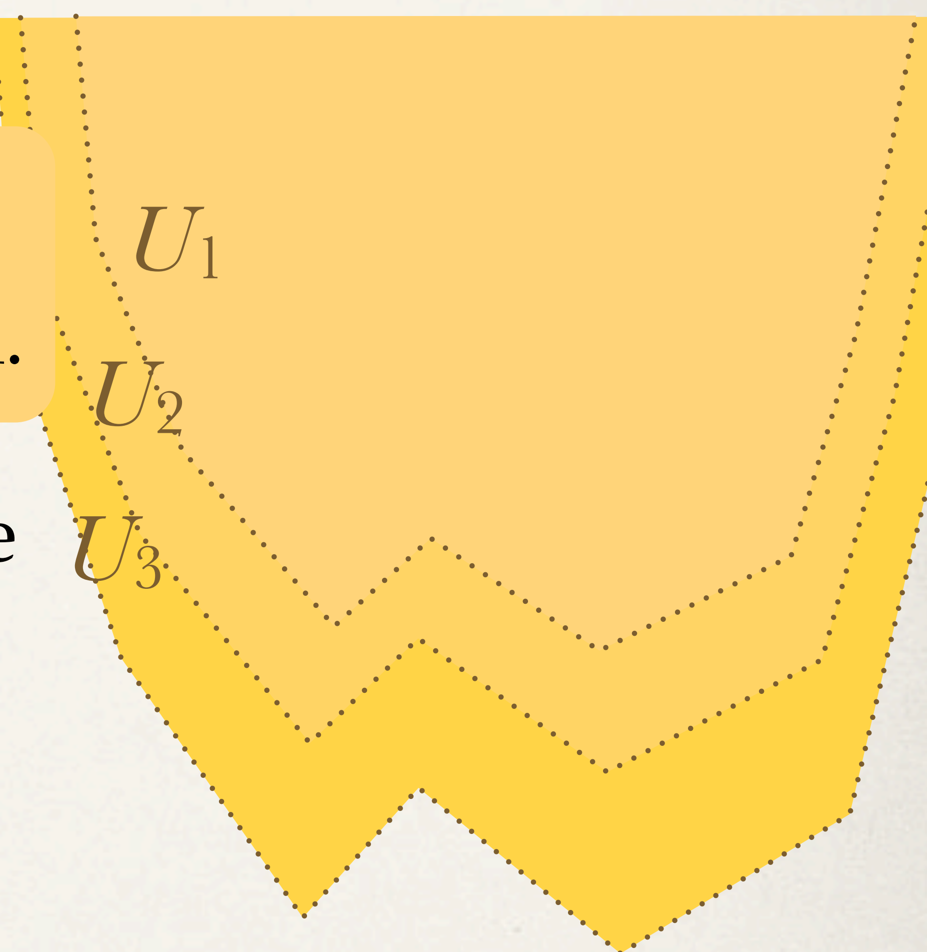


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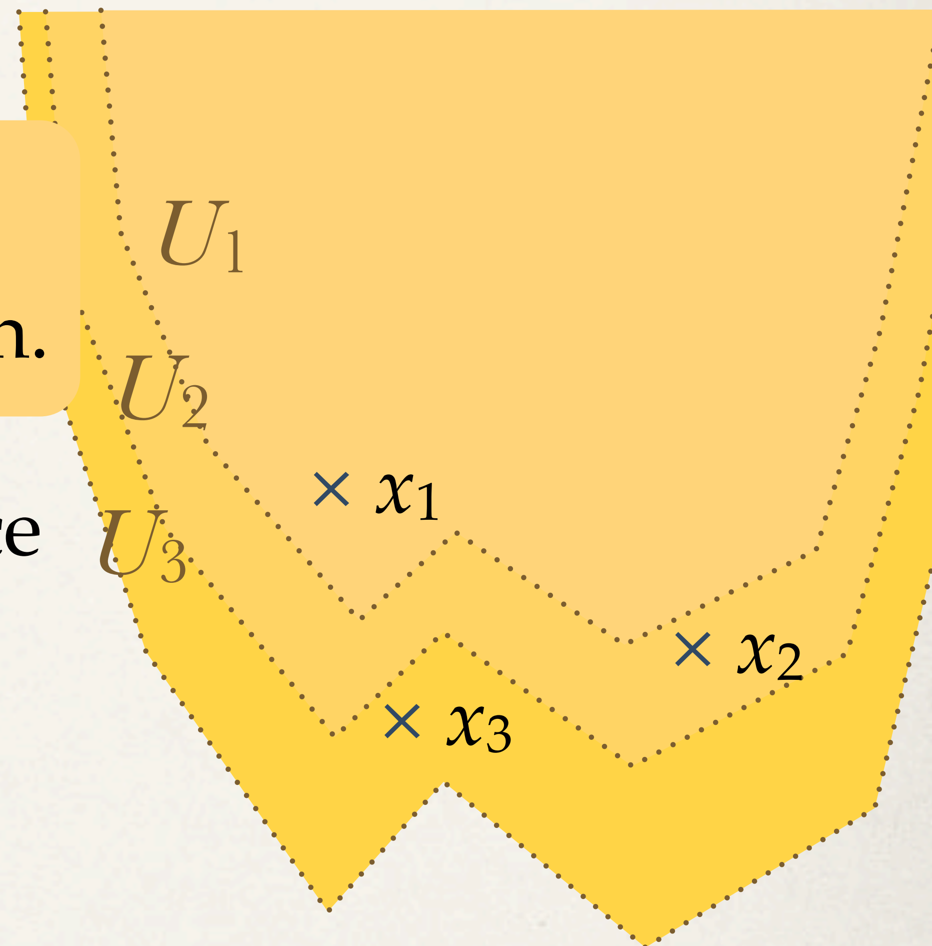


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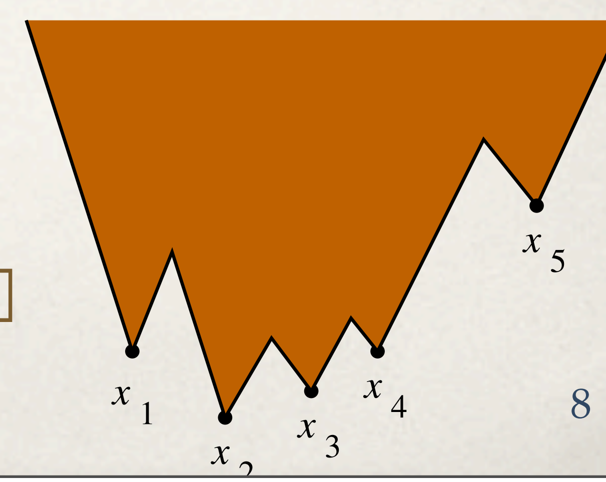
# Noetherian + Alexandroff

- **Prop.** Let $(X, \leq)$ be wqo.
  With its Alexandroff topology, $X$ is Noetherian.

- There are also Noetherian spaces that are not Alexandroff:
  — spectra of rings, with the Zariski topology
  — powersets (see later)

- **Conversely**, the qo sets $(X, \leq)$ that are Noetherian
  in their Alexandroff topology
  are exactly the **wqo** sets.

- *Proof*. From $(x_n)_{n\in\mathbb{N}}$ define $U_n = \uparrow\{x_1, ..., x_n\}$.
  This stabilizes at $n$: $U_{n-1} = U_n$, so $x_n \in \uparrow\{x_1, ..., x_{n-1}\}$. $\square$



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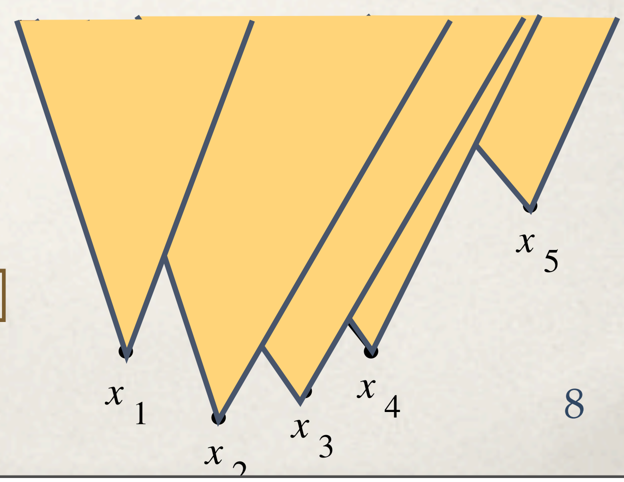


# Basic constructions

- **Prop.** (1) Every **wqo** is Noetherian in its Alexandroff topology
  (2) The **spectrum** of a Noetherian ring is Noetherian
  (3) Finite **products** of Noetherian spaces are Noetherian
  (4) Finite **coproducts** of Noetherian spaces are Noetherian
  (5) **Subspaces** of Noetherian spaces are Noetherian
  (6) Topologies **coarser** than a Noetherian topology are Noetherian
  (7) Continuous **images** of Noetherian spaces are Noetherian
  (in particular, quotients)

- We shall see other constructions that preserve Noetherianness.

- We need additional characterizations of Noetherianness.

# Cluster points

- **Prop.** $X$ is Noetherian iff every net $(x_i)_{i \in I}$ *contains* a cluster point $x_i$.
  (The important point is: the cluster point $x_i$ *belongs* to the net.)

- *Proof.*
  ($\Rightarrow$) If $X$ Noetherian, then subspace $K=\{x_i \mid i \in I\}$ is compact,
  hence $(x_i)_{i \in I}$ has a cluster point **in *K*.**
  ($\Leftarrow$) Let $U$ be open in $X$.
  Every net $(x_i)_{i \in I}$ inside $U$ has a cluster point in $U$, *viz.* some $x_i$.
  So $U$ is compact. $\square$

- **Note:** in Alexandroff spaces, $x_i$ cluster point means that for some $i$, infinitely many entries $x_j$ are above $x_i$. (Take the open $\uparrow x_i$.)
  ... hence all sequences are **good**.

# Self-convergent nets

- A net $(x_i)_{i \in I}$ is **self-convergent** iff it converges to **every** $x_i$.
  (A very much non-Hausdorff notion!)

- **Thm.** $X$ is Noetherian iff every net $(x_i)_{i \in I}$ has a self-convergent subnet.

- *Proof.* ($\Rightarrow$) Let $J$ be $\{i \in I \mid x_i \text{ is a cluster point of the net}\}$.
  By previous Prop., $J$ is non-empty.
  Check: $J$ is cofinal and directed in $I$; so $(x_j)_{j \in J}$ is a subnet.
  By Kelley's Theorem, $(x_j)_{j \in J}$ has a further subnet that is an ultranet.
  Check that this ultranet is self-convergent.
  ($\Leftarrow$) Obvious, using previous Prop. $\square$

- In Alexandroff spaces, $(x_i)_{i \in I}$ self-convergent iff eventually monotone
  .... hence all sequences are **perfect**.

# Ultrafilters

- A similar characterization ($\lim \mathcal{U}$ =set of limits of $\mathcal{U}$):
- **Thm.** X is Noetherian iff every ultrafilter $\mathcal{U}$ is compact: $\lim \mathcal{U} \in \mathcal{U}$.
- *Proof*. ($\Rightarrow$) Let $U$ be (open) complement of $\lim \mathcal{U}$.
  If $\lim \mathcal{U}$ not in $\mathcal{U}$, then $U$ is in $\mathcal{U}$ (ultrafilter).
  Since $U$ is compact, $\mathcal{U}$ has a limit in $U$.
  So $\lim \mathcal{U}$ intersects $U$: contradiction.

  ($\Leftarrow$) Fix an open $U$. Let $\mathcal{U}$ be an arbitrary ultrafilter containing $U$.
  Since $\lim \mathcal{U} \in \mathcal{U}$, $\lim \mathcal{U} \cap U \in \mathcal{U}$, so $\lim \mathcal{U} \cap U \neq \emptyset$.
  Hence $\mathcal{U}$ has a limit in $U$: $U$ is compact. $\square$

# Application: finite products

- Well-known: finite products of Noetherian spaces are Noetherian. Here is a simple proof.
  (Warning: I'm lying a bit about what a subnet is.)

- Let $X$, $Y$ be Noetherian.
  Let $(x_i, y_i)_{i \in I}$ be a net in $X \times Y$.
  Extract a self-convergent subnet $(x_j)_{j \in J}$.
  From $(y_j)_{j \in J}$ extract a further self-convergent subnet $(y_k)_{k \in K}$.
  Then $(x_k, y_k)_{k \in K}$ is a self-convergent subnet of the original net.

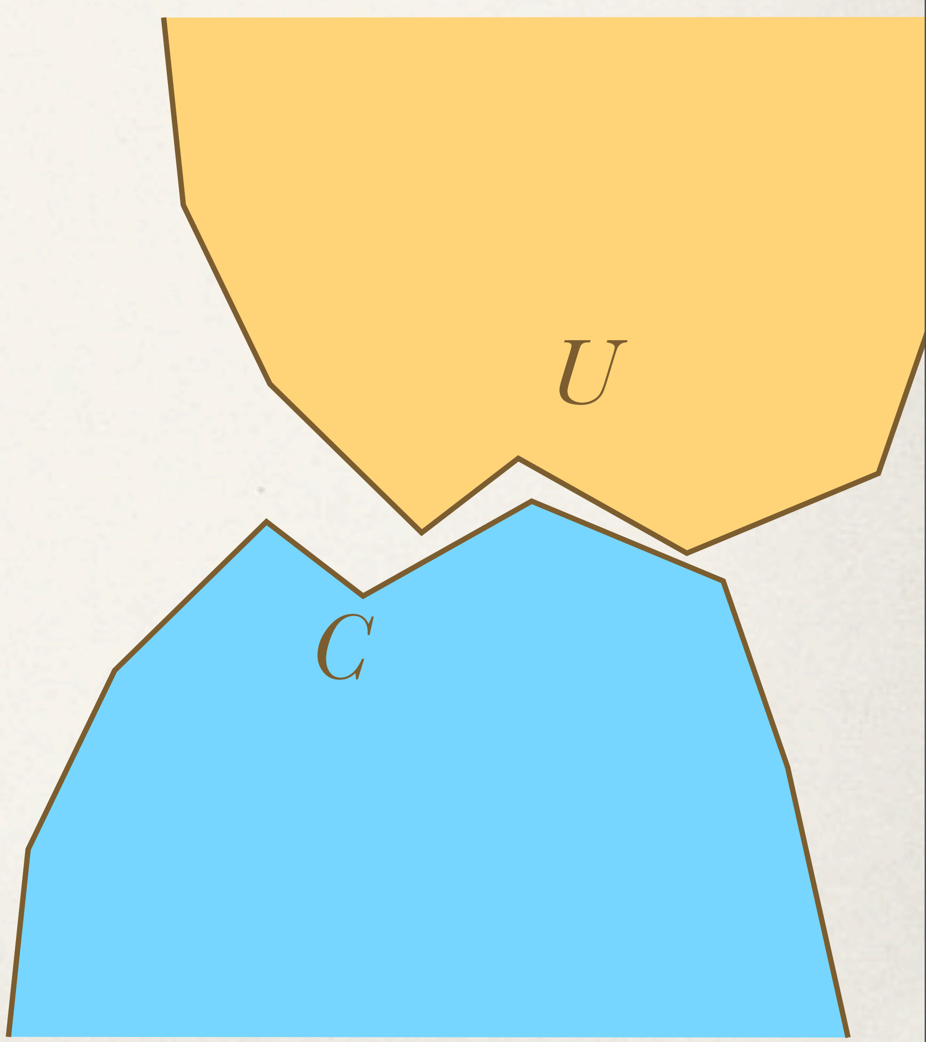
- This is a topological version of the Ramsey argument behind the classical wqo proofs.

# Stone duality in a nutshell

- There is a functor **O** : **Top** → **Frame**$^{op}$ that:
  — maps each space $X$ to its frame **O**$X$ of opens
  — maps $f$:$X \to Y$ to **O**$f$ : **O**$Y \to$ **O**$X$ : $V \mapsto f^{-1}(V)$.
- **O** is left-adjoint to a functor **pt** : **Frame**$^{op}$ → **Top**.
- **S**=**pt O** is the **sobrification** functor.
- **Defn.** A space $X$ is **sober** iff it is of the form **pt** $L$ for some frame $L$
  iff it is of the form **S**$Y$ for some space $Y$
  iff $X$=**S**$X$ (all that, up to iso.)

# The specialization quasi-ordering

- **Defn** (specialization, ≤). In a space $X$, $x \leq y$
  iff every open containing $x$ also contains $y$
  iff $x \in \mathrm{cl}(\{y\})$.
- $X$ is $T_0$ iff ≤ is antisymmetric (an ordering).
- Every open $U$ is upwards-closed.
  Every closed set $C$ is downwards-closed.
- The closure $\mathrm{cl}(\{x\})$ is $\downarrow x = \{z \mid z \leq x\}$.



# Sober spaces

- Call $C$ **irreducible closed** iff closed and:
  if $C \subseteq \cup_{i=1}^n C_i$ then $C \subseteq C_i$ for some $i$.
  E.g., $\downarrow x = \mathrm{cl}(\{x\})$ is irreducible closed, for every point $x$.

- **Fact.** $X$ is sober iff $T_0$ and all irreducible closed sets are of this form.

- All Hausdorff spaces are sober, but there are more (e.g., continuous and quasi-continuous dcpos in domain theory).

# Sobrification

- The sobrification functor can be described more concisely as:
  — $\mathbf{S}X$ = {irreducible closed subsets of $X$}
  Opens $\Diamond U = \{C \mid C \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$
  — For $f : X \to Y$ to, $\mathbf{S}f : \mathbf{S}X \to \mathbf{S}Y : C \mapsto \mathrm{cl}(f(C))$.
  — $X$ **embeds** into $\mathbf{S}X$ through $\eta : X \to \mathbf{S}X : x \mapsto \downarrow x$.

- **Fact.** $X$ is Noetherian iff $\mathbf{S}X$ is Noetherian.

- *Proof.* $\Diamond : \mathbf{O}X \to \mathbf{OS}X$ iso, and Noetherianness is a property of opens (ascending sequences of opens stabilize). □

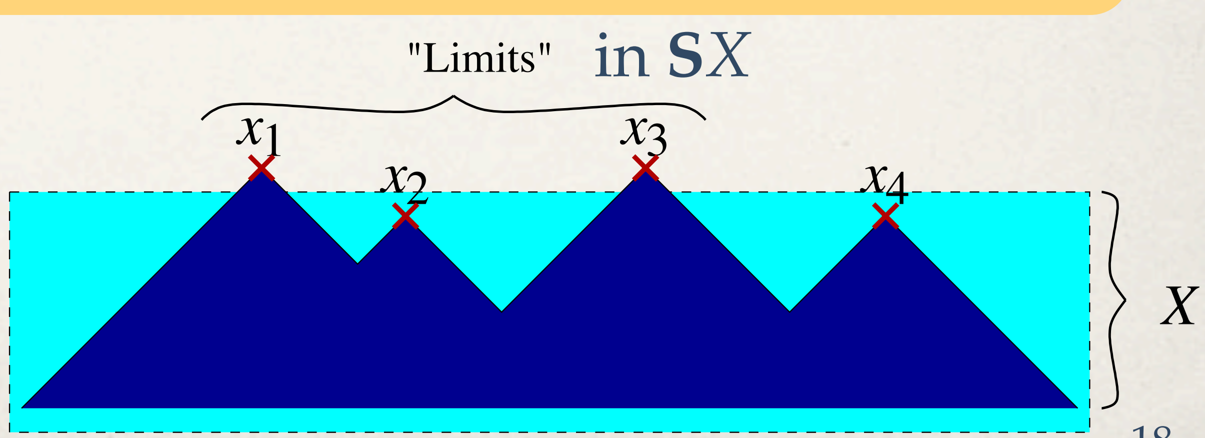
- **Fact.** The Noetherian sober spaces $X$ are the Stone duals $\mathbf{pt}\ L$ of distributive lattices $L$ with the ascending chain condition.

# Sober Noetherian spaces

- An **order-theoretic** characterization.
  Call sets of the form $\downarrow\{x_1, \ldots, x_n\}$ **finitary**.

- **Thm.** A sober space $X$ is Noetherian iff:
  (1) $\leq$ is **well-founded**, and
  (2) the set of lower bounds of any finite set is **finitary**.
  Then:
  (3) The topology of $X$ is the upper topology of $\leq$
  (4) **The closed sets are the finitary sets.**

- *Proof.* Folklore,
  or see (JGL 2013)
  or (Dickmann,Schwartz&Tressl). □



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- Characterizations of Noetherian spaces (half of them well-known)
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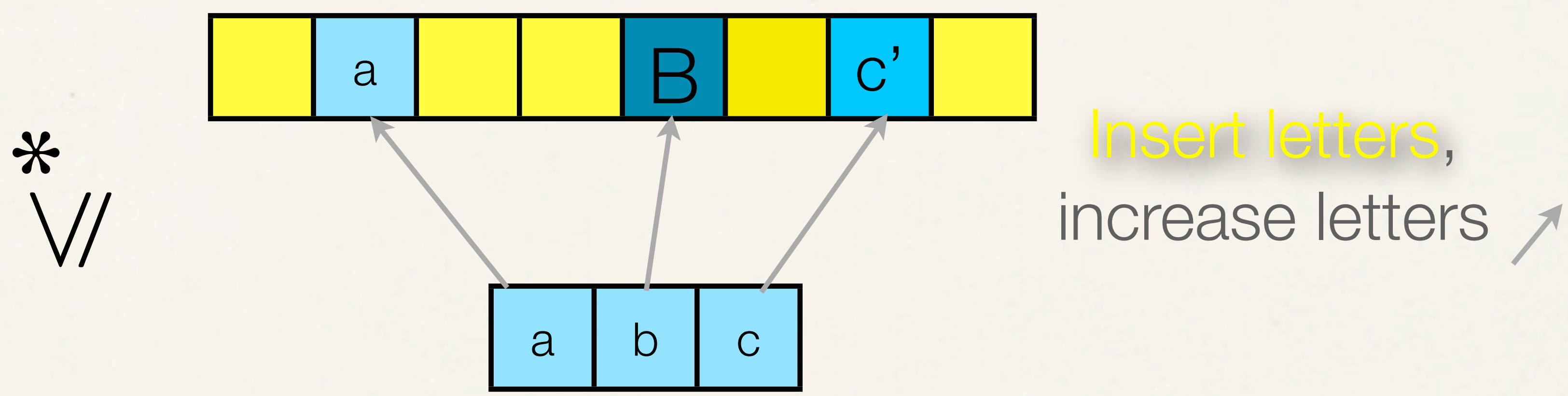
# Higman's Lemma




- **Lemma** (Higman 1954). If $X, \leq$ is wqo, then so is the qoset $X^*$ of finite words under **subword** relation $\leq^*$.

- **Thm** (Topological Higman Lemma, JGL 2013). If $X$ is Noetherian, then so is $X^*$ with the **subword** topology.

- My aim here is to give you a proof of that, imitating Nash-Williams' classic proof (1963) of Higman's Lemma.

# The subword topology

- **Defn.** For opens $U_1, U_2, \ldots, U_n$ in $X$, let $[U_1\, U_2 \ldots\, U_n]$ be set of words
  $\ldots\, a_1 \ldots\, a_2 \ldots$ (etc.) $\ldots\, a_n \ldots$
  with $a_1 \in U_1$, $a_2 \in U_2$, ..., $a_n \in U_n$.
  The **subword topology** on $X^*$ is generated by those sets $[U_1\, U_2 \ldots\, U_n]$.
- Specialization quasi-ordering is $\leq^*$
- If $X$ is Alexandroff, then subword topology=Alexandroff on $X^*$,
  so Higman's Lemma is a special case
  of the topological Higman Lemma.

# Bad sequences

- Let $X$ be a topological space, with subbase $\mathcal{B}$.
  A **bad sequence** is a sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{B}$ such that no $U_n$ is included in a union $\bigcup_{m<n} U_n$ of previous elements.

- **Lemma.** If $X$ is not Noetherian, then (whatever the subbase) it has a bad sequence.

- *Proof*. Let $U$ be non-compact open.
  By Alexander's Subbase Lemma, $U$ has a cover $(U_i)_{i\in I}$ by elements of $\mathcal{B}$ that has no finite subcover.
  Pick some $U_{i1}$. Some point $x_1$ in $U$ is not in $U_{i1}$.
  Pick $U_{i2}$ containing $x_1$. Some point $x_2$ in $U$ is not in $U_{i1} \cup U_{i2}$.
  Pick $U_{i3}$ containing $x_2$. Some point ... etc. $\square$

# Minimal bad sequences

- Assume additionally a well-founded $\sqsubseteq$ ordering on $\mathcal{B}$.
  A **bad sequence** $(U_n)_{n\in\mathbb{N}}$ is **minimal** iff every $\sqsubseteq$-lexicographically smaller sequence $(V_n)_{n\in\mathbb{N}}$ is **good** (i.e., not bad).
  [i.e., $V_0=U_0$, $V_1=U_1$, ..., $V_{n-1}=U_{n-1}$, and $V_n \sqsubset U_n$ (strictly) for some $n$]

- **Lemma.** If $X$ is not Noetherian, then (whatever $\mathcal{B}$ and $\sqsubseteq$) it has a minimal bad sequence.

- *Proof.* Find $U_0$ $\sqsubseteq$-minimal so that it starts a bad sequence.
  Given $U_0$, find $U_1$ $\sqsubseteq$-minimal so that $U_0$, $U_1$ start a bad sequence.
  Given $U_0$, $U_1$, find $U_2$ $\sqsubseteq$-minimal so that ... etc. $\square$

- **Note:** Similar to wqos, where bad sequences are sequences of **points**.

# Proof plan

- On $X^*$, let $\mathcal{B}$ consist of the subbasic opens $[U_1\ U_2 \ldots\ U_m]$.
  Let $[U_1\ U_2 \ldots\ U_m] \sqsubseteq [V_1\ V_2 \ldots\ V_n]$ iff
  there is a (strictly) increasing map $f$:$\{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$
  such that $U_k = V_{f(k)}$.
  (I.e., the $V_p$s are obtained by inserting new opens in the list of $U_k$s.)

- If $X^*$ is not Noetherian, then extract some minimal bad sequence.

- Using the zoom-in Lemma (next slide), find a smaller sequence: that one must be good, leading to a contradiction.

# The zoom-in Lemma

- **Lemma.** Let $X$ be Noetherian, and $a_n \in U_n$ open for each $n \in \mathbb{N}$. There is a subsequence $(a_{n(k)})_k$ s.t. $a_{n(k)} \in U_{n(0)} \cap \ldots \cap U_{n(k)}$ for every $k$.

- *Proof*. Pick a cluster point $a_{n(0)}$ (inside the sequence itself).

  Infinitely many $a_n$s with $n>n(0)$ are in $U_{n(0)}$, forming a subsequence. Pick a cluster point $a_{n(1)}$ from that subsequence.

  Infinitely many $a_n$s with $n>n(1)$ from that subsequence are in $U_{n(0)} \cap U_{n(1)}$, forming a sub-subsequence. Pick a cluster point $a_{n(2)}$ from that sub-subsequence... etc. □

# Proving the topological Higman Lemma

- **Thm** (Topological Higman Lemma, JGL 2013). If $X$ is Noetherian, then so is $X^*$ with the **subword** topology.

- *Proof* (1/3).
  Imagine $X^*$ is not Noetherian, and let
  $\mathcal{U}_n = [U_{n1}\ U_{n2} \ldots U_{nm} \ldots]$ form a minimal bad sequence.
  Pick a word $w_n$ in $\mathcal{U}_n$ that is in no previous $\mathcal{U}_m$.
  Let $\mathcal{R}_n = [U_{n2} \ldots U_{nm} \ldots]$ be « $\mathcal{U}_n$ without its first open $U_{n1}$».

  By definition, $w_n = l_n\ a_n\ r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  By zoom-in, extract $(a_{n(k)})_k$ s.t. $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$ for every $k$.
  By minimality, $\mathcal{U}_0$, $\mathcal{U}_1$, ..., $\mathcal{U}_{n(0)-1}$, $\mathcal{R}_{n(0)}$, $\mathcal{R}_{n(1)}$, ... is **good**. So, for some $k$:

$$\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)-1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k-1)}.$$

# Proving the topological Higman Lemma

- **Thm** (Topological Higman Lemma, JGL 2013). If $X$ is Noetherian, then so is $X^*$ with the **subword** topology.

- *Proof* (2/3). Recall: $\mathcal{U}_n = [U_{n1}\ U_{n2} \ldots U_{nm} \ldots]$, $w_n$ in $\mathcal{U}_n$, in no previous $\mathcal{U}_m$.
  $\mathcal{R}_n = [U_{n2} \ldots U_{nm} \ldots]$
  $w_n = l_n\ a_n\ r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$
  $\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)\text{-}1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k\text{-}1)}$.

- Note $r_{n(k)} \in \mathcal{R}_{n(k)}$.

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  $\mathcal{R}_n = [U_{n2} \ldots\ U_{nm} \ldots]$
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  $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$
  $\underline{\mathcal{R}_{n(k)}} \subseteq \underline{\mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)\text{-}1}} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k\text{-}1)}$.

- Note $r_{n(k)} \in \mathcal{R}_{n(k)}$. **Case 1:** $r_{n(k)} \in \mathcal{U}_m$ for some $m$ in 0, ..., $n(0)$-1.
  Then the larger $w_{n(k)}$ is in $\mathcal{U}_m$, too. (Opens are upwards-closed.)
  Impossible since $\mathcal{U}_m$ is previous ($m$<$n(k)$).

# Proving the topological Higman Lemma

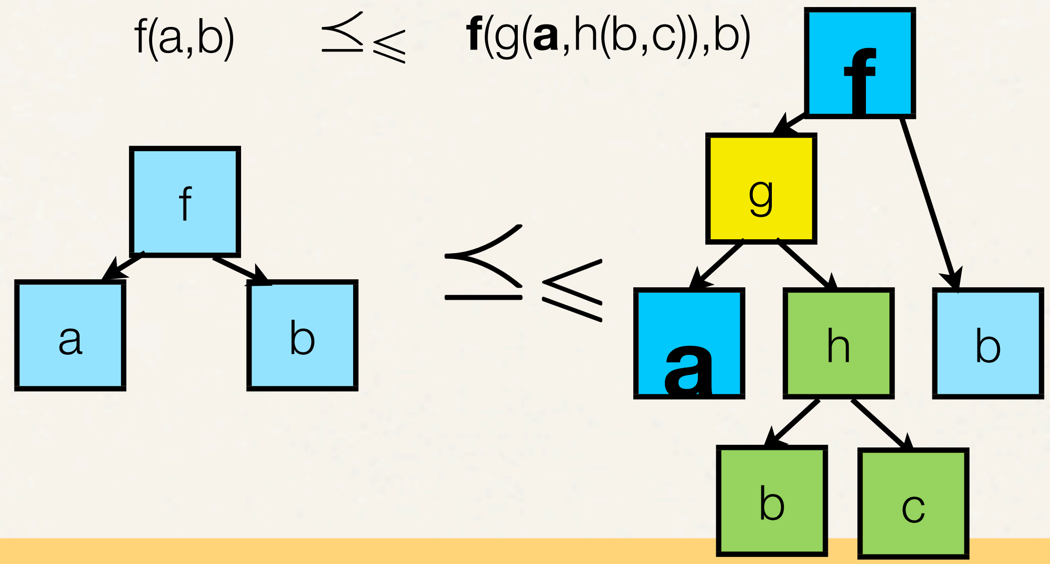
- **Thm** (Topological Higman Lemma, JGL 2013). If $X$ is Noetherian, then so is $X^*$ with the **subword** topology.

- *Proof* (3/3). Recall: $\mathcal{U}_n = [U_{n1}\ U_{n2} \ldots U_{nm} \ldots]$, $w_n$ in $\mathcal{U}_n$, in no previous $\mathcal{U}_m$.
  $\mathcal{R}_n = [U_{n2} \ldots U_{nm} \ldots]$
  $w_n = l_n\ a_n\ r_n$ where $a_n \in U_{n1}$, $r_n \in \mathcal{R}_n$.
  $a_{n(k)} \in U_{n(0)1} \cap \ldots \cap U_{n(k)1}$
  $\mathcal{R}_{n(k)} \subseteq \mathcal{U}_0 \cup \ldots \cup \mathcal{U}_{n(0)\text{-}1} \cup \mathcal{R}_{n(0)} \cup \ldots \cup \mathcal{R}_{n(k\text{-}1)}$.

- Note $r_{n(k)} \in \mathcal{R}_{n(k)}$. **Case 2:** $r_{n(k)} \in \mathcal{R}_{n(j)}$ for some $j$ in 0, ..., $k$-1.
  Note that $a_{n(k)} \in U_{n(j)1}$.
  Hence $w_{n(k)} = l_{n(k)}\ a_{n(k)}\ r_{n(k)}$ is in $\mathcal{U}_{n(j)}$, too.
  Impossible since $\mathcal{U}_{n(j)}$ is previous ($j<k$). $\square$

# Kruskal's Theorem

f(a,b) $\preceq_{\leqslant}$ **f**(g(**a**,h(b,c)),b)




assuming
f $\leqslant$ **f**
a $\leqslant$ **a**

- **Thm** (Kruskal 1960). If $X, \leq$ is wqo, then so is the qoset $\mathcal{T}(X)$ of finite trees labeled by $X$ under **homeomorphic embedding** relation $\preceq_{\leq}$.
- **Thm** (Topological Kruskal Theorem, JGL 2013). If $X$ is Noetherian, then so is $\mathcal{T}(X)$ with the **tree** topology.
- Admitted. Slightly more complex.

# Powersets

- Let $\mathbb{P}(X)$ come with the lower Vietoris topology, with subbase $\Diamond U = \{A \mid A \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.
- **Thm** (JGL, 2007). If $X$ is Noetherian, then so is $\mathbb{P}(X)$.
- *Proof.* If $\mathbb{P}(X)$ not Noetherian, let $(\Diamond U_n)_{n \in \mathbb{N}}$ be a bad sequence:

  no $\Diamond U_n$ is included in $\bigcup_{m<n} \Diamond U_m = \Diamond \bigcup_{m<n} U_m$.

  Since $\Diamond$ is monotonic, no $U_n$ is included in $\bigcup_{m<n} U_m$.
  Therefore $(U_n)_{n \in \mathbb{N}}$ is bad: contradiction. $\square$
- Specialization qo: $A \leq^\flat B$ iff every $a \in A$ is below some $b \in B$.

# Powersets, or: beyond wqos

- Let $\mathbb{P}(X)$ come with the lower Vietoris topology, with subbase $\Diamond U = \{A \mid A \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.

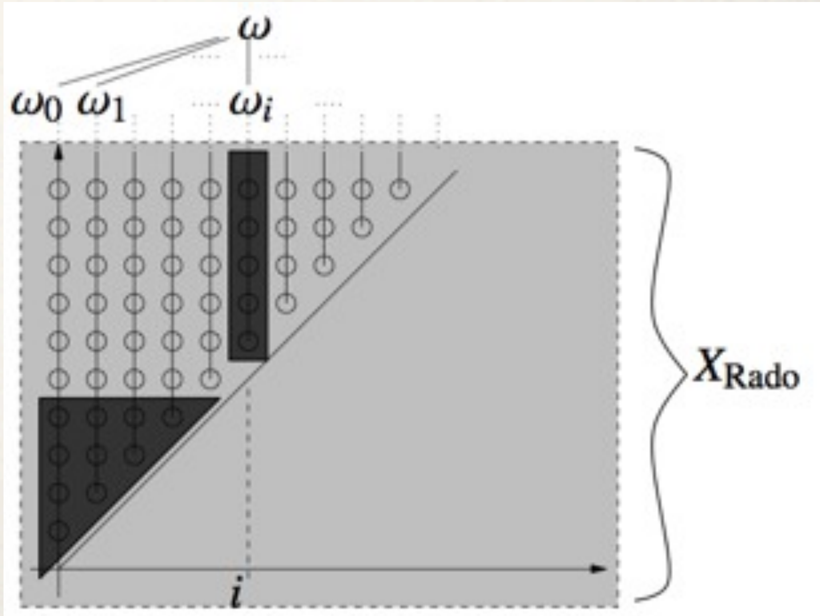
- **Thm** (JGL, 2007). If $X$ is Noetherian, then so is $\mathbb{P}(X)$.

- Specialization qo: $A \leq^\flat B$ iff every $a \in A$ is below some $b \in B$.

  Pretty remarkable, since:

- **Prop** (Rado, 1957). There are wqos $X, \leq$ such that $\mathbb{P}(X), \leq^\flat$ is **not** wqo.

  $X_{\text{Rado}} = \{(i,j) \mid i<j\}$, $(i,j) \sqsubseteq (k,l)$ iff $i=k$ and $j \leq l$ or $j<k$.



# A catalogue of Noetherian spaces

| | | | |
|---|---|---|---|
| $D$ | $::=$ | $A$ | finite poset |
| | $\mid$ | $\mathbb{N}$ | natural numbers |
| | $\mid$ | $D_1 \times D_2 \times \ldots \times D_n$ | products |
| | $\mid$ | $D_1 + D_2 + \ldots + D_n$ | sums |
| | $\mid$ | $\mathcal{S}(D)$ | sobrification |
| | $\mid$ | $\mathbb{P}(D)$ | powerset |
| | $\mid$ | $\mathbb{P}^*(D)$ | non-empty powerset |
| | $\mid$ | $\mathcal{H}(D)$ | extended Hoare powerspace |
| | $\mid$ | $\mathcal{H}_\emptyset(D)$ | Hoare powerspace |
| | $\mid$ | $\mathrm{Spec}(R)$ | spectrum of a ring |
| | $\mid$ | $\mathbb{C}^k$ | complex vector space (Zariski) |
| | $\mid$ | $D^*$ | words / embedding |
| | $\mid$ | $D^{\circledast}$ | multisets |
| | $\mid$ | $\triangleright_{n=1}^{+\infty} D_n$ | words / prefix |
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Noetherian,
not wqo

# Outline

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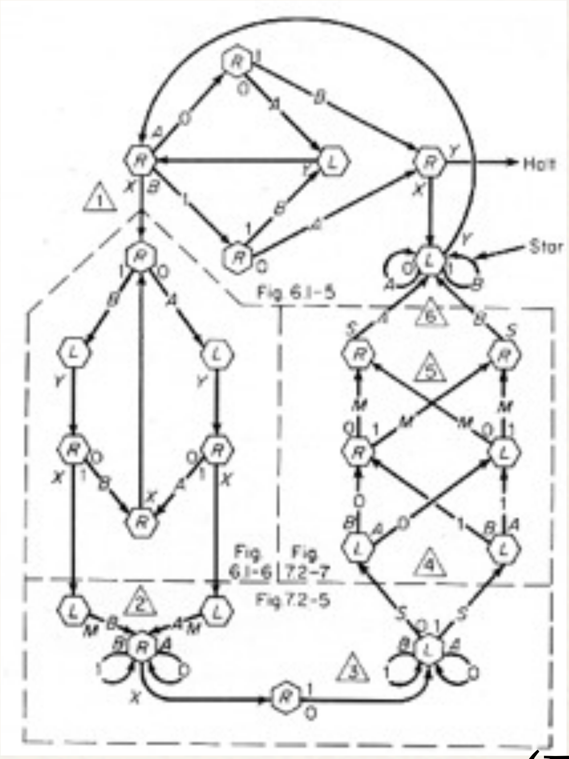
# Verification

- How do you ensure a software/hardware system **Sys** is **correct**?
- **Testing**: fine and useful, but not exhaustive
- **Verification**: given a desirable property **P**, check that

  **Sys** ⊧ **P**

- That check should be done by an **algorithm**.

**Sys**



$\models \Box\,(E \Rightarrow \Diamond F)$

**P**

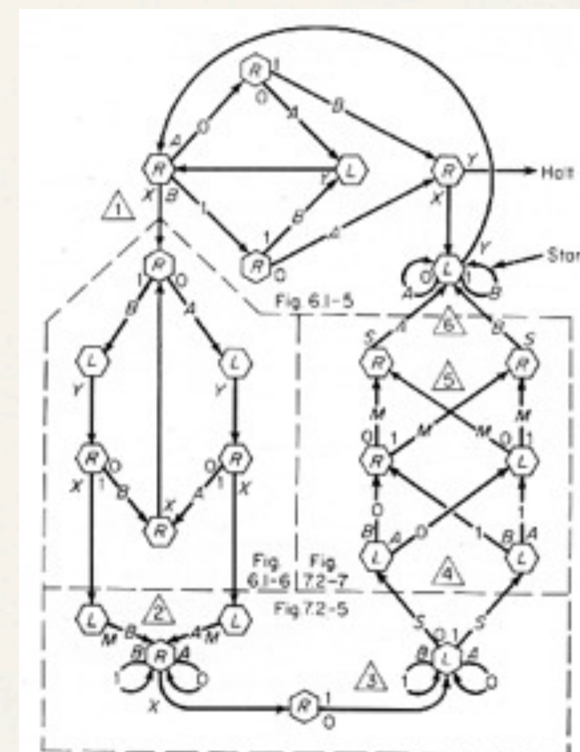
# Verification

- A paradigmatic case is given by:
  — **Sys** is a transition system
  (a directed graph, vertices=states)
  — **P** is a (non-)reachability property
  «can **Sys** evolve from an initial state *s*
  to a state in the set **Bad**?»

- Verification is **undecidable** in general.
  Decidable for **Sys** finite.
  But most systems are infinite.

- Classes of infinite-state systems/properties
  for which verification would be decidable?

**Sys**




$\models \neg\Diamond$**Bad**

**Bad**=

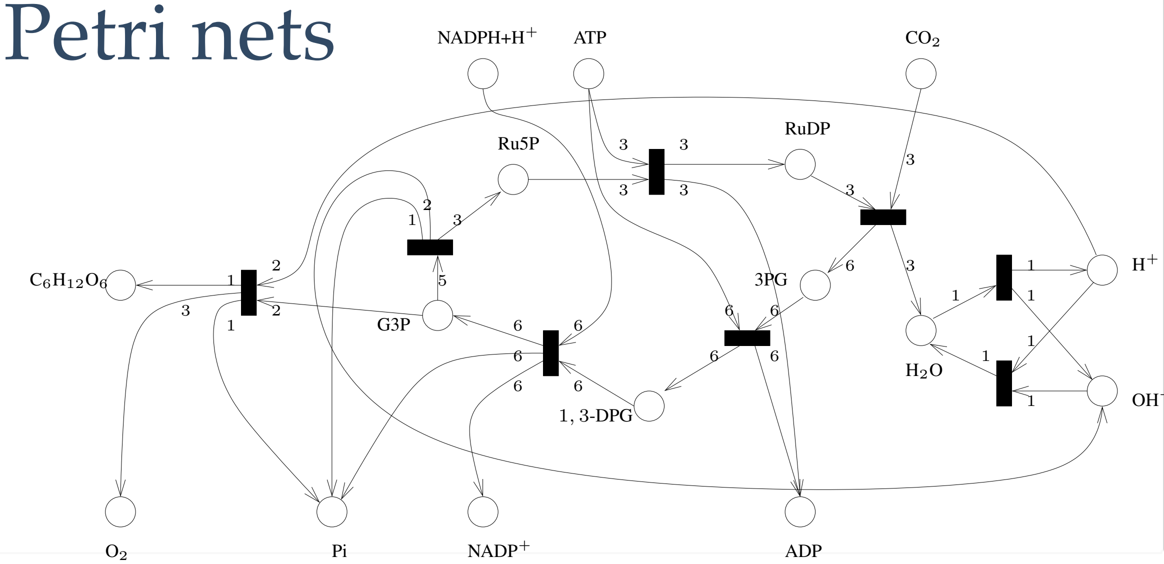
# Well-Structured Transition Systems (WSTS)

- A transition system: state space *X*, transition relation $\xrightarrow{\delta}$
- with a **wqo** ≤
- satisfying **monotonicity**

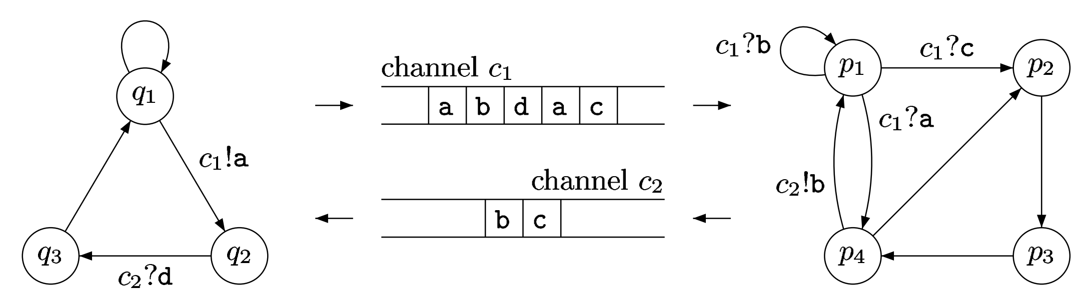
$$\begin{tikzcd} x \arrow[r, "\leq"] \arrow[d, "\delta"'] & x' \arrow[d, dotted, "\delta"] \\ y \arrow[r, dotted, "\leq"] & y' \end{tikzcd}$$

(Finkel 1990,
Abdulla,Čerāns,
Jonsson&Tsay 2000,
Finkel&Schnoebelen 2001)

## Petri nets



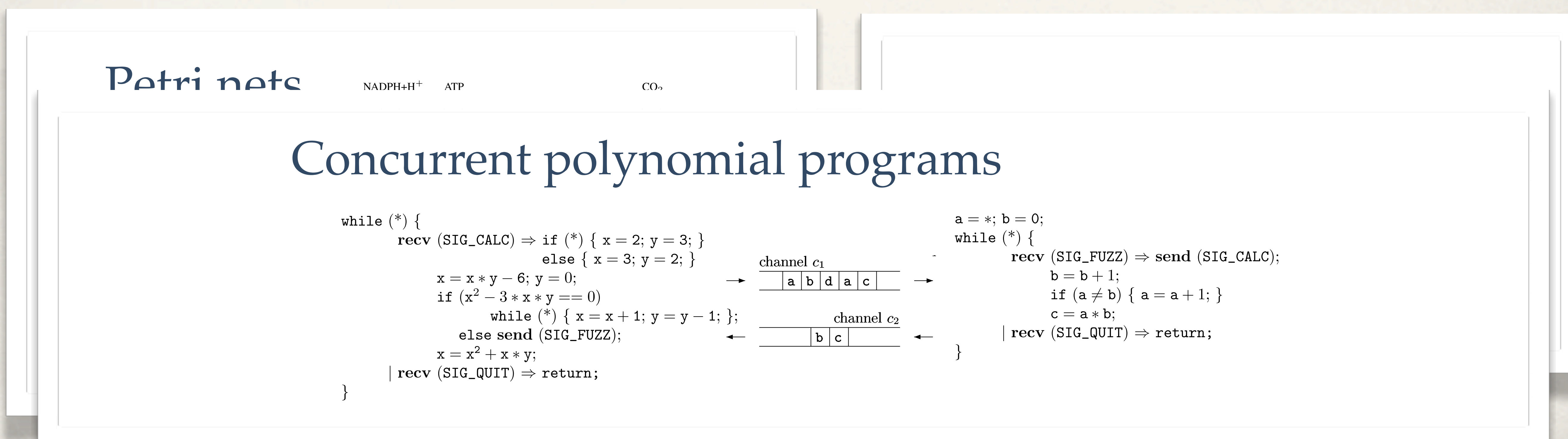

## Lossy channel systems




... and many other examples

# Topological WSTS

- A transition system: state space *X*, transition relation $\xrightarrow{\delta}$
- with a **Noetherian** topology
- satisfying **lower semi-continuity**: for every open *U*, $\delta^{-1}(U)$ open.



# The standard backward algorithm

- **Defn** (coverability).
  **INPUT**: state $s$, and open state set **Bad**
  **QUESTION:** $s \rightarrow_\delta^* \mathbf{Bad}$?

- **Prop.** Given an effective topological WSTS, coverability is **decidable**.

- *Proof*. The function `pre*` computes
  $$U_0 = \mathbf{Bad},\ U_{n+1} = U_n \cup \delta^{-1}(U_n).$$
  This **terminates** because $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n \subseteq ..$ stabilizes (Noetherianness).
  At the end, $U_n = \{s \mid s \rightarrow_\delta^* \mathbf{Bad}\}$. □

```
fun pre* U =
    let V = pre U
    in
       if V⊆U
          then U
       else pre* (U ∪ V)
    end;

fun coverability (s, bad) =
    s in pre* (bad);
```

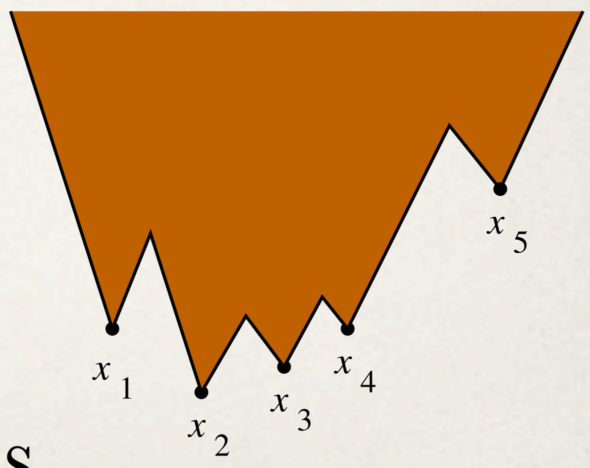
(Don't be fooled by the simplicity of the algorithm: complexity is not even primitive recursive in general
—and I mean the complexity of the problem, independently of the algorithm.)

# Effective?

- By an **effective** topological WSTS, we mean one where:
  — opens $U$ are **representable** by some data structure
  — the inclusion test $U \subseteq V$ is **decidable**
  — one-step predecessors $\delta^{-1}(U)$ of open sets are **computable**.

- When the Noetherian state space is Alexandroff (a wqo), there is a standard representation of open (upwards-closed) sets:

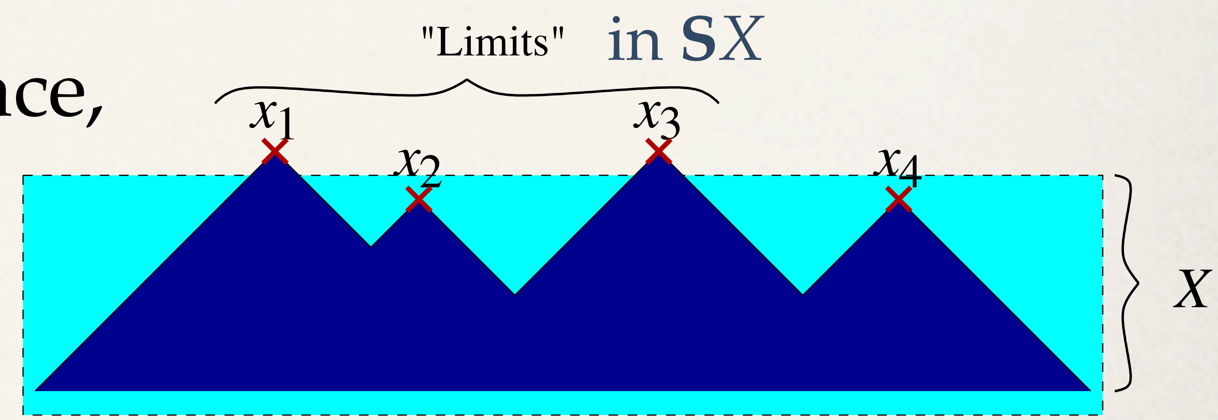
- **Prop.** In a wqo, every upwards-closed subset is the upward closure $\uparrow\{x_1, \ldots, x_n\}$ of finitely many points.




- No longer true in more general Noetherian spaces.

# Sobrification and closed sets

- Recall that, in a sober Noetherian space, every closed set $C$ is **finitary**:
$$C = \downarrow\{x_1, \ldots, x_n\}.$$




- Provides alternate representation of open sets $U$ in Noetherian $X$:
— $U$ is also open in larger space **S**$X$
— represent $U$ by its closed complement in **S**$X$,
— i.e., by finite sets of **points** in **S**$X$.

- ➩ Find computable **representations** of **points** of **S**$X$.

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# A simple case

- Consider $\mathbb{N}$ with the (Alexandroff topology of) its ordering $\leq$.
- Its closed subsets are $\emptyset$, $[n]=\{0,1,\ldots,n\}$ and the whole of $\mathbb{N}$.
- All except $\emptyset$ are irreducible. Hence:
- **Prop.** A representation for $\mathbf{S}\mathbb{N}$ is $\mathbb{N}_\omega$, i.e., $\mathbb{N}$ plus a top element $\omega$.
- This is an effective representation.
- I'm not giving the topology on $\mathbb{N}_\omega$:
  this must be the upper topology of its ordering.

# Products

- Another simple case.
  We know that $\mathbf{S}(\prod_i X_i) = \prod_i \mathbf{S}X_i$, up to iso (R.-E. Hoffmann 1979).
  Hence:

- **Prop.** A representation of $\mathbf{S}(X_1 \times \ldots \times X_n)$ is
  the Cartesian product of representations for $\mathbf{S}X_i$.

- This is effective,
  provided the representations for $\mathbf{S}X_i$ are.

# Words and regular expressions

- **Thm** (Finkel&JGL 2009). A representation for $\mathbf{S}(X^*)$ is the space of **word products**,
  i.e., **regular expressions** of the form:
  $$R_1\ R_2\ \ldots\ R_n$$
  where each $R_i$ is of the form:
  — $(\downarrow a)^?$ with $a \in \mathbf{S}(X)$, or
  — $(\downarrow\{a_1, \ldots, a_k\})^*$ with $a_1, \ldots, a_k \in \mathbf{S}(X)$

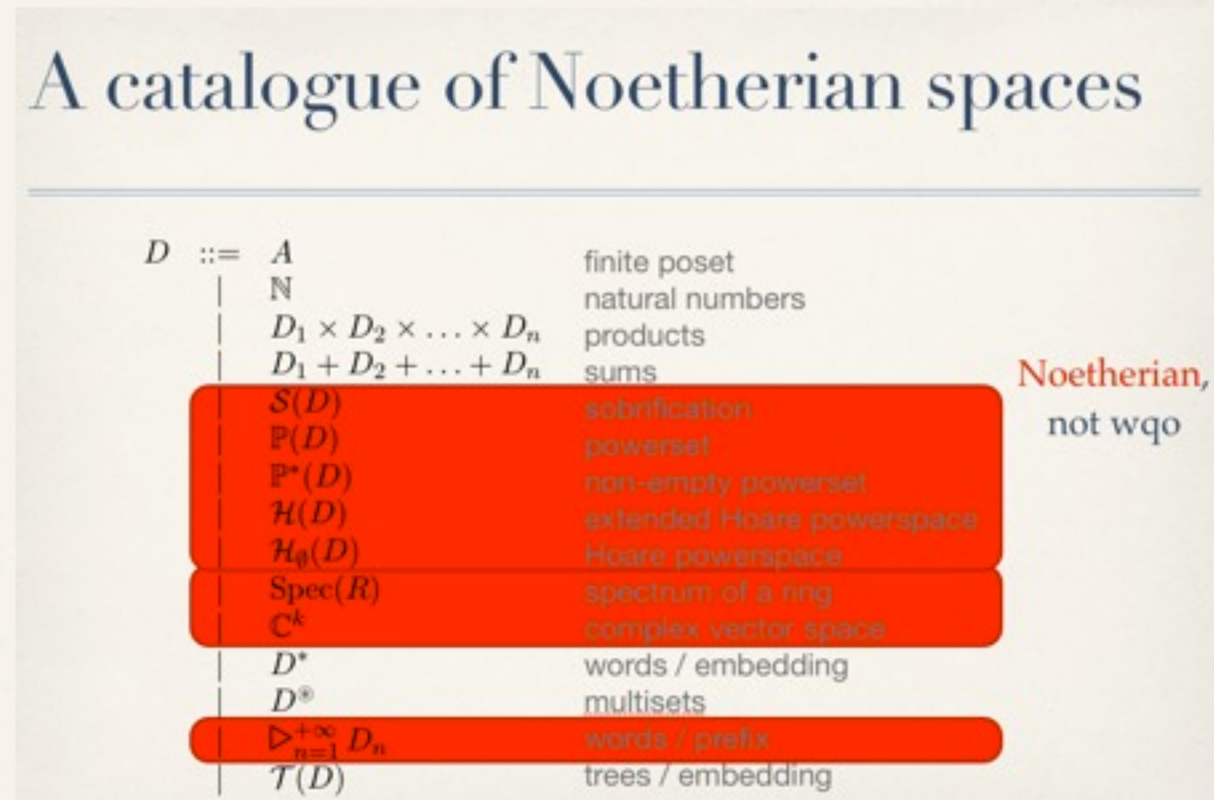
- Proof omitted. Again, effectivity is preserved.
  Was already known for wqos (Kabil&Pouzet 1992).

- Embedding $X^* \to \mathbf{S}(X^*)$ maps $a_1a_2\ldots a_n$ to $(\downarrow a_1)^?\ (\downarrow a_2)^?\ \ldots\ (\downarrow a_n)^?$
  Limit elements include $(\downarrow a_1)^?\ (\downarrow\{a_2, a_3\})^*\ (\downarrow a_1)^?$, for example.

# Representations

- **Thm** (Finkel&JGL, unpublished).
  For **all** the spaces $X$ in our catalogue of Noetherian spaces,
  $\mathbf{S}X$ has effective representations.

- …



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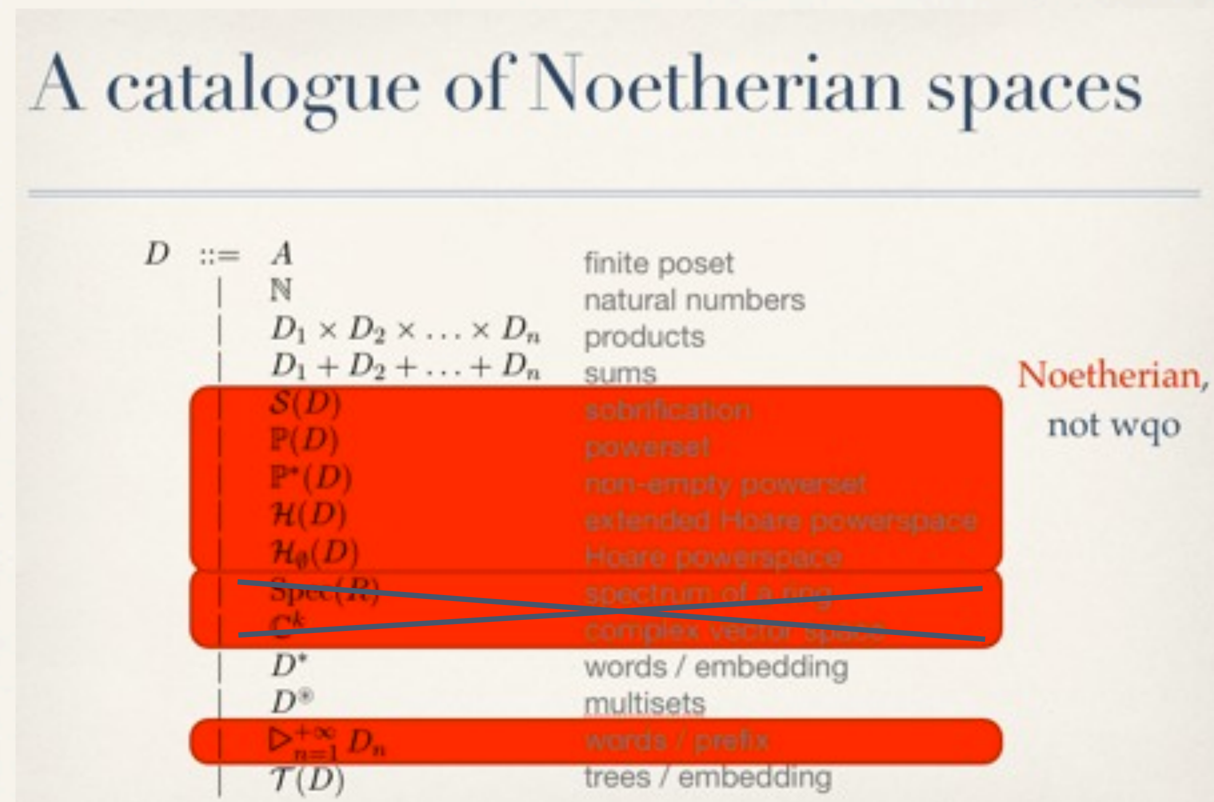

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- **Thm** (Finkel&JGL, unpublished).
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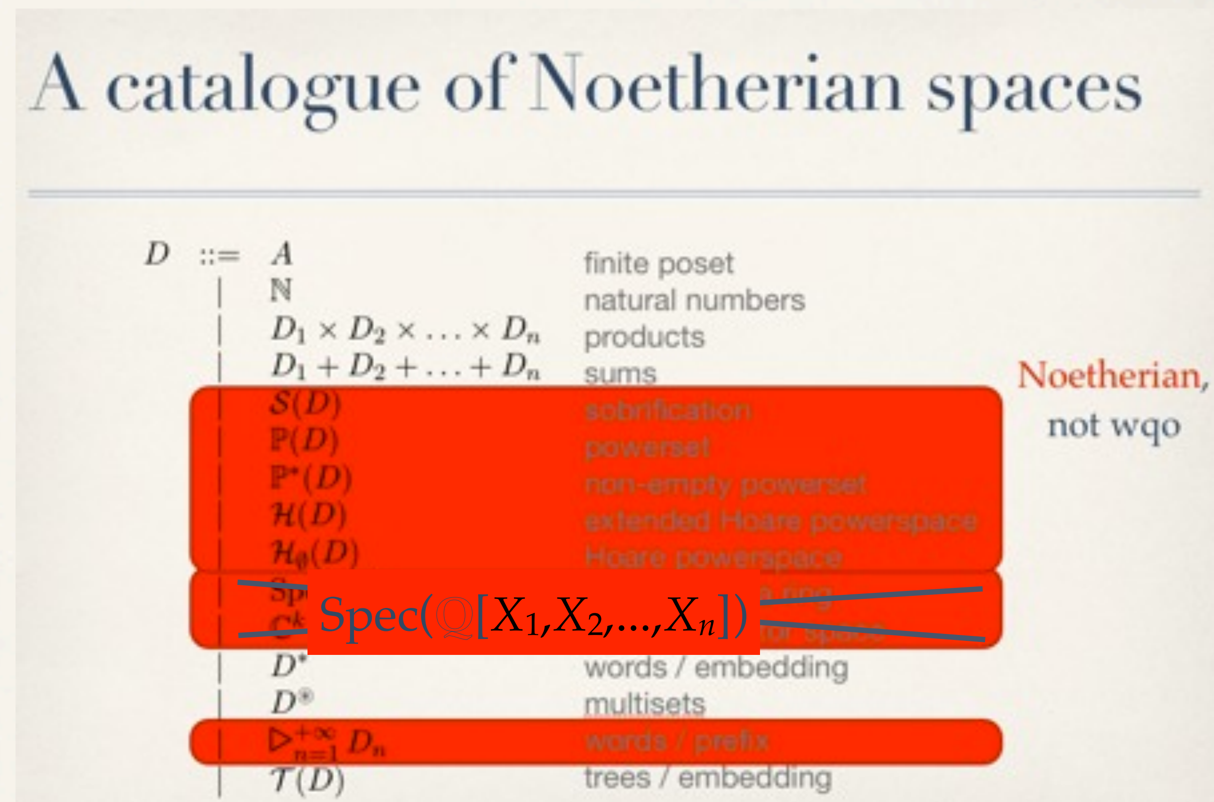
- ... up to a small change: replace Spec($R$) and $\mathbb{C}^k$ (Zariski) with some concrete spectrum, say, Spec($\mathbb{Q}[X_1,X_2,\ldots,X_n]$).



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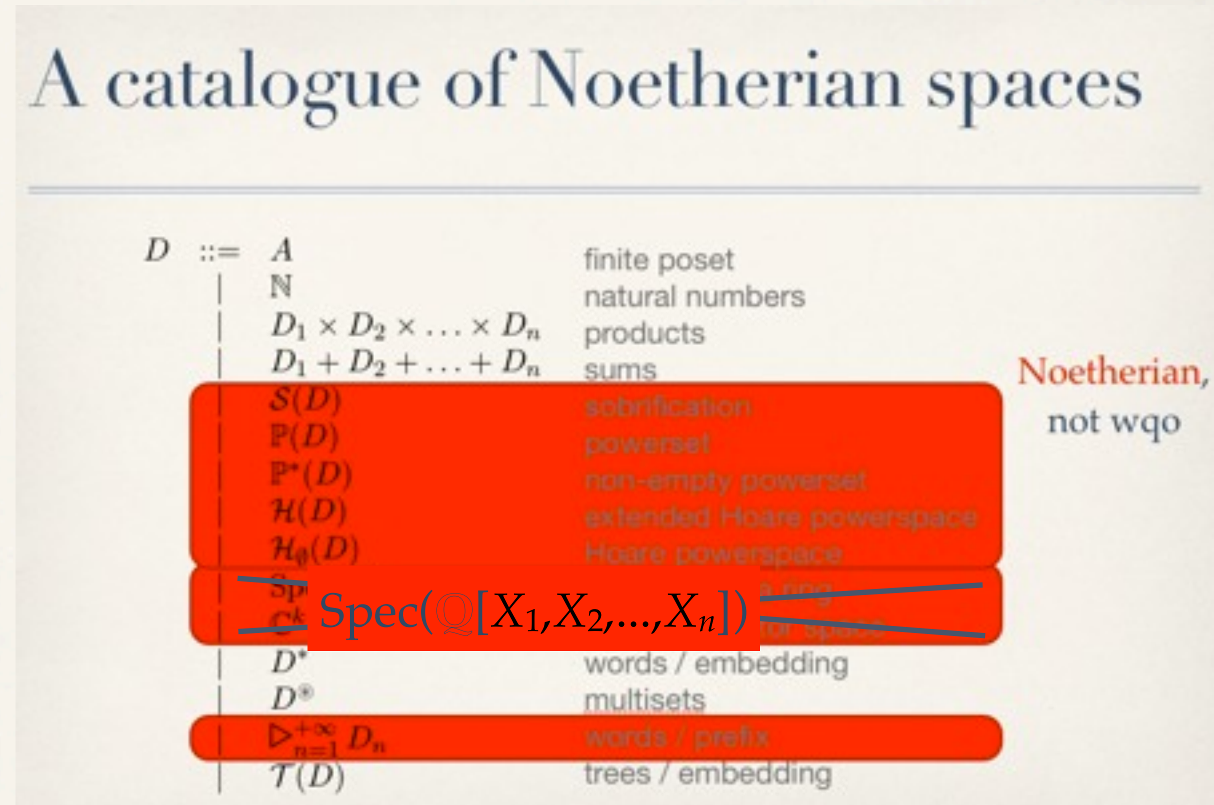
- Including (infinite) powersets!



# Representing powersets?

- Recall $\mathbb{P}(X)$ has subbase $\Diamond U = \{A \mid A \cap U \neq \emptyset\}$, $U \in \mathbf{O}X$.
  Let $\mathbb{F}(Y)$ be the set of **finite** subsets of $Y$, with subspace topology.

- **Prop.** $\mathbf{S}(\mathbb{P}(X)) = \mathbb{F}(\mathbf{S}X)$, up to iso.
  Hence a representation for $\mathbf{S}(\mathbb{P}(X))$ is given by finite sets of elements from a representation of $\mathbf{S}X$.

- *Proof*. Let $\mathbf{H}(X)$ be subspace of closed subsets of X.
  $\mathrm{cl} : \mathbb{P}(X) \to \mathbf{H}(X)$ is a quasi-iso, i.e. $\mathbf{O}(\mathrm{cl}) : \mathbf{OH}(X) \to \mathbf{O}\mathbb{P}(X)$ is iso.
  (Exercise: use $A \cap U \neq \emptyset$ iff $\mathrm{cl}(A) \cap U \neq \emptyset$.)
  Hence $\mathbf{S}(\mathbb{P}(X)) = \mathbf{S}(\mathbf{H}(X))$.
  Since $\mathbf{O}X = \mathbf{OS}X$, $\mathbf{H}(X) = \mathbf{H}(\mathbf{S}(X))$.
  Now recall that all elements of $\mathbf{H}(\mathbf{S}(X))$ are finitary. $\square$

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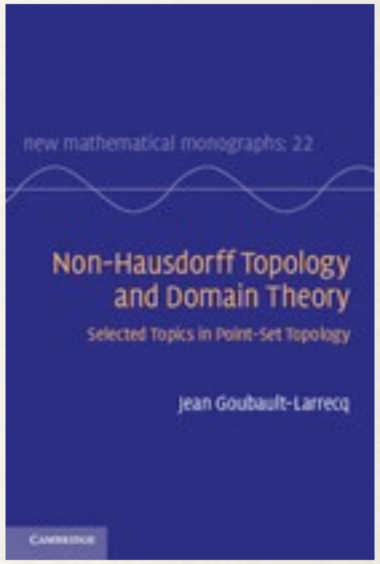
# Conclusion

- There is more to Noetherian spaces than algebraic geometry.
  A deep connection with **wqos**.
  Basic theory in (JGL 2013), Section 9.7.




- Any topological analogue of **better** quasi-orderings?
- Any topological analogue of the Robertson-Seymour theorem (for undirected finite graphs with labels in a Noetherian space)?
- Any topological analogue of the theory of **maximal order types** of wqos? (Hint: ordinal height of $\mathbf{H}(X)$, of $\mathbf{S}X$.) Application to complexity of WSTS algorithms (à la Schnoebelen, Schmitz, Halfon).