Jean Goubault-Larrecq

Randomized complexity classes

Today: **RP**, **coRP**, and **ZPP** (what a zoo!)

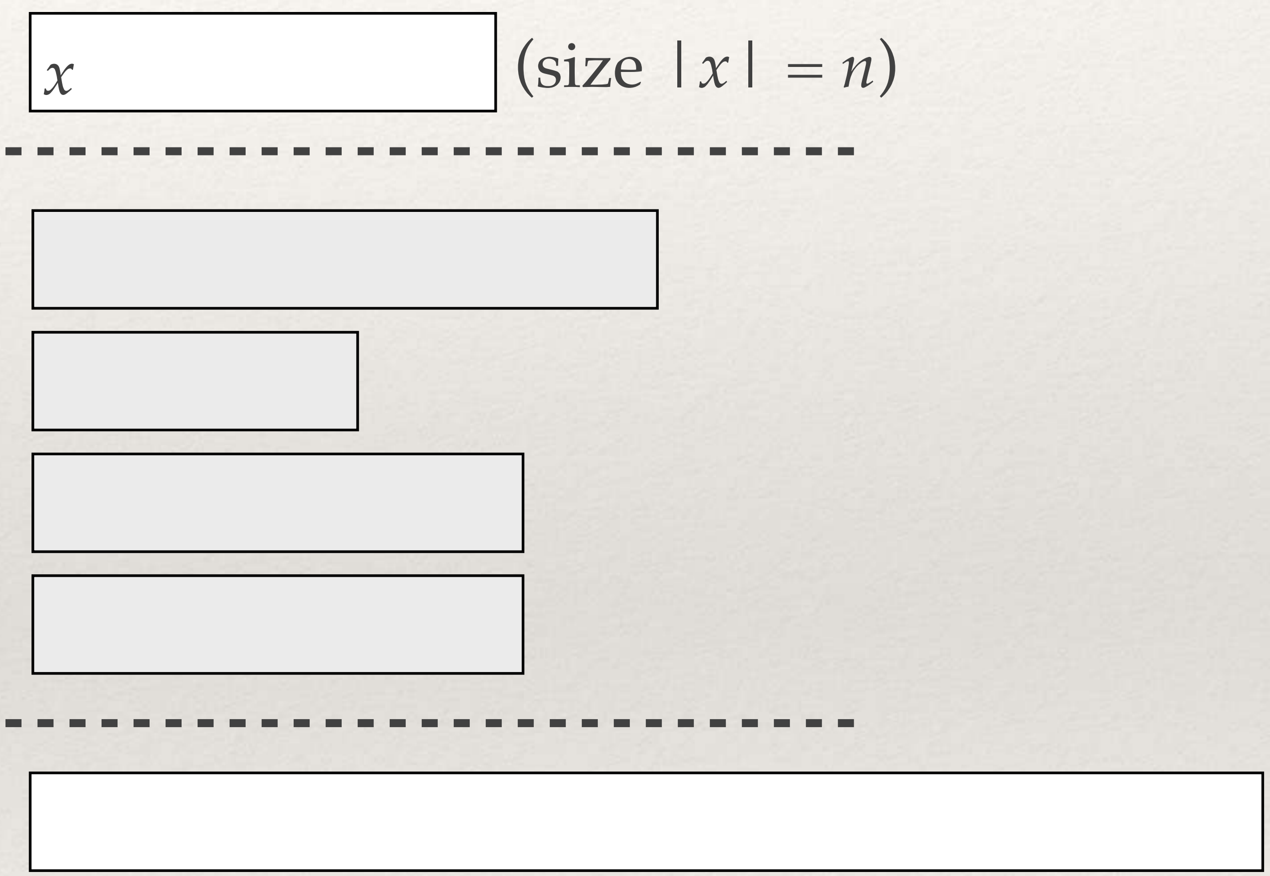
Today

- Randomized Turing machines
- One-sided error: **RP**, **coRP**
- No error: **ZPP**
- Next time: **two**-sided error **BPP**

Randomized Turing machines

Ordinary Turing machines

- One **read-only** input tape



- As many **work tapes** as you need (but only a constant number!)
- (Possibly) one **write-only** output tape

Drawing strings at random

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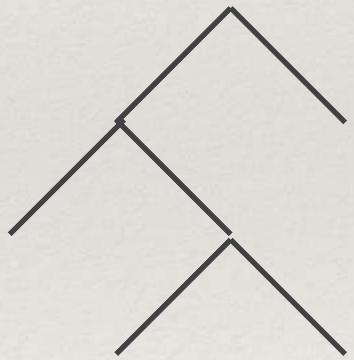
  (we won't do that; hard to do it right)



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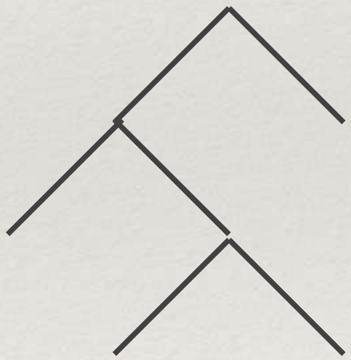
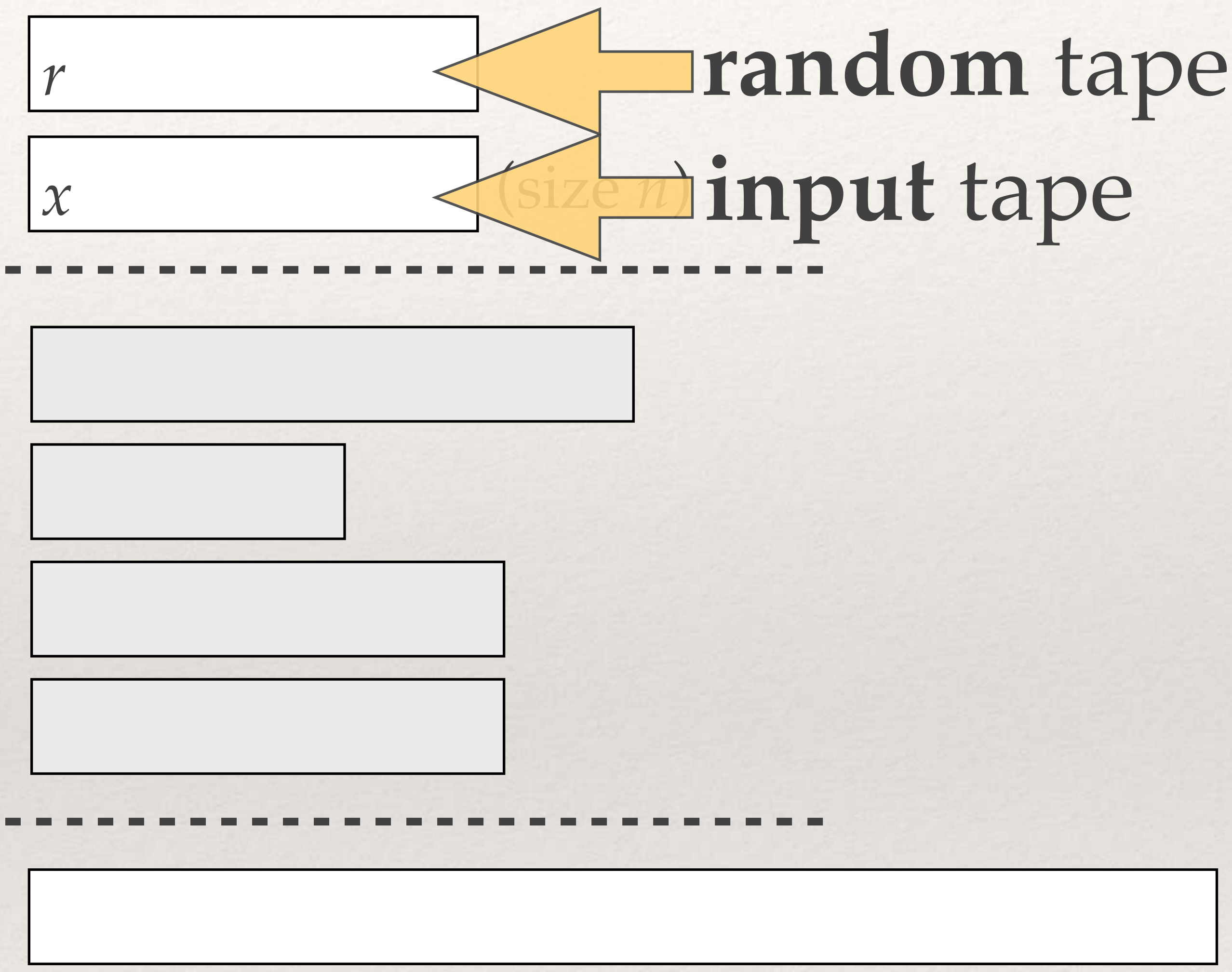


- Choice 2: … next slide

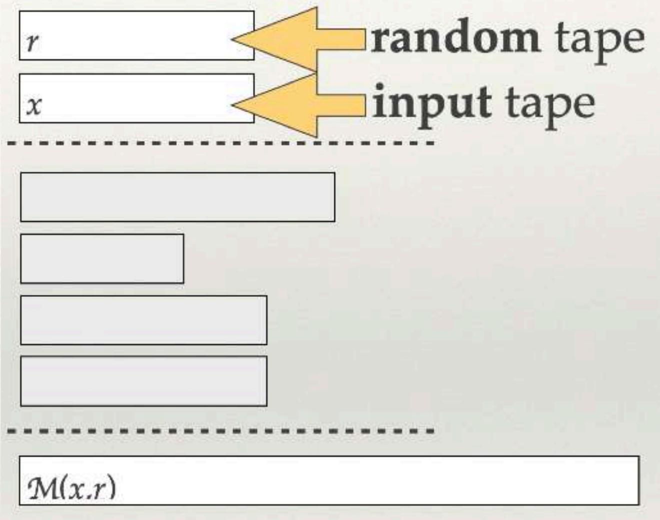
Randomized Turing machines

- Two ~~One~~ **read-only** tapes
- As many **work tapes** as you need (but only a constant number!)
- (Possibly) one **write-only** output tape



Technical points 1/2

- We draw the random tape r **uniformly at random**



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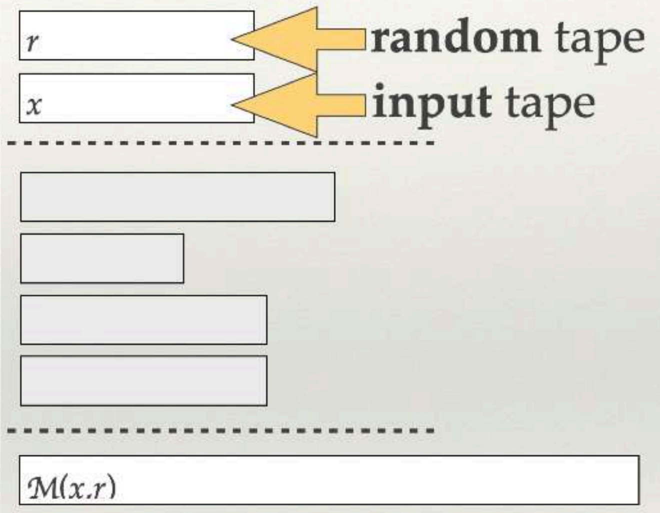

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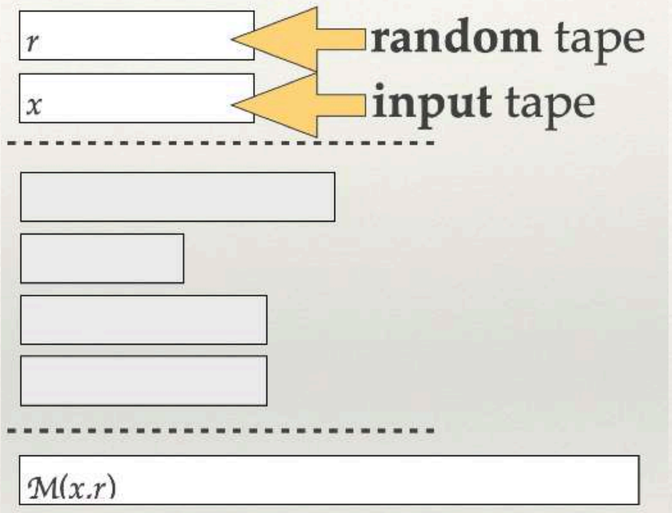
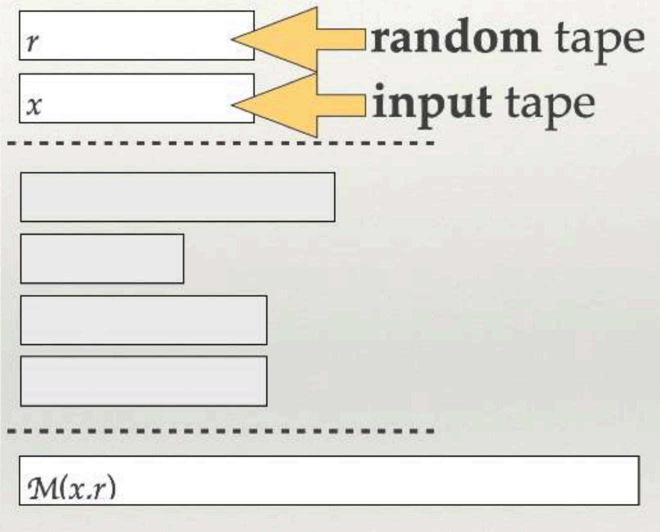


- We draw the random tape r **uniformly at random**
- We will be interested in **probabilities**, e.g.
  $$\Pr_r\,[\mathcal{M}(x,r)\text{ accepts}]$$
- Random tape must not just be read-only: we impose that **no bit on r is ever read twice** (otherwise bits read are not independent)

Technical points 2/2

- $\Rightarrow$ we need r to contain at least $f(n)$ bits, where $f(n)$ is an upper bound on the **time** taken by the TM.



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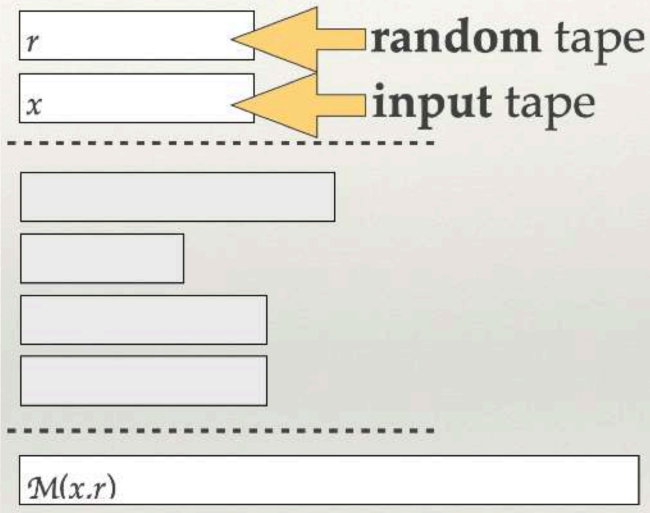

- ⇒ we need r to contain at least $f(n)$ bits, where $f(n)$ is an upper bound on the **time** taken by the TM.
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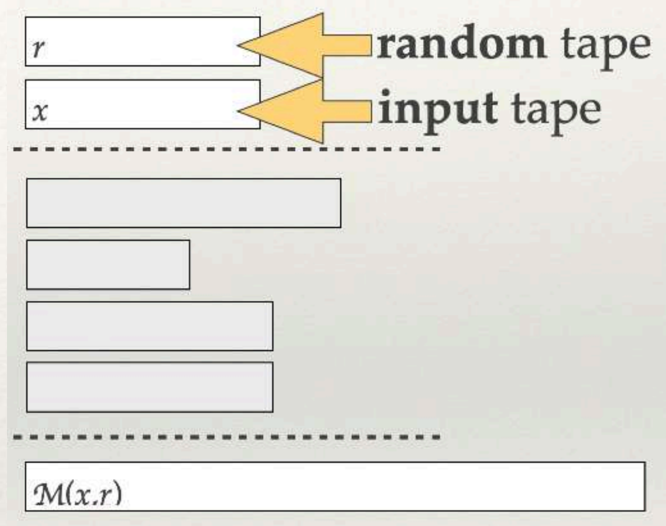

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- We will always assume that r is **large enough**
- OK for classes defined by **worst-case time**, will cause problems for classes defined with no a priori upper bound on time (e.g., **ZPP**)

Our first probabilistic class: **RP**

(also sometimes known as the class of *Monte Carlo* languages)

http://fr.casino-jackpot.com/wp-content/uploads/2018/04/casino-monaco.jpg

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Perhaps paradoxically, that means that we make **no** error if $\mathcal{M}(x,r)$ **accepts**
(so please do not confuse acceptance with being in the language!)

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… but if we wanted to define « **RP**-machines », those would be machines $\mathcal{M}$ such that, for every x,
— either $\Pr_r\,[\mathcal{M}(x,r)$ accepts$] \geq 1/2$
— or $\Pr_r\,[\mathcal{M}(x,r)$ accepts$] = 0$

coRP

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- In **NP** [Pratt 1975]
- Can also be solved efficiently with **randomization**…

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- An experiment… (next slide)

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- $\Rightarrow$ 87 is **not prime** (definitely)

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r	r^{86}
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	13

r	r^{86}
11	34
12	57
13	82
14	22
15	51
16	82
17	28
18	63
19	13
20	52

r	r^{86}
21	6
22	49
23	7
24	54
25	16
26	67
27	33
28	1
29	58
30	30

r	x^{86}
31	4
32	67
33	45
34	25
35	7
36	78
37	64
38	52
39	42
40	34

r	r^{86}
41	28
42	24
43	22
44	22
45	24
46	28
47	34
48	42
49	52
50	64

r	r^{86}
51	78
52	7
53	25
54	45
55	67
56	4
57	30
58	58
59	1
60	33

r	r^{86}
61	67
62	16
63	54
64	7
65	49
66	6
67	52
68	13
69	63
70	28

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71	82
72	51
73	22
74	82
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84	9
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- Is 87 prime?
- The probability (over r) of error is:
  $$2/84 \approx 0.024$$

r	r^{86}	r	r^{86}	r	r^{86}	r	x^{86}	r	r^{86}	r	r^{86}	r	r^{86}	r	r^{86}	r	r^{86}
		11	34	21	6	31	4	41	28	51	78	61	67	71	82	81	36
2	4	12	57	22	49	32	67	42	24	52	7	62	16	72	51	82	25
3	9	13	82	23	7	33	45	43	22	53	25	63	54	73	22	83	16
4	16	14	22	24	54	34	25	44	22	54	45	64	7	74	82	84	9
5	25	15	51	25	16	35	7	45	24	55	67	65	49	75	57	85	4
6	36	16	82	26	67	36	78	46	28	56	4	66	6	76	34		
7	49	17	28	27	33	37	64	47	34	57	30	67	52	77	13		
8	64	18	63	28	1	38	52	48	42	58	58	68	13	78	81		
9	81	19	13	29	58	39	42	49	52	59	1	69	63	79	64		
10	13	20	52	30	30	40	34	50	64	60	33	70	28	80	49		

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- Else, will fail with (hopefully) **high probability** (0.024 in the example, looks good); but…

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- Infinitely many of them [Alford, Granville, Pomerance 1994]:
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- Infinitely many of them [Alford, Granville, Pomerance 1994]:
  561, 1105, 1729, 2465, 2821, 6601 , 8911, 10585, 15841, etc.
- Frustrating: if p is not prime and passes at least **one** Fermat test, then it passes at least **half** of them…

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- Hence, if p is prime and odd (so $p-1 = 2^k q$, q odd):

(read from right to left : ←)

r^q	…	r^($2^{i-1} q$)	r^($2^i q$)	…	r^($2^{k-1} q$)	r^($2^k q$)	mod p
(don't care …	…. don't care)	-1	1	…	1	1	for some i, or:

The Miller-Rabin test (1/2)

- We use another basic fact: if p is prime, then
  the only square roots of 1 mod p are 1 and –1
- Hence, if p is prime and odd (so p–1 = $2^k q$, q odd):

(read from right to left : ←)

r^q	…	r^($2^{i-1}q$)	r^($2^i q$)	…	r^($2^{k-1}q$)	r^($2^k q$)	mod p
(don't care …	…. don't care)	–1	1	…	1	1	for some i, or:

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1	…	1	1	…	1	1	

The Miller-Rabin test (2/2)

- On input p,
  draw r at random:
  — if the test shown here:
  succeeds, then **accept** (p probably prime)
  — otherwise **reject** (p definitely not prime)

Hence, if p is prime and odd (so $p–1 = 2^k\, q$, q odd):

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- Probability of error $\leq 1/4$. Excellent! Hence:
- **Theorem. PRIMALITY** is in **coRP**.
- (Superseded by [AKS04]…
  but Miller-Rabin works in log space, not [AKS04]!)

To know more

Notes on Primality Testing
And Public Key Cryptography
Part 1: Randomized Algorithms
Miller–Rabin and Solovay–Strassen Tests

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February 27, 2019

https://www.cis.upenn.edu/~jean/RSA-primality-testing.pdf

Error reduction

- What is so special about error $1/2$?

- A language L is in **RP** if and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):
- if $x \in L$ then $\Pr_r [\mathcal{M}(x,r) \text{ accepts}] \geq 1/2$
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error $= 1 - 1/2$
($= 1/2$ here)

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error = 1 – 1/2 (= 1/2 here)

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Error reduction: the easy direction

- Clearly, if $\eta \leq \varepsilon$ then
  $\mathbf{RP}(\eta) \subseteq \mathbf{RP}(\varepsilon)$

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error = ε

- Note: $\mathbf{RP}(0)=\mathbf{P}$ (believed $\neq$ $\mathbf{RP}$)

  $\mathbf{RP}(1)=\{$*all languages*$\}$ (why?)

The hard direction: repeating experiments

- Let $L \in \mathbf{RP}(\varepsilon)$, $0<\eta<\varepsilon<1$
- On input x, let us do the following (at most) K times:

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  - Otherwise, proceed and loop.
- At the end of the loop, **reject**.

> - A language L is in **RP**(ε) and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):
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Remember: if $\mathcal{M}(x, r)$ accepts, then x **must** be in L.

Repeating experiments (pretty) formally

- We have defined a **new** randomized TM

  $\mathcal{M}'(x, r[1]\#\ldots\#r[K])$ by:

- `for` i=1 `to` K:
  - If $\mathcal{M}(x, r[i])$ accepts, then exit the loop and **accept**;
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Remember: if $\mathcal{M}(x, r[i])$ accepts, then x **must** be in L.

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Acceptance: 1. if $x \in L$

- If $x \in L$ (recall L in **RP**(ε)), then letting $r=r[1]\#\ldots\#r[K]$, $Pr_r(\mathcal{M}'(x, r)$ rejects)

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- If $x \in L$ (recall L in **RP**(ε)), then letting r=r[1]#…#r[K], $\Pr_r(\mathcal{M}'(x, r)$ rejects)
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  - $= \Pi_{i=1..K} \Pr_{r[i]}(\mathcal{M}(x, r[i])$ rejects) **(independence)**

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  - $\leq \varepsilon^K$
- $\Rightarrow$ If $x \in L$ then $\Pr_r(\mathcal{M}'(x, r)$ accepts) $\geq 1-\varepsilon^K$

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Acceptance: 2. if $x \notin L$; Complexity

- If $x \in L$ (recall L in **RP**(ε)) then $\Pr_r(\mathcal{M}'(x, r)\text{ accepts}) \geq 1-\varepsilon^K$

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- If $\mathcal{M}$ runs in time $p(n)$, then $\mathcal{M}'$ runs in time $O(Kp(n))$

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- If $\mathcal{M}$ runs in time $p(n)$, then $\mathcal{M}'$ runs in time $O(Kp(n))$
- Hence L is in **RP**(ε^K)

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- We have shown that every language L in **RP**(ε)

  is in **RP**(ε^K)

  (for any $\varepsilon \in [0,1]$, $K \geq 1$)

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- If $0<\eta<\varepsilon<1$,
  choose K **large enough**
  so that $\varepsilon^K \leq \eta$
  (explicitly, $K \geq \eta / \log \varepsilon$)

- A language L is in **RP**(ε) and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):
- if $x \in L$ then $\Pr_r [\mathcal{M}(x,r) \text{ accepts}] \geq 1-\varepsilon$
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error = ε

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The hard direction: the end

- We have shown that every language L in **RP**(ε) is in **RP**(ε^K) (for any $\varepsilon \in [0,1]$, $K \geq 1$)
- If $0<\eta<\varepsilon<1$, choose K **large enough** so that $\varepsilon^K \leq \eta$ (explicitly, $K \geq \eta / \log \varepsilon$)
- Then L is in **RP**(η). □

- A language L is in **RP**(ε) and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):
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error $= \varepsilon$

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- … or with error $\varepsilon = 0.99999999$!

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error = ε

Can we do even better?

- Hence we define the same class with error $\varepsilon = 0.000000000001$
- … or with error $\varepsilon = 0.99999999$!
- Can we make ε go to 0 as $n \to \infty$?

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error = ε

The hard direction **revisited**

- If $x \in L$ (recall L in **RP**(ε)) then $\Pr_r(\mathcal{M}'(x, r) \text{ accepts}) \geq 1-\varepsilon^K$
- If $x \notin L$, then $\mathcal{M}'(x, r)$ accepts for no r
- If $\mathcal{M}$ runs in time $p(n)$, then $\mathcal{M}'$ runs in time $O(Kp(n))$
- Hence L is in **RP**(ε^K).

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=$O(q(n)p(n))$, still **polynomial time**

The hard direction **revisited**

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- If $x \notin L$, then $\mathcal{M}'(x, r)$ accepts for no r
- If $\mathcal{M}$ runs in time $p(n)$, then $\mathcal{M}'$ runs in time $O(Kp(n))$
- Hence L is in $\mathbf{RP}(\varepsilon^K)$.

error $\varepsilon^K = \varepsilon^{q(n)}$
(exponentially small)

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The hard direction **revisited**

- Let ε=1/2. We have proved:
- **Theorem. RP=RP**$(1/2^{q(n)})$
  for every polynomial $q(n)$.
- I.e., error can be made
  exponentially small.
- (Note: **RP**(ε) called $\cup_{p(n)}$ **RTIME**$(p(n),p(n),0,\varepsilon)$
  in the notes: ignore the complication)

- A language L is in **RP**(ε) and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):
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- (Note: $\mathbf{RP}(\varepsilon)$ called $\cup_{p(n)}$ $\mathbf{RTIME}(p(n),p(n),0,\varepsilon)$ in the notes: ignore the complication)

- Exercise: show that, conversely:
- **Theorem.** $\mathbf{RP}=\mathbf{RP}(1-1/q(n))$
  for every polynomial $q(n)$.
- I.e., error can be assumed « polynomially large » as well

Relation to ordinary classes

- **Theorem. P $\subseteq$ RP $\subseteq$ NP.**
- *Proof.* First,

  $$\mathbf{P} = \mathbf{RP}(0) \subseteq \mathbf{RP}(1/2) = \mathbf{RP}$$

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  - If $x \in L \Rightarrow$ for some r, $\mathcal{M}(x, r)$ accepts (in fact, for at least half of them!)

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  - If $x \in L \Rightarrow$ for some r, $\mathcal{M}(x, r)$ accepts (in fact, for at least half of them!)
  - If $x \notin L \Rightarrow$ for no r.
- Hence $L = \{x \mid \exists r, \mathcal{M}(x, r) \text{ accepts}\}$ is in **NP**. $\square$

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Our second probabilistic class: **ZPP**

(also known as the class of
Las Vegas languages)

https://www.agoda.com/fr-fr/paris-las-vegas_8/hotel/las-vegas-nv-us.html?cid=1844104

ZPP

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- Not clear that those two definitions are equivalent, right?

ZPP

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  <u>two</u> poly-time rand. TMs $\mathcal{M}_1$ and $\mathcal{M}_2$ such that:
  - if $x \in L$ then $\mathcal{M}_1(x,r)$ accepts for every r [no error]
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- Let us define **ZPP′** (for now) as the class of languages L which we can decide in **average** polynomial-time with probability **zero** of making a mistake.
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- we will need something called **Markov's inequality** too
- … but before that, we explain why (intuitively) **ZPP** $\subseteq$ **ZPP′**.

Deciding L in **ZPP = RP ∩ coRP** with no error

- Assume $\mathcal{M}_1$ and $\mathcal{M}_2$ such as here→
- Now run the following on input x:

  `forever`:

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then x cannot be in L (sure)

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— in which case we loop
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why?
(if you tell me that this is even $\leq 1/4$, you are wrong)

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We will see that this implies that the machine terminates in **≤ 2 turns** of the loop **on average**

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- Requires **measure theory**:
  there is a unique measure μ on $\{0,1\}^\omega$
  with σ-algebra generated by cylinders $w.\{0,1\}^\omega$
  such that $\mu(w.\{0,1\}^\omega) = 1/2^{|w|}$ (Carathéodory)

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  such that $\mu(w.\{0,1\}^\omega) = 1/2^{|w|}$ (Carathéodory)
- We will happily ignore this.

Rejection sampling

- A classic probabilistic procedure (**rejection sampling**):

  `forever`:

  compute something (with some random data r), x;

  `if` $P(x)$ holds: stop and return x

- **Trick.** If:

  — the random bits are independent across turns of the loop

  — and $P(x)$ holds with **prob.** $\geq \alpha$ at each turn

  then rejection sampling terminates in

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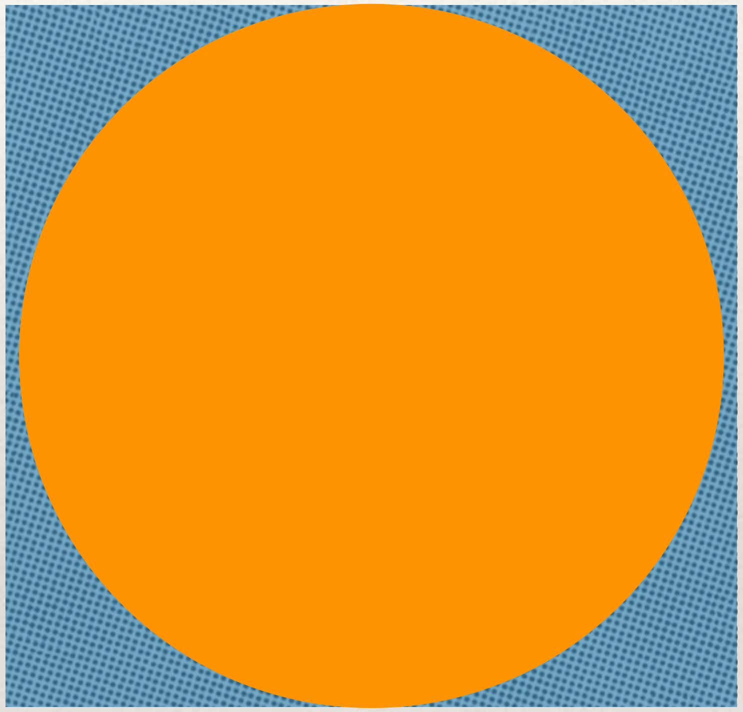
- $E(X) = \sum_{n\geq 1} n.\Pr(X=n) = \sum_{n\geq 1} \Pr(X\geq n) \leq \sum_{n\geq 1} (1-\alpha)^{n-1} = 1/\alpha.$ $\square$

Expectation (average)

Rejection sampling: a typical application

- Draw a point inside the disc:
- Repeatedly draw a point inside the inscribing square
  - If it is in the disc, return it.



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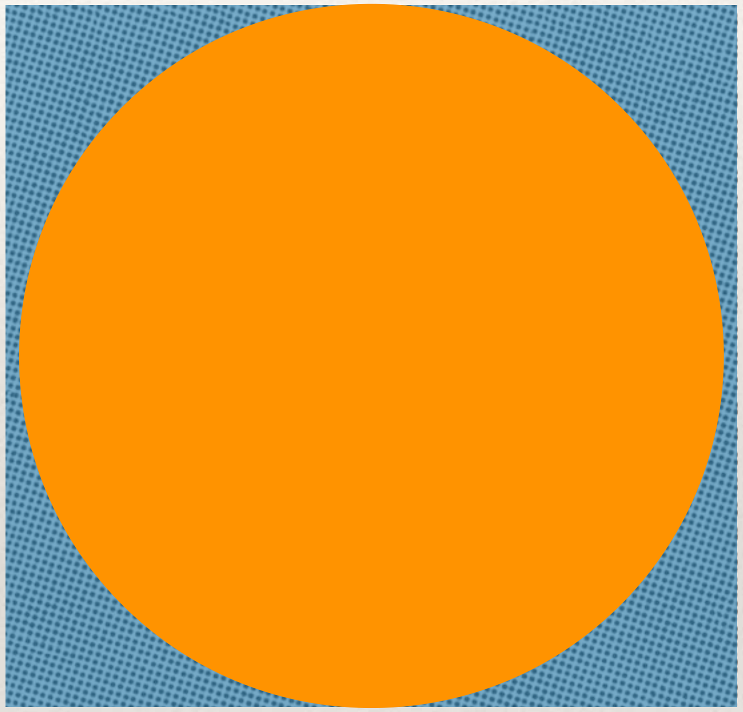

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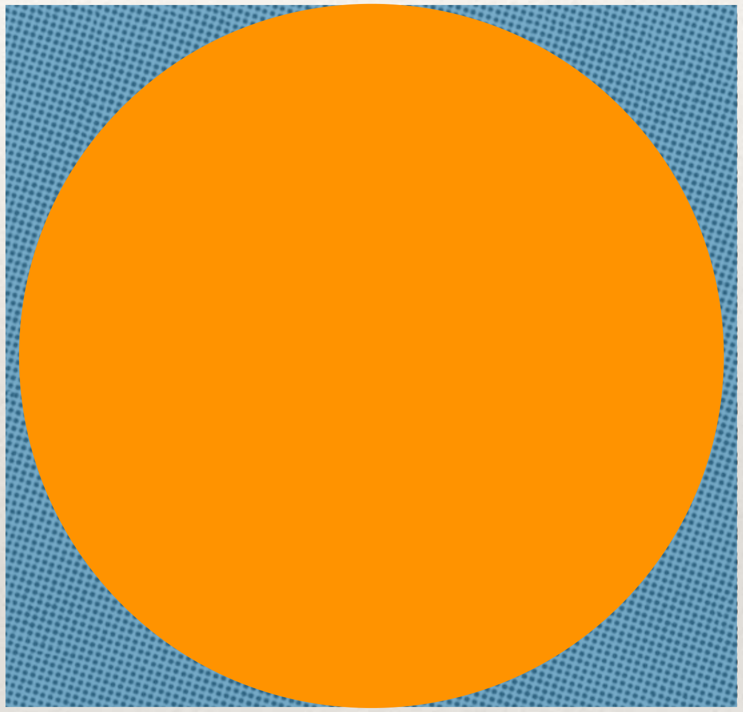

- (Used as first step in the **Box-Muller** procedure drawing two independent numbers with a normal distribution)

Deciding L in **ZPP** = **RP** ∩ **coRP** with no error

- Assume $\mathcal{M}_1$ and $\mathcal{M}_2$ such as here:
- Now run the following on input x:

`forever:`
    `if` $\mathcal{M}_1(x,\ldots)$ rejects: stop and **reject**
    `if` $\mathcal{M}_2(x,\ldots)$ accepts: stop and **accept**

I.e., L is in **ZPP** iff there are
two poly-time rand. TMs $\mathcal{M}_1$ and $\mathcal{M}_2$ such that:

- if $x \in L$ then $\mathcal{M}_1(x,r)$ accepts for every r [no error]
  $\mathcal{M}_2(x,r)$ accepts with prob.$\geq 1/2$
- if $x \notin L$ then $\mathcal{M}_1(x,r)$ accepts with prob.$\leq 1/2$
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then x cannot be in L (sure)

then x must be in L (sure)

Hence this machine **never makes any mistake**

It may be that $\mathcal{M}_1(x,\ldots)$ accepted and $\mathcal{M}_2(x,\ldots)$ rejected,
— in which case we loop
— and that happens with probability $\leq 1/2\ldots$
(two cases: x in L, x not in L)

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This is **rejection sampling**:
stops in ≤ 2 turns on average
hence in **polytime on average.**

Markov's inequality

> Let us define **ZPP′** (for now) as the class of languages L which we can decide in **average** polynomial-time with probability **zero** of making a mistake.
>
> I claim that **ZPP** = **ZPP′**.

- Hence:
  ZPP (= **RP** $\cap$ **coRP**) $\subseteq$ **ZPP′**
- In order to show the reverse inclusion, we use:
- **Theorem (Markov's inequality).**
  Let X be a **non-negative real-valued** random variable with **finite** expectation $E(X)$. For every $a>0$:
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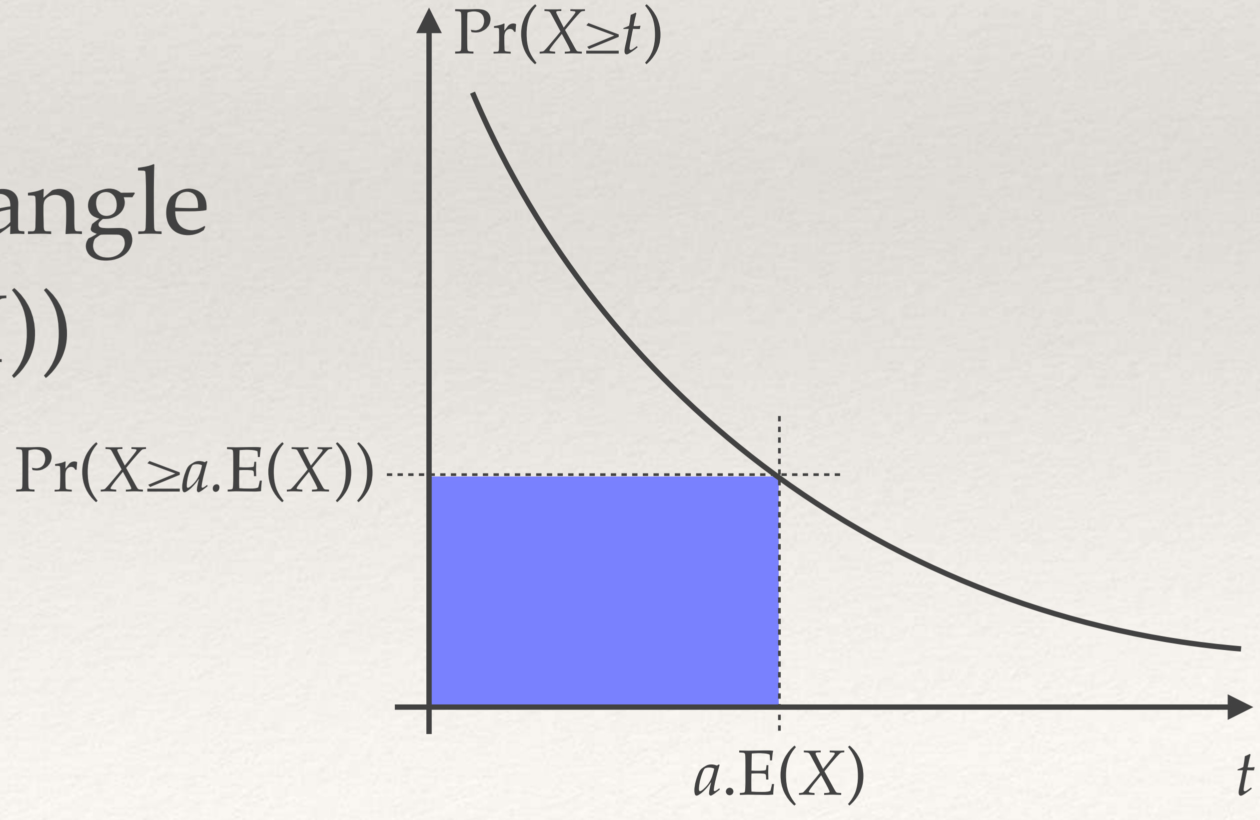
- *Proof.* $E(X) = \int_t \Pr(X \geq t)\, dt$
  $\geq$ area of the blue rectangle
  $= a\,.\,E(X)\,.\,\Pr(X \geq a.E(X))$
  Then divide out by $a\,.\,E(X)$. □



The reverse inclusion **ZPP' ⊆ ZPP**

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- Let L in **ZPP'**, decided by $\mathcal{M}$ running in **average** poly. time $p(n)$ with **no** error.
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&\leq 1/a = 1/2 && (\text{Markov})
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- Hence L is in **coRP**.

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Symmetrically:

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- Hence L is in ~~**coRP**~~ **RP.**

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I claim that **ZPP** = **ZPP'**. Recall **ZPP** = **RP** ∩ **coRP**

Let L in **ZPP'**, decided by $\mathcal{M}$ running in **average** poly. time $p(n)$ with **no** error.

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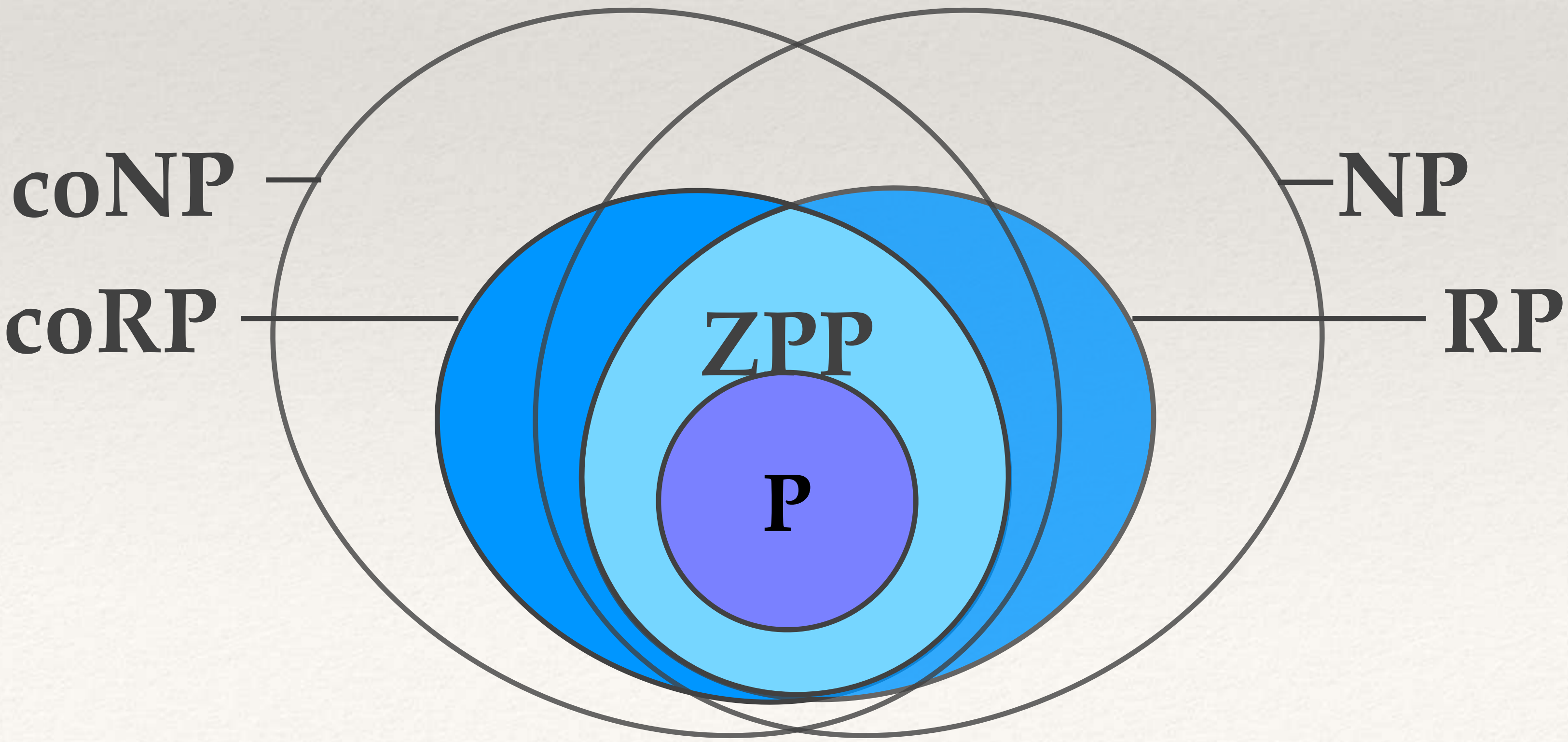
- If $x \notin L \Rightarrow \mathcal{M}_2(x,r)$ must ~~accept~~ reject.
- Hence L is in ~~coRP~~ **RP.**

Hence L is both in **RP** and in **coRP**, namely in **ZPP**. □

Summary on **ZPP**

- **Definition. ZPP** = **RP** ∩ **coRP**
- **Theorem. ZPP** is the class of languages L which we can decide in **average** polynomial-time with probability **zero** of making a mistake.



Next time…

BPP: Bounded Prob. of Error Polynomial time

- A language L is in **BPP** if and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):
- if $x \in L$ then $\Pr_r[\mathcal{M}(x,r) \text{ accepts}] \geq 2/3$
- if $x \notin L$ then $\Pr_r[\mathcal{M}(x,r) \text{ accepts}] \leq 1/3$.

two-sided error:
$\Pr_r[\mathcal{M}(x,r) \text{ errs}] \leq 1/3$