Jean Goubault-Larrecq

Randomized complexity classes

Today: **Sipser's coding lemmas**, and consequences

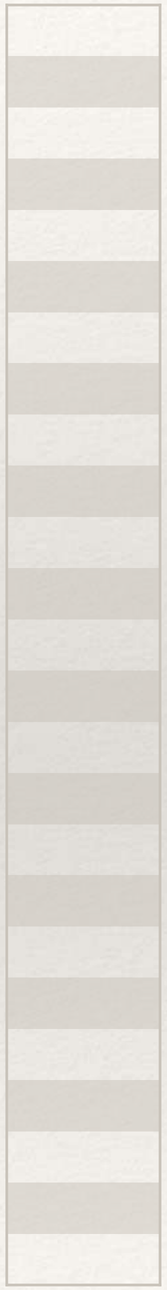
Today

- Sipser's coding lemmas
- **AM** is in the polynomial hierarchy
- The Goldwasser-Sipser theorem:
  public coins≡private coins
- The Boppana-Håstad-Zachos theorem:
  Graph Isomorphism is most certainly not **NP**-complete.

Sipser's coding lemmas

Hash tables

- Store elements of type σ (e.g. strings)
  In general, associative array $\sigma \rightarrow \tau$



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  Hash function $h : \sigma \to [0, N-1]$
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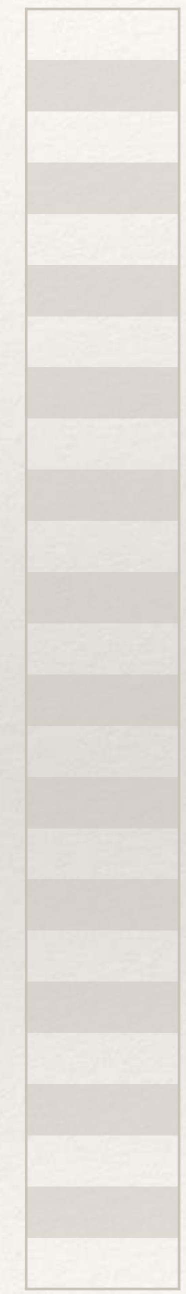

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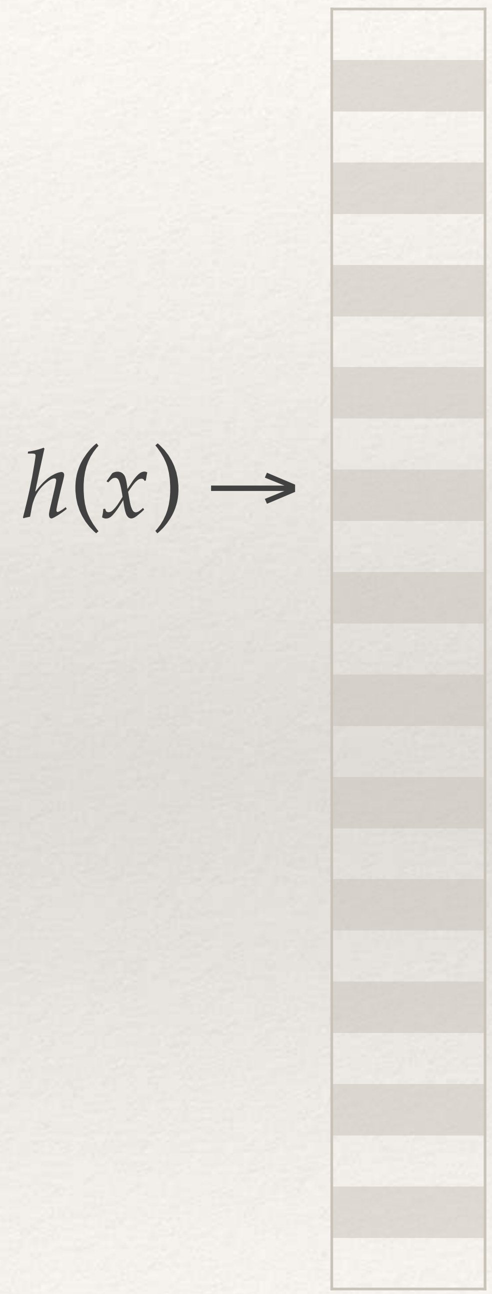

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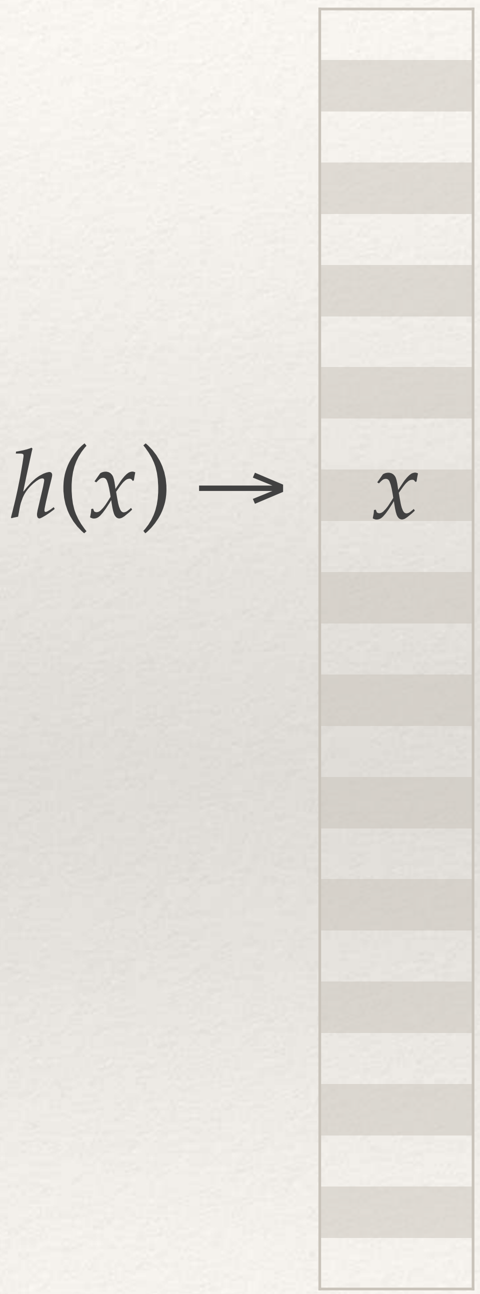

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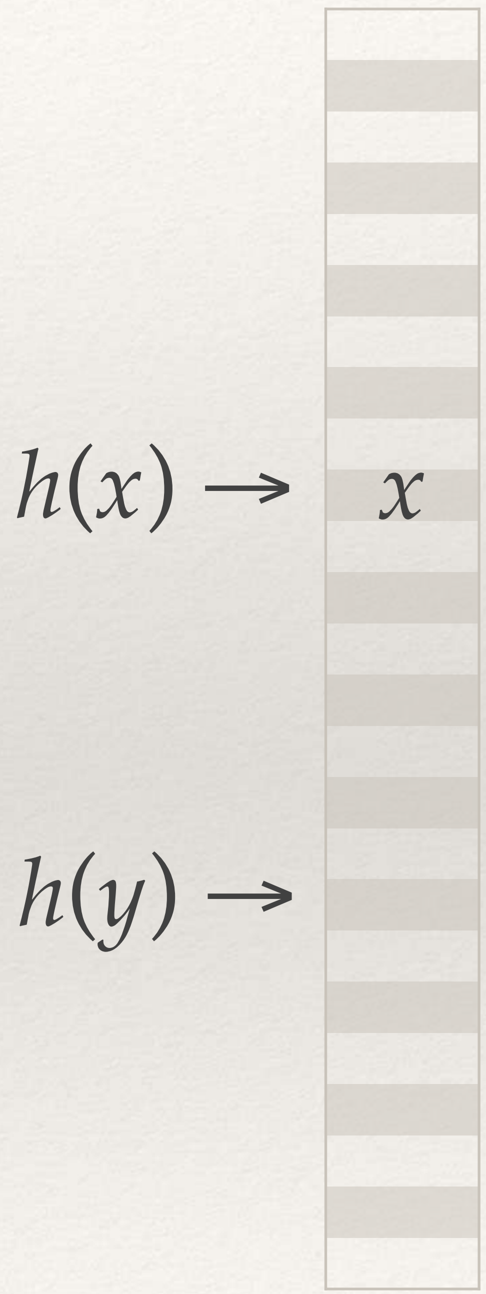

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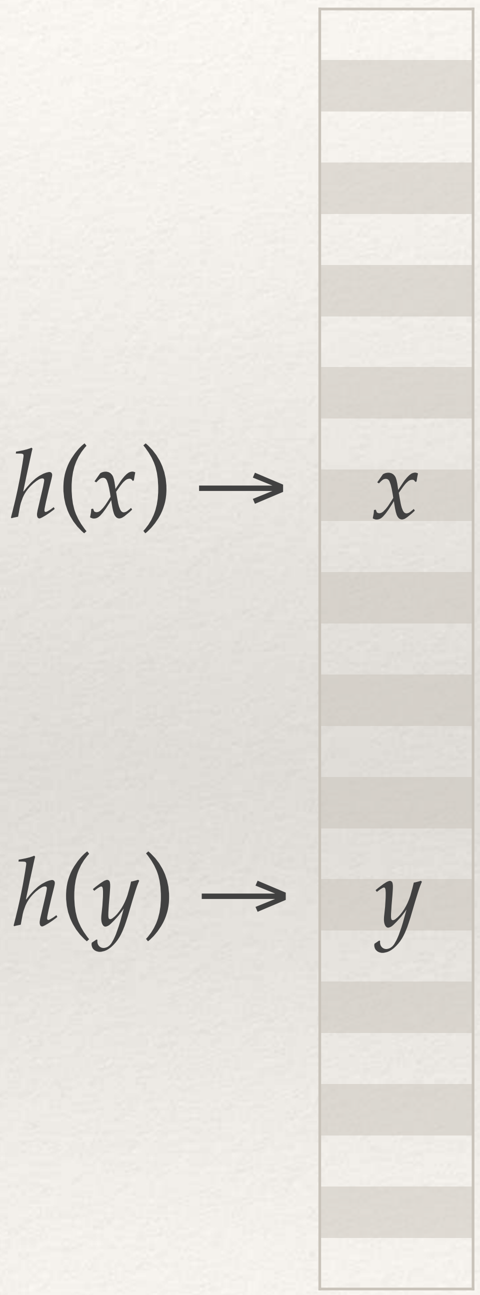

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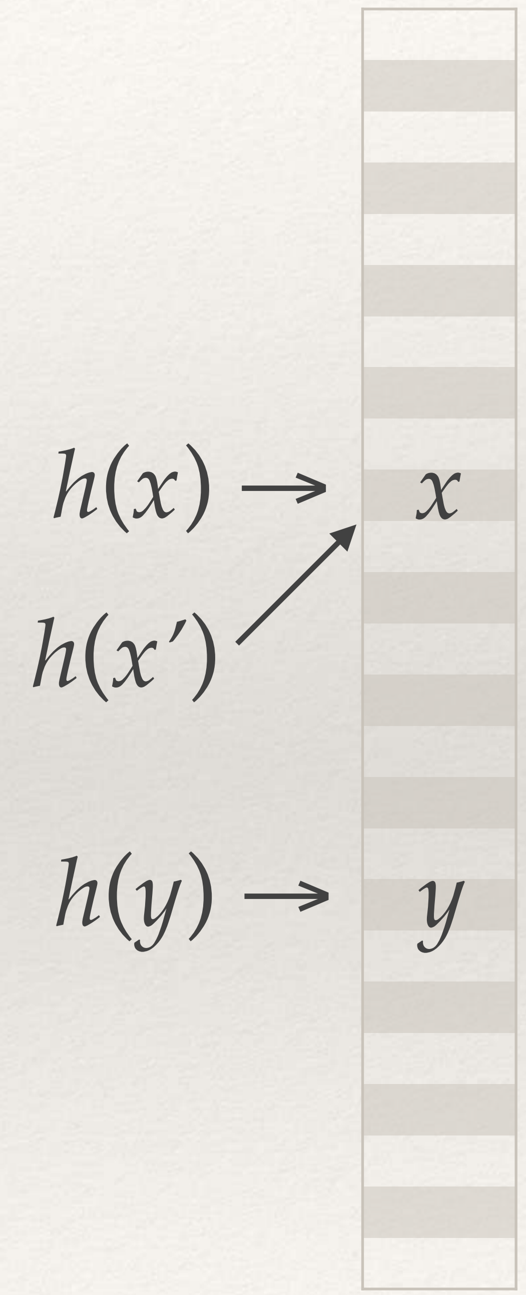
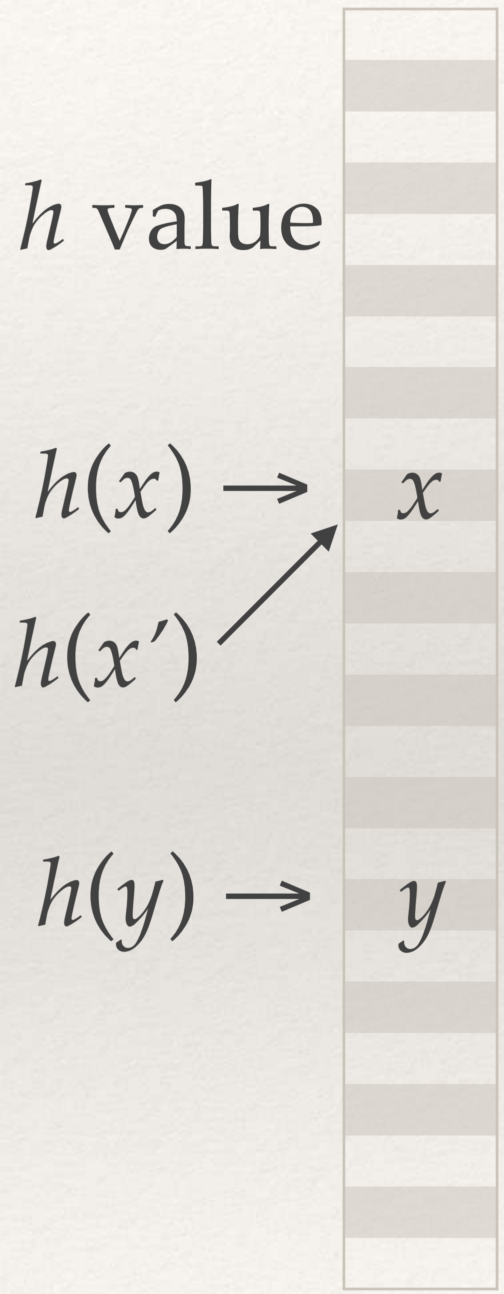
- **Collision:** element x of σ
  such that there is an element $x' \neq x$ of σ
  with $h(x)=h(x')$



Collisions

- In practice, one avoids collisions by:
  — storing **lists** of data x with the same h value instead of just elements
  — **resizing** the table (increasing N) in case of collisions
- But how can we ensure that N is large enough so that there are **no** collisions? How do we choose h?



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- In practice, one avoids collisions by:
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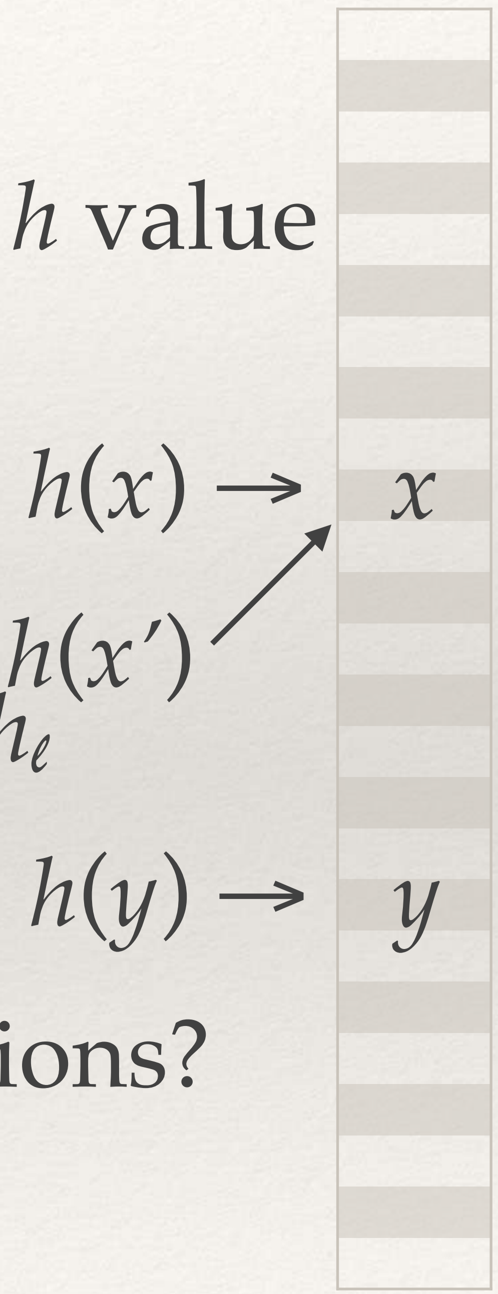
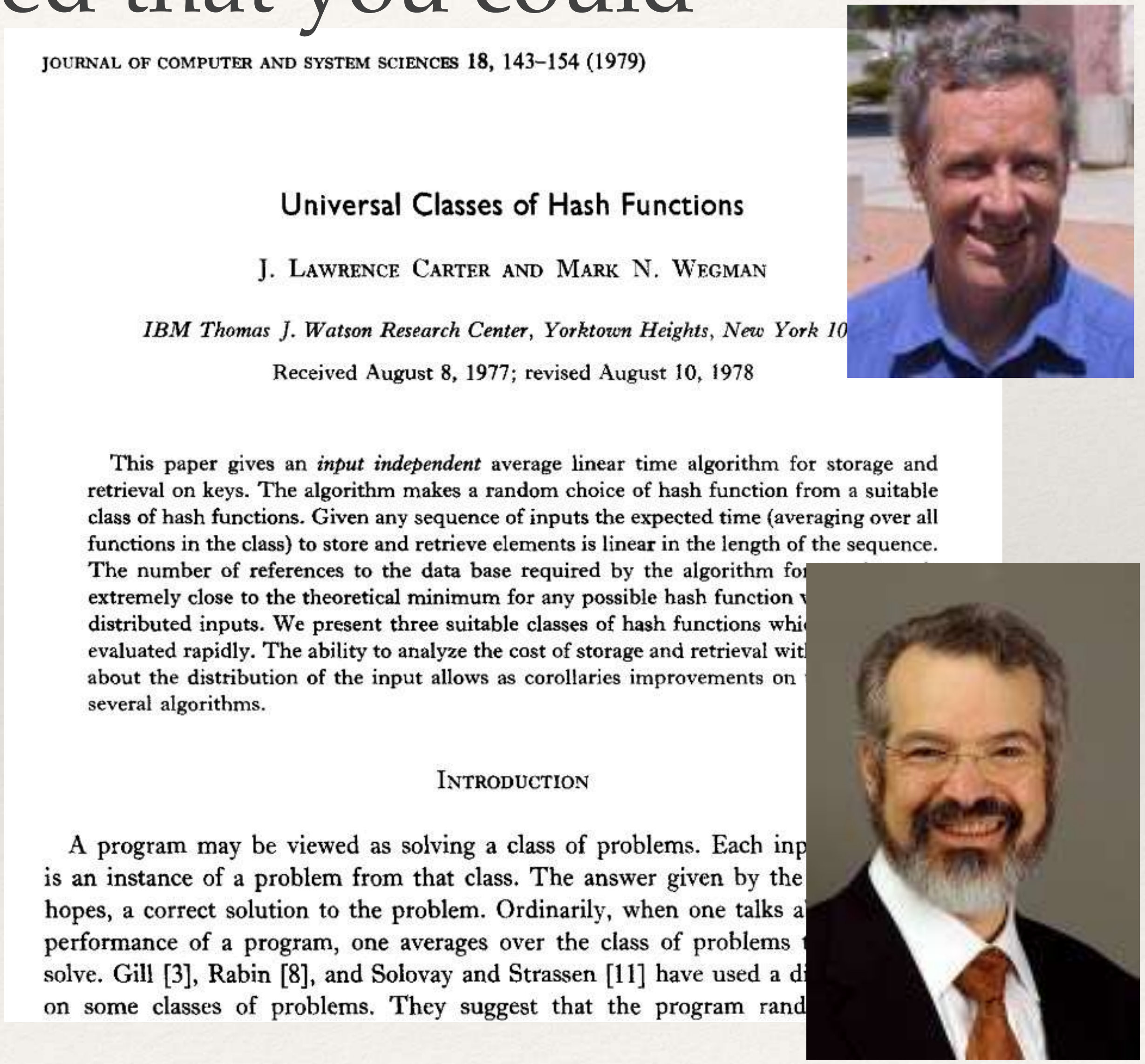
  — using **several** hash functions $h_1, \ldots, h_\ell$



- Still, how can we ensure that N is large enough so that there are **no** collisions?
  How do we choose $H \stackrel{\text{def}}{=} (h_1, \ldots, h_\ell)$?

Universal hash functions

- Carter and Wegman realized that you could **draw** $H \stackrel{\text{def}}{=} (h_1, \ldots, h_\ell)$ **at random** from certain so-called universal classes…


JOURNAL OF COMPUTER AND SYSTEM SCIENCES 18, 143–154 (1979)

Universal Classes of Hash Functions

J. LAWRENCE CARTER AND MARK N. WEGMAN

IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10

Received August 8, 1977; revised August 10, 1978

This paper gives an *input independent* average linear time algorithm for storage and retrieval on keys. The algorithm makes a random choice of hash function from a suitable class of hash functions. Given any sequence of inputs the expected time (averaging over all functions in the class) to store and retrieve elements is linear in the length of the sequence. The number of references to the data base required by the algorithm fo extremely close to the theoretical minimum for any possible hash function distributed inputs. We present three suitable classes of hash functions whi evaluated rapidly. The ability to analyze the cost of storage and retrieval wit about the distribution of the input allows as corollaries improvements on several algorithms.

INTRODUCTION

A program may be viewed as solving a class of problems. Each inp is an instance of a problem from that class. The answer given by the hopes, a correct solution to the problem. Ordinarily, when one talks a performance of a program, one averages over the class of problems solve. Gill [3], Rabin [8], and Solovay and Strassen [11] have used a d on some classes of problems. They suggest that the program rand

- and there are very simple such classes!

https://cse.ucsd.edu/sites/cse.ucsd.edu/files/faculty/carter17-115x150.jpg
https://www.ithistory.org/sites/default/files/honor-roll/Mark%20N.%20Wegman.jpg

Linear hash functions

- Let $\sum$={0,1} $= \mathbb{Z}/2\mathbb{Z}$
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- … i.e., h is given by a **matrix of bits** $B = (b_{ij})_{i=1..m',\, j=1..m}$:
$$h(x_1,\ldots,x_m) = (b_{i1}x_1 + \ldots + b_{im}x_m)_{i=1..m'}$$

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- For computer geeks, each row $(b_{i1},\ldots,b_{im})$ is a **mask**
  and $b_{i1}x_1 + \ldots + b_{im}x_m$ is a **parity check**
  = exclusive or of the bits x_j at those positions j / $b_{ij}=1$

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- **(Coding lemma I):** if X is sufficiently small, then drawing $H \stackrel{\text{def}}{=} (h_1, \ldots, h_\ell)$ at random, with **high probability** there will be no collision in X

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- **(Coding lemma II):** if X is too large, then whichever $H \stackrel{\text{def}}{=} (h_1, \ldots, h_\ell)$ you take, there will **definitely** be a collision in X.

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The definition of collisions

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- If such an x exists, we say that X **has a collision for** H.

Sipser's coding lemma I (X small)

- **Lemma (3.15).** Let $X \subseteq \Sigma^m$. Assume $|X| \leq 2^{m'-1}$. Let $\ell \geq m'$.
  Then $\Pr_H(X \text{ has a collision for } H) \leq 1/2^{\ell-m'+1}$.

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- Indeed, y has a non-zero coordinate y_i (hence $y_i=1$)
  Let $t \stackrel{\text{def}}{=} (0, \ldots, 0, 1, 0, \ldots, 0)$ with the only 1 at position i.
  Then $z \mapsto z \oplus t$ (flip bit i) is a **bijection**
  of $\{z \in \Sigma^m \mid z\,.\,y=0\}$ onto $\{z \in \Sigma^m \mid z\,.\,y=1\}$.

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- In particular, if $x \neq y_j$, then $\Pr_z(z \,.\, x=z \,.\, y_j) = 1/2$
  (take $y \stackrel{\text{def}}{=} x-y_j$).

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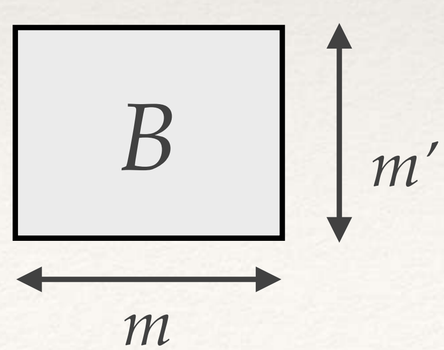
- Hence $\Pr_h(h(x)=h(y_j))$
  $= \Pr_{B\ (m' \times m)\ \text{matrix}}(B.x = B.y_j)$
  $= \Pr_{B\ (m' \times m)\ \text{matrix}}(\forall \text{row } z \text{ of } B,\ z \,.\, x = z \,.\, y_j) = 1/2^{m'}$



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  $= \Pr_H(\exists y_1, \ldots, y_\ell \in X-\{x\}, \bigwedge_{j=1}^{\ell} h_j(x)=h_j(y_j))$

- $\leq \Sigma_{y_1, \ldots, y_\ell \in X-\{x\}} \Pi_{j=1}^{\ell} \Pr_h(h(x)=h(y_j))$ (sum bound+independence)

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  $= \Pr_H(\exists y_1, \ldots, y_\ell \in X-\{x\}, \bigwedge_{j=1}^{\ell} h_j(x)=h_j(y_j))$
- $\leq \sum_{y_1, \ldots, y_\ell \in X-\{x\}} \prod_{j=1}^{\ell} \Pr_h(h(x)=h(y_j))$ (sum bound+independence)
- $\leq (|X|-1)^\ell / 2^{\ell m'} < 1/2^\ell$

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- $\leq |X|/2^\ell \leq 1/2^{\ell-m'+1}$. $\square$

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- **Lemma (3.16).** Let $X \subseteq \Sigma^m$. Assume $|X| > \ell . 2^{m'}$.
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- Hence card $X \leq \ell.\text{card } \Sigma^{m'} = \ell.2^{m'}$. □

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Only needs gap $\ell.2^{m'}/2^{m'-1} =$
$2\ell = \mathrm{poly}(n)$

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- We start by showing that for every language $L \in$ **AM**, we can require **perfect soundness** (=no error if $x \in L$).

$$\mathbf{AM} \subseteq \Pi^{p}_{2}$$

AM with perfect soundness (1/4)

- Let L be in **AM**. For some $D \in \mathbf{P}$,
— if $x \in L$ then $(\mathrm{E}r, \exists y, x\#r\#y \in D) \geq 1\text{–}1/2^n$ (« large »)
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Can we really do this in
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  Then $\Pr_H(X \text{ has a collision for } H) \leq 1/2^{\ell-m'+1}$.

  Lemma (3.16). Let $X \subseteq \Sigma^m$. Assume $|X| > \ell.2^{m'}$, where $\ell \geq m'$.
  Then X (definitely) has a collision for H.

- To apply Sipser I, need $|R| \leq 2^{m'-1}$ if $x \notin L \Rightarrow$ e.g., require $m'-1 \geq m-n$ (1)
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AM with perfect soundness (2/4)

- Let L be in **AM**. For some $D \in \mathbf{P}$,
  — if $x \in L$ then $(\mathrm{E}r, \exists y, x\#r\#y \in D) \geq 1-1/2^n$ (« large »)
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- $m=q(n)$ (assume $m \geq n$): determine m', ℓ
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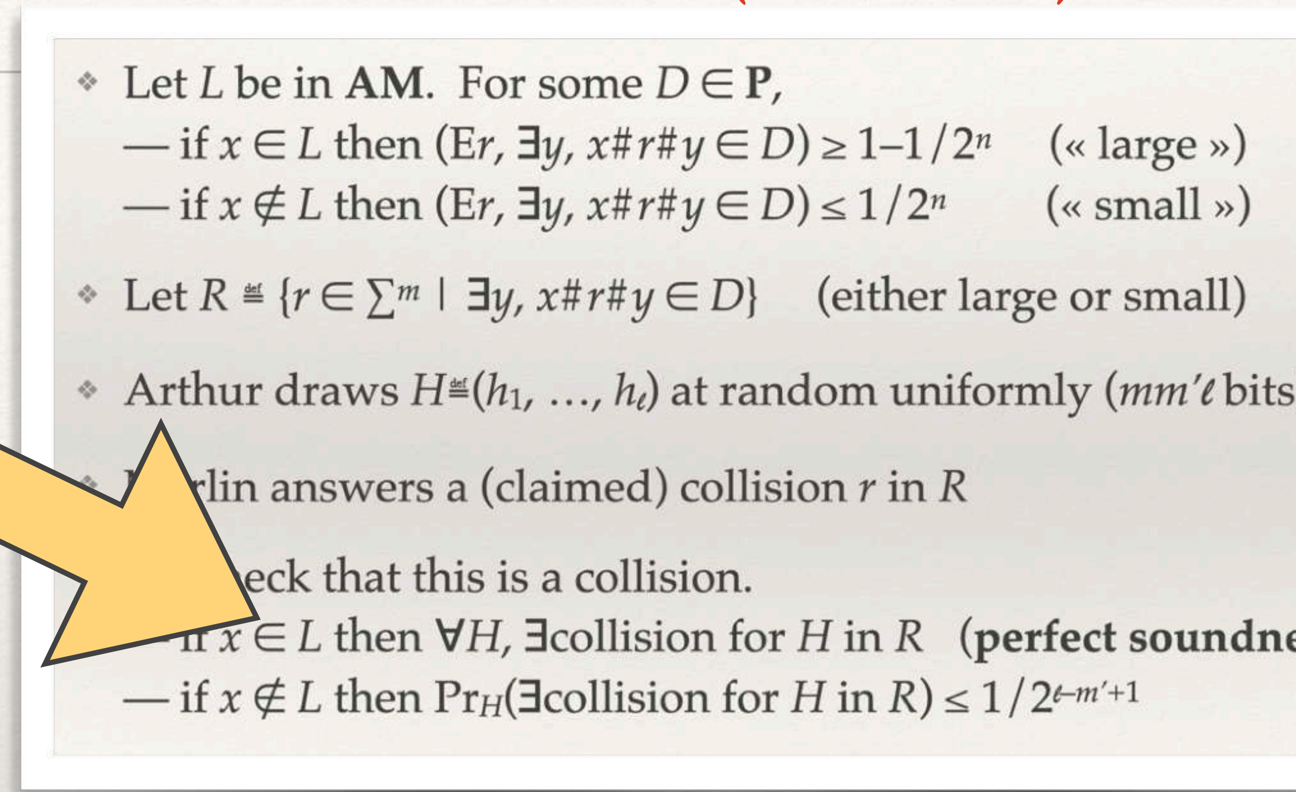
E.g., $m' \stackrel{\text{def}}{=} m-n+1$ (for (1)), $\ell \stackrel{\text{def}}{=} m'+g(n)$ (for (3), (4))
((2) OK for n large enough, otherwise tabulate)

AM with perfect soundness (3/4)

- Now m, m', ℓ=poly(n)
  Can we check that r is a collision in R in poly time?



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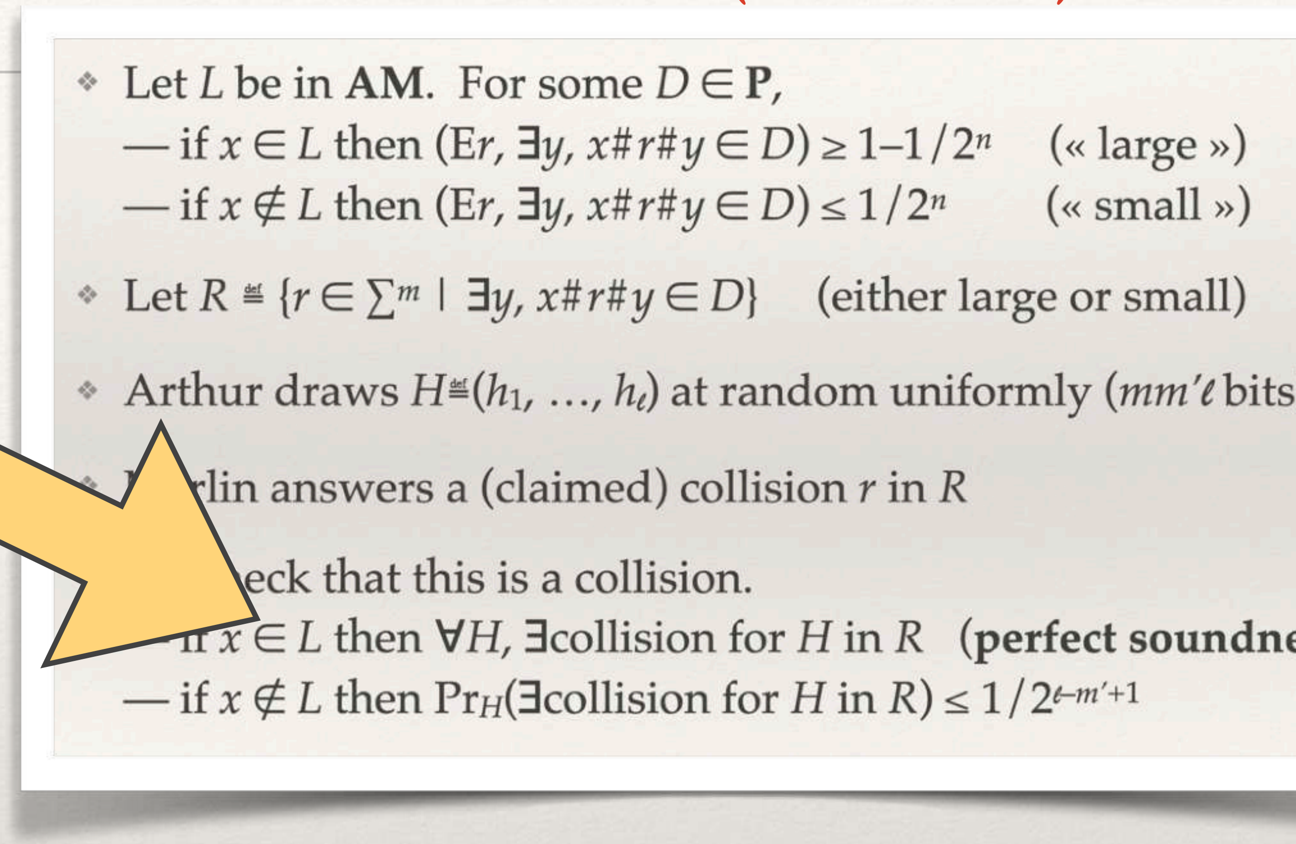

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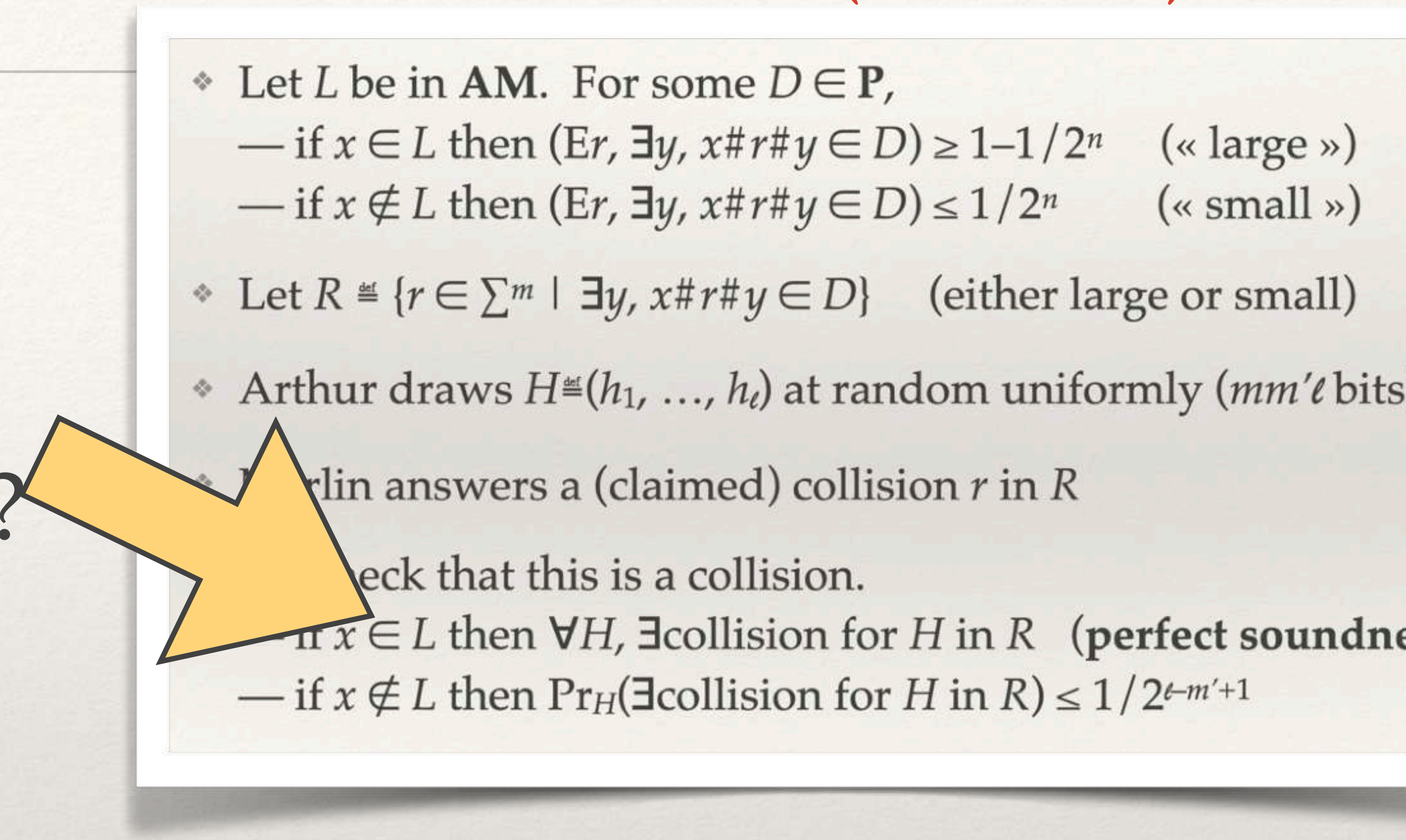
- Instead…



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in poly time!

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Prop (3.18). Every $L \in \mathbf{AM}$ can be decided with an **AM** game with perfect soundness (no error if $x \in L$)

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- Proof. $x \in L$ iff $\forall H$, $\exists$collision for H in R (with proofs!) and proofs of collisions can be checked in poly time. □

Graph Non-Isomorphism is in **AM**

Reminder: Graph Non-Isomorphism

- **GNI** $\overset{\text{def}}{=}$ complement of **GI**: in **coNP**,
  not known to be in **P** or **coNP**-complete

GI

INPUT: 2 graphs $G_1=(V, E_1)$, $G_2=(V, E_2)$ (with the same V)
$V = \{1, \ldots, N\}$
QUESTION: are G_1, G_2 isomorphic?

- **Prop. GNI is in IP**[1].
- *Algorithm.*
  — Arthur draws $i \in \{1,2\}$, $\pi \in \mathbf{S}_N$ at random uniformly,
  sends $q \overset{\text{def}}{=} \pi.G_i$
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Note: it is crucial here that i remains secret!
This is not an **AM** game

GNI is in AM

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  Let $X_i \stackrel{\text{def}}{=} \{\text{graphs } G \text{ on } V \text{ such that } G \equiv G_i\}$,
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QUESTION: are G_1, G_2 isomorphic?

The main problem is this:
$|X_i|$ can vary wildly,
— from $N!/2$ (if G_i is a chain)
— to 1 (if G_i is a complete graph)

GNI is in AM

- **Idea:** (that fails, but we will fix this later)
  Let $X_i \stackrel{\text{def}}{=} \{\text{graphs } G \text{ on } V \text{ such that } G \equiv G_i\}$,
  $X \stackrel{\text{def}}{=} X_1 \cup X_2$
- Imagine $|X_1| \approx |X_2| \approx K$
- If $(G_1, G_2) \in$ **GNI**,
  i.e. if $G_1 \not\equiv G_2$ then $|X| \approx 2K$ (X is large)
- Otherwise $|X| \approx K$ (X is small)
- We test which is the case using Sipser
  (All random bits in H are public $\Rightarrow$ in **AM**.)

GI

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The main problem is this:
$|X_i|$ can vary wildly,
— from $N!/2$ (if G_i is a chain)
— to 1 (if G_i is a complete graph)

Oops, gap is only 2:
not enough for Sipser, but we will see later how to increase it.

Building sets of uniform size

- X_i is the **orbit** of G_i under the group action of $\mathbf{S}_N$ on $\boldsymbol{G}_N$

GI

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- Let $V = \{1, \ldots, N\}$ set of vertices,
  $\boldsymbol{G}_N \stackrel{\text{def}}{=}$ directed graphs on V,
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- $\mathbf{S}_N$ acts on $\boldsymbol{G}_N$ by: $\forall \pi \in \mathbf{S}_N, \forall G=(V,E) \in \mathbf{S}_N$,
  $\pi.G \stackrel{\text{def}}{=} (V, \{(\pi(u), \pi(v)) \mid (u, v) \in E\}$
- Two graphs
  $G_1=(V, E_1)$, $G_2=(V, E_2)$ (with the same V)
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- The **orbit-stabilizer thm:** $|\text{orbit}| \,.\, |\text{stabilizer}|$ $=$ order of the group $\mathbf{S}_N$
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- … independently of i.

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- Hence let $X'_i \overset{\text{def}}{=} X_i \times \phi_i = \{(G,\pi) \mid G \equiv G_i \text{ et } \pi.G_i = G_i\}$, $X \overset{\text{def}}{=} X'_1 \cup X'_2$

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The power trick (repeating experiments virtually)

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  if $G_1 \not\equiv G_2$ then $|X| = (2N!)^k$
- Otherwise $|X| = (N!)^k$
- Gap is now 2^k.
  Now take k so large that this exceeds 2ℓ.

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GNI is in AM

- Arthur draws $H \stackrel{\text{def}}{=} (h_1, \ldots, h_\ell)$ at random uniformly ($mm'\ell$ bits)

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- Merlin answers a (claimed) collision x in X (with proofs!)

- We check (the proofs) that this is a collision.
— if $(G_1, G_2) \in$ **GNI** then $\forall H$, $\exists$collision for H in X
— if $(G_1, G_2) \notin$ **GNI** then

$$\Pr_H(\exists \text{collision for } H \text{ in } R) \leq 1/2^{\ell - m' + 1}$$

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We need to tune m, m', ℓ and k so that the error is $\leq 1/2^{g(n)}$

Determining m, m', ℓ, and k

- $m=k \times$ (size(graph)+size(permutation))
  $= O(k(N^2+N \log N)) = O(kN^2) = O(kn)$

Let $X'_i \stackrel{\text{def}}{=} X_i \times \phi_i = \{(G,\pi) \mid G \equiv G_i$ et $\pi.G_i = G_i\}$,
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Determining m, m', ℓ, and k

Note: size(graph)=N^2 (adjacency matrix)
size(input)=n=$2N^2$

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- Use Sipser I and II:

Lemma (3.15). Let $X \subseteq \Sigma^m$. Assume $|X| \leq 2^{m'-1}$. Let $\ell \geq m'$.
Then $\Pr_H(X \text{ has a collision for } H) \leq 1/2^{\ell-m'+1}$.

Lemma (3.16). Let $X \subseteq \Sigma^m$. Assume $|X| > \ell . 2^{m'}$, where $\ell \geq m'$.
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- We wish error to be $\leq 1/2^{g(n)}$ $\Rightarrow$ require $\ell-m'+1 \geq g(n)$ (3)

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- Other constraints: $\ell \geq m'$ (4), all of m, m', ℓ, k polynomial in n.

Determining m, m', ℓ, and k

Note: size(graph)=N^2 (adjacency matrix)
size(input)=n=$2N^2$

- $m=k$ × (size(graph)+size(permutation))
  $= O(k(N^2+N \log N)) = O(kN^2) = O(kn)$

Let $X'_i \stackrel{\text{def}}{=} X_i \times \phi_i = \{(G,\pi) \mid G \equiv G_i$ et $\pi.G_i = G_i\}$,
$X \stackrel{\text{def}}{=} (X'_1 \cup X'_2)^k$

— if $(G_1, G_2) \in$ **GNI** then $\forall H$, $\exists$collision for H in X
— if $(G_1, G_2) \notin$ **GNI** then
$\Pr_H(\exists$collision for H in $R) \leq 1/2^{\ell-m'+1}$

- Use Sipser I and II:

Lemma (3.15). Let $X \subseteq \Sigma^m$. Assume $|X| \leq 2^{m'-1}$. Let $\ell \geq m'$.
Then $\Pr_H(X$ has a collision for $H) \leq 1/2^{\ell-m'+1}$.

Lemma (3.16). Let $X \subseteq \Sigma^m$. Assume $|X| > \ell.2^{m'}$, where $\ell \geq m'$.
Then X (definitely) has a collision for H.

- Sipser I: need $|X| \leq 2^{m'-1}$ if $(G_1, G_2) \notin$ **GNI** ⇒ require $m'-1 \geq k.\log_2(N!)$ (1)
- Sipser II: need $|X| > \ell.2^{m'}$ if $(G_1, G_2) \in$ **GNI** ⇒ require $m'+\log_2 \ell < k(1+\log_2(N!))$ (2)
- We wish error to be $\leq 1/2^{g(n)}$ ⇒ require $\ell-m'+1 \geq g(n)$ (3)
- Other constraints: $\ell \geq m'$ (4), all of m, m', ℓ, k polynomial in n.

E.g., $k \stackrel{\text{def}}{=} N$, $m' \stackrel{\text{def}}{=} 1+k.\log_2(N!)$ (for (1)), $\ell \stackrel{\text{def}}{=} m'+g(n)$ (for (3), (4))
((2) OK for n large enough, otherwise tabulate)

Checking (proofs of) collisions

- Explicitly, Merlin sends:
  — an element x
  — a **proof** that x is in X
  — elements $x_1, \ldots, x_\ell$
  — proofs that each x_j is in X
- Then we will check the proofs
  + $x \neq x_j$ and $h_j(x) = h_j(x_j)$ for each $j = 1..\ell$.

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- A **proof** that $x \stackrel{\text{def}}{=} ((G'_1, \pi_1), \ldots, (G'_k, \pi_k))$ is in X is:
  — for each $i = 1..k$, a permutation π'_i / $\pi'_i.G'_i = G_1$ or G_2
  — Checking it means checking $\pi'_i.G'_i = G_1$ or G_2,
  and also $\pi_i.G'_i = G'_i$, for each i.

GNI is in AM… and more

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  For every $k \geq 1$, $\mathbf{IP}[k] \subseteq \mathbf{AM}[k+1]$.

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- I will omit the proof, see the lecture notes.
- So $\mathbf{AM} \subseteq \mathbf{IP}[1] \subseteq \ldots \subseteq \mathbf{IP}[k] \subseteq \mathbf{AM}[k+1] = \mathbf{AM}$
- **Corl.** For every $k \geq 1$, $\mathbf{IP}[k] = \mathbf{AM}[k] = \mathbf{AM}$. (!)

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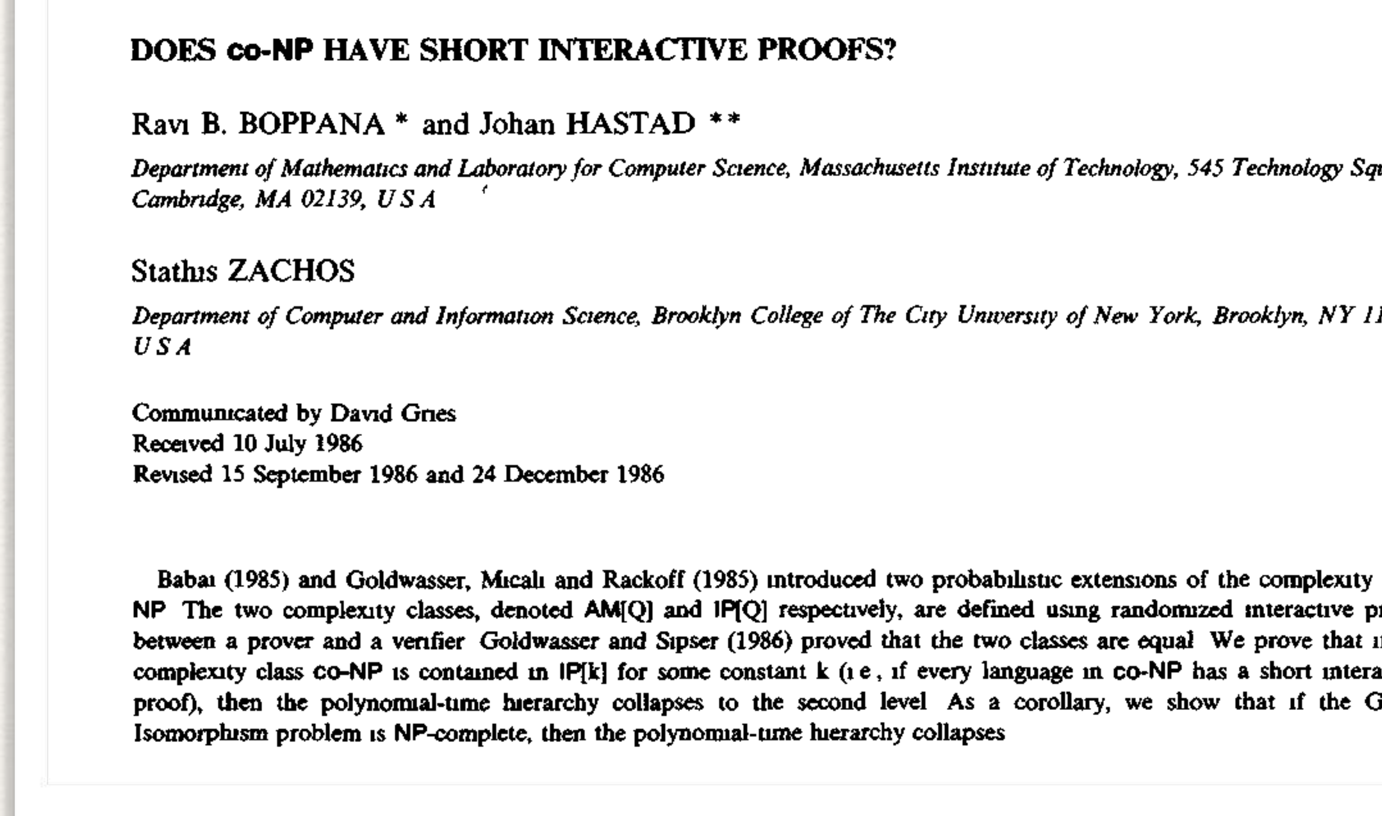

DOES co-NP HAVE SHORT INTERACTIVE PROOFS?

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Babaı (1985) and Goldwasser, Mıcalı and Rackoff (1985) ıntroduced two probabılıstıc extensıons of the complexıty NP The two complexıty classes, denoted AM[Q] and IP[Q] respectıvely, are defıned usıng randomızed ınteractıve pr between a prover and a verıfıer Goldwasser and Sıpser (1986) proved that the two classes are equal We prove that ı complexıty class co-NP ıs contaıned ın IP[k] for some constant k (ı e , ıf every language ın co-NP has a short ıntera proof), then the polynomıal-tıme hıerarchy collapses to the second level As a corollary, we show that ıf the G Isomorphısm problem ıs NP-complete, then the polynomıal-tıme hıerarchy collapses

The Boppana-Håstad-Zachos theorem

https://math.mit.edu/images/profile/boppana_ravi.png

https://www.ae-info.org/attach/User/Hastad_Johan/scaled-0x200_haastad_johan_small_ae.jpg

https://alchetron.com/cdn/stathis-zachos-44e8a09d-57b0-4dd6-8d38-c8273e19ee3-resize-750.jp

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- *Proof.* Let $L \in \Sigma^p_2$. We will show that L is in Π^p_2.
  $L=\{x \mid \exists y, (x,y) \in L'\}$, for some $L' \in$ **coNP**.

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- Hence $L' \in$ **AM**. There is a D in **P** such that:
  — if $(x,y) \in L'$ then ($\mathrm{E}r, \exists z, x\#y\#r\#z \in D$) large
  — if $(x,y) \notin L'$ then ($\mathrm{E}r, \exists z, x\#y\#r\#z \in D$) small

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  — if $x \notin L$ then $(\exists y, \mathrm{E}r, \exists z, x\#y\#r\#z \in D)$ small
- Hence $L \in$ **MAM** $=$ **AM** (Babai) $\subseteq \Pi^p{}_2$. $\square$

The BHZ theorem, and Graph Isomorphism

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- Remember that **GNI** is in **AM**, as we have just shown.

The BHZ theorem, and Graph Isomorphism

- **Corl (3.21).** If **GI** is **NP**-complete
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- *Proof.* **AM** is **closed under poly time reductions**.
- Remember that **GNI** is in **AM**, as we have just shown.
- Hence if **GI** is **NP**-complete,
  then **GNI** is **coNP**-complete,
  hence **coNP** $\subseteq$ **AM**.
  Now apply the previous theorem. □

Graph Isomorphism

- **Corl (3.21).** If **GI** is **NP**-complete
  then **PH** collapses at level 2.
- Remember that **GI** is not known to be in **P**,
  and not known to be **NP**-complete.
- The BHZ theorem shows that the latter is unlikely.
- Note: Babai gave a super polynomial time algo for GI in 2015 (still does not solve the question, but what a progress!); builds on a lot of things, including BHZ.

Next time…

IP and PSPACE

- **IP** and **AM** with **polynomially many rounds** (the classes **IP** and **ABPP**)
- Shamir's theorem: **ABPP=IP=PSPACE** (!)