Jean Goubault-Larrecq

Randomized complexity classes

Today: **BPP** (part 1)

Today

- Two-sided error: **BPP**
- Error reduction, voting, Chernoff's bound
- The Sipser-Gács-Lautemann theorem

Our third probabilistic class: **BPP**

(also sometimes known as the class of *Metropolis* languages, although some speak of Monte Carlo here again)

http://fr.casino-jackpot.com/wp-content/uploads/2018/04/casino-monaco.jpg
https://upload.wikimedia.org/wikipedia/commons/5/56/Nicholas_Metropolis_cropped.PNG

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two-sided error: $\Pr_r\,[\mathcal{M}(x,r)$ errs$] \leq 1/3$

Examples

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PolyMath

The complexity class BPP

Examples

The problem of determining whether a multivariate polynomial vanishes is in BPP. The idea of the randomized algorithm is to compute the polynomial at a small number of randomly chosen points. For a non-zero polynomial the probability that it vanishes at all those points decreases rapidly with the number of points, and so if it vanishes at all those points we can say with some confidence that the polynomial vanishes everywhere. This problem is also in co-RP, since if the polynomial really does vanish everywhere, then the algorithm is guaranteed to output 1.

It would be good to have more examples. In particular, it would be nice to have an example that isn't obviously in RP or co-RP.

https://asone.ai/polymath/index.php?title=The_complexity_class_BPP

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Error reduction

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- What is so special about error $1/3$?
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- **Theorem.** $\forall \varepsilon \in]0, 1/2[$, **BPP** = **BPP**($\varepsilon$).
- Note: **BPP**=**BPP**(1/3) (def.)
  BPP(ε)={*all languages*} if $\varepsilon \geq 1/2$…
  BPP(0)=**P**

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error = ε

The easy cases: error amplification(!)

- Clearly, if $\eta \leq \varepsilon$ then
  $\mathbf{BPP}(\eta) \subseteq \mathbf{BPP}(\varepsilon)$
- Note: **BPP**(0)=**P** (sometimes believed ≠ **BPP**)
  BPP(ε)={*all languages*} for every $\varepsilon \geq 1/2$
- In the middle, hence, we will see that all the intermediate **BPP**(ε) ($\varepsilon \in]0, 1/2[$) are equal to **BPP**.

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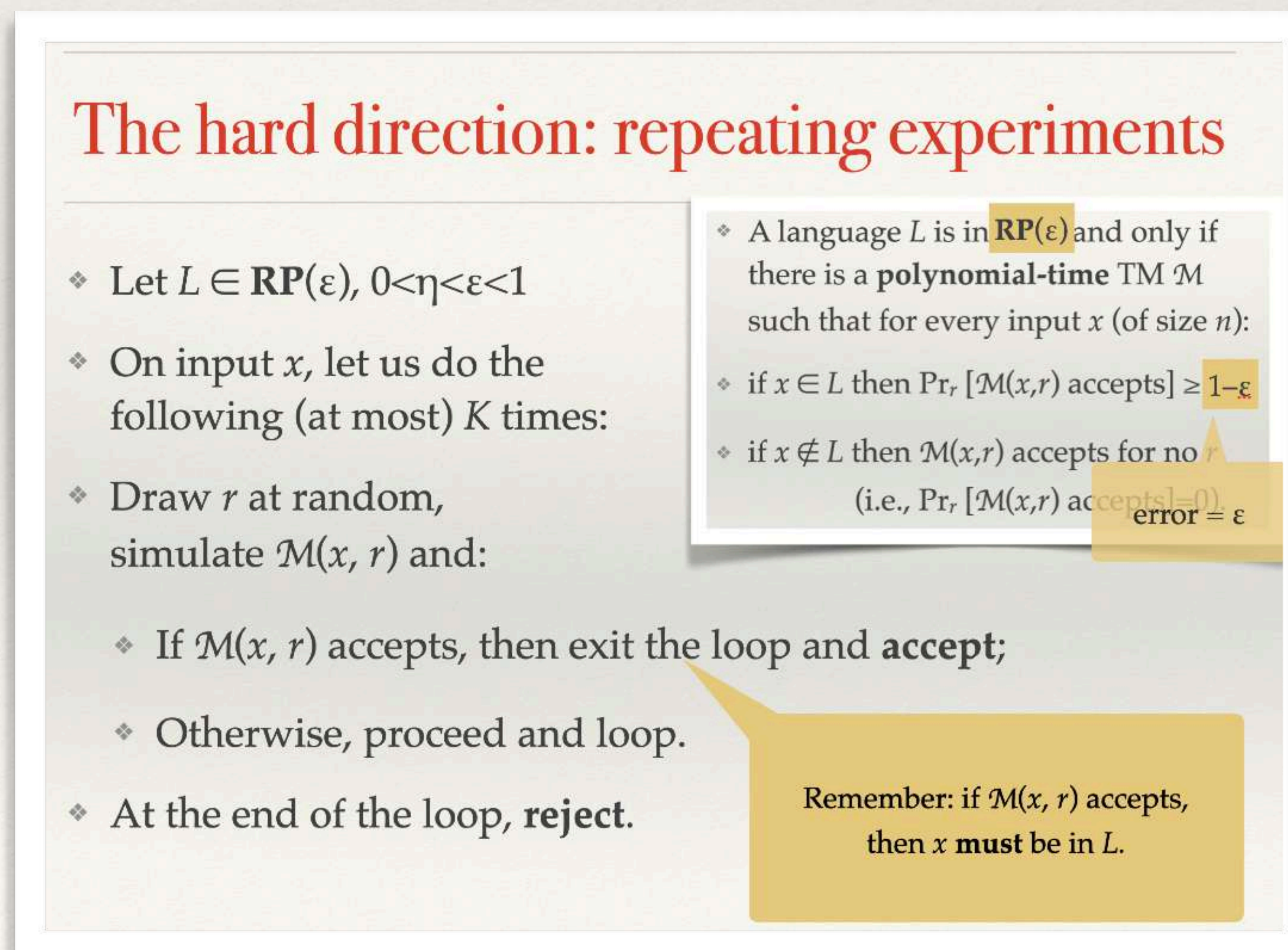

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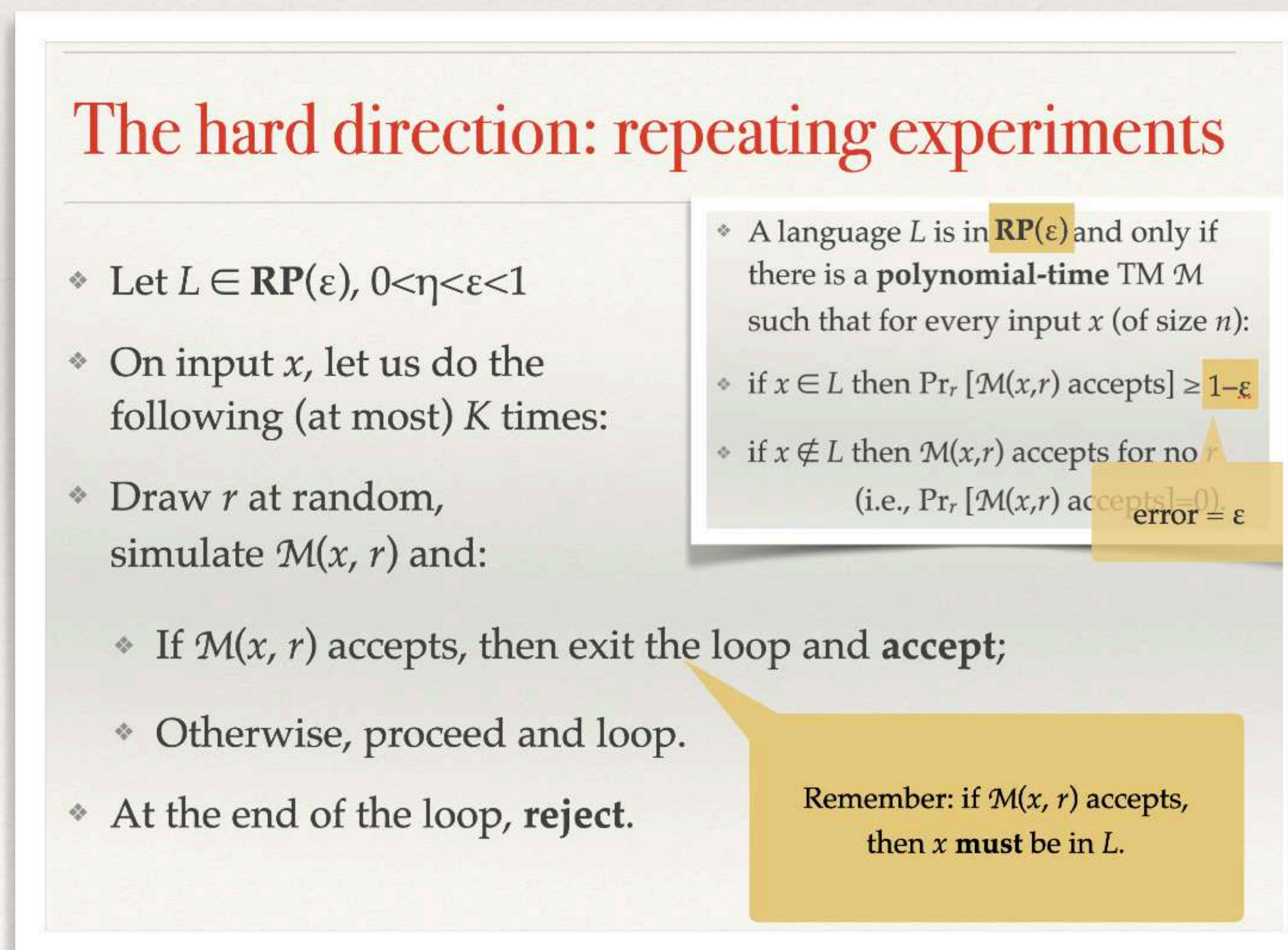
- Hence we must proceed differently



Majority voting

- Imagine running $\mathcal{M}(x,r)$ for various values of r, and **tallying the votes**

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accept

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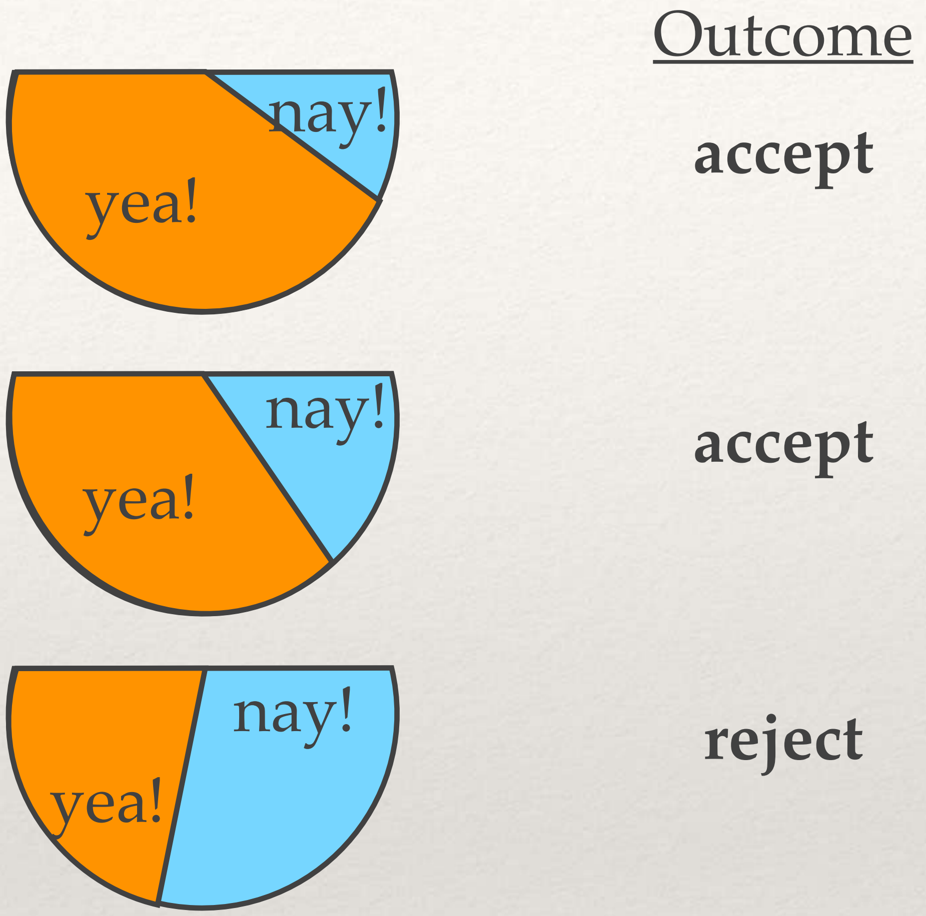

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Majority voting

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- Redo the vote N times (here N=4)
- Here 3 accepts / 1 reject ⇒ majority is for **acceptance**

Outcome

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Outcome

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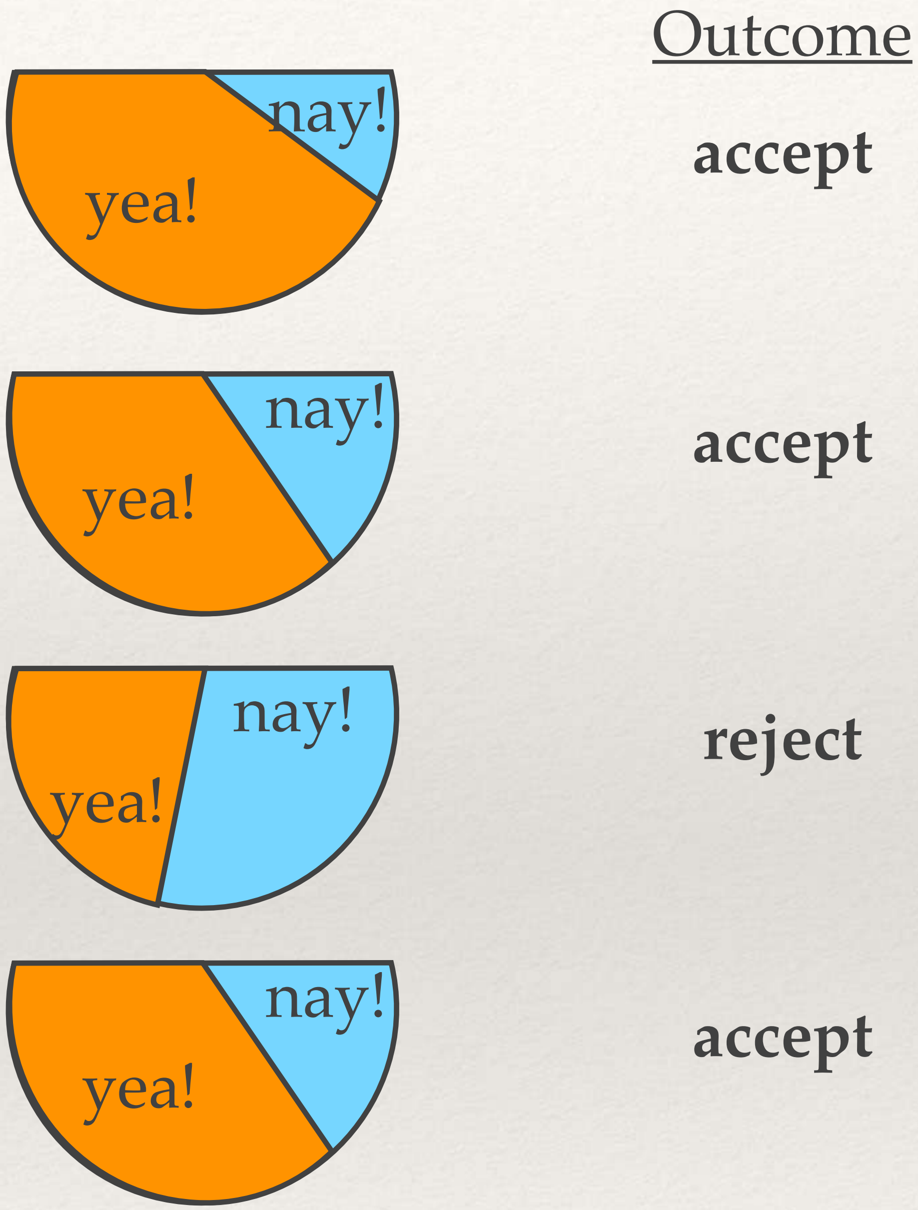
- … but how high?



Chernoff's bound

- Intuitive contents:
  Imagine Pr(**yes**) = p
  Then Pr(proportion of **yes**es among N voters is close to p)
  goes to 1 **exponentially fast** as $N \to \infty$.

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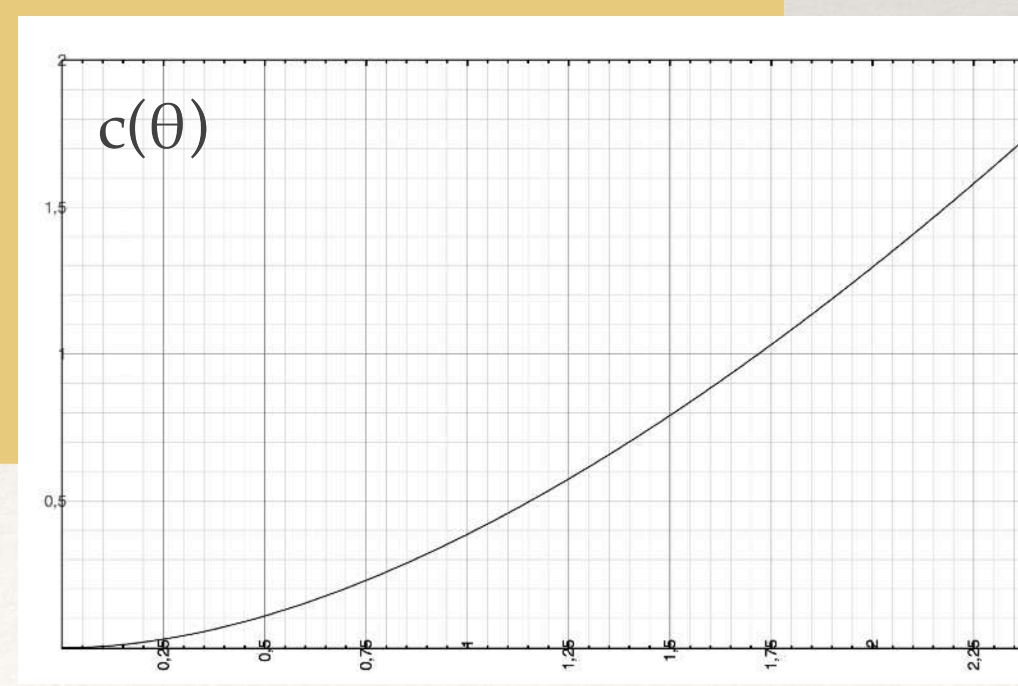

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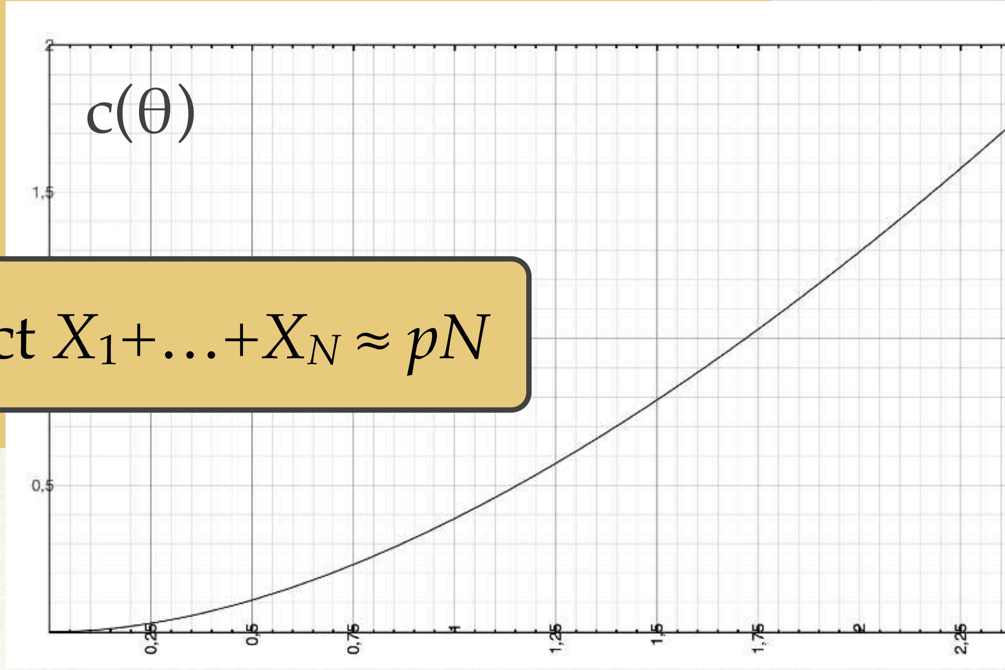

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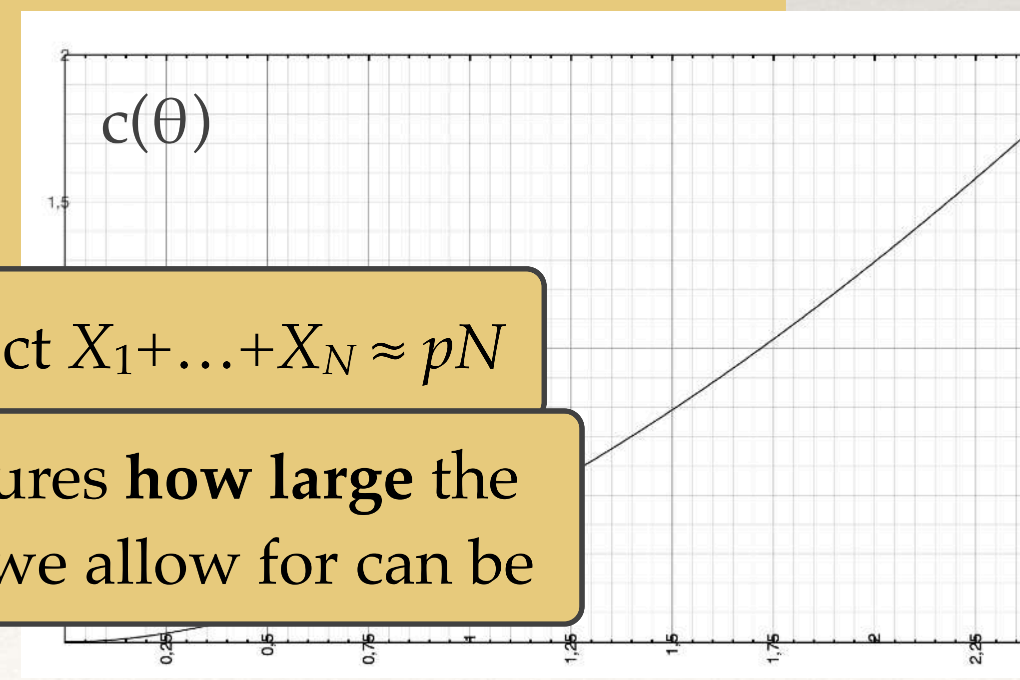

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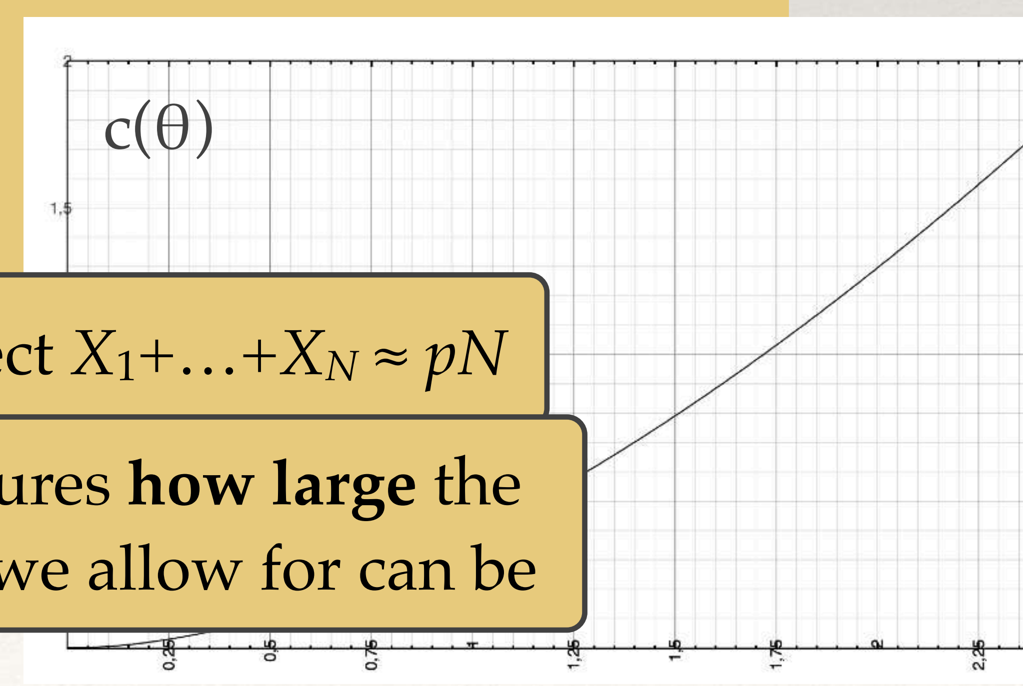

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For all practical purposes, $c(\theta) \approx \theta^2/3$

Proof of Chernoff's bound (1/4)

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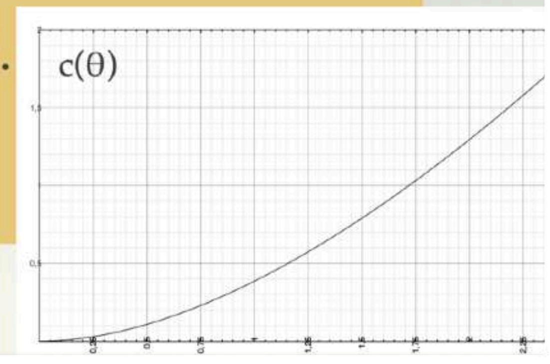

- Let $t, a > 0$ to be fixed later
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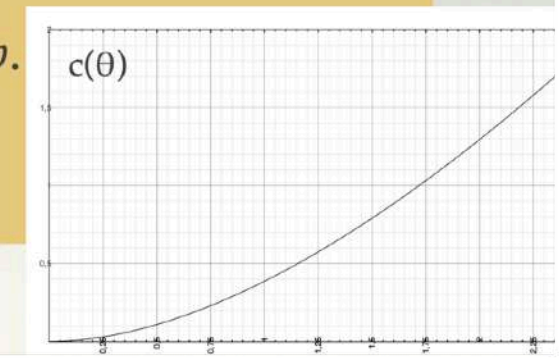
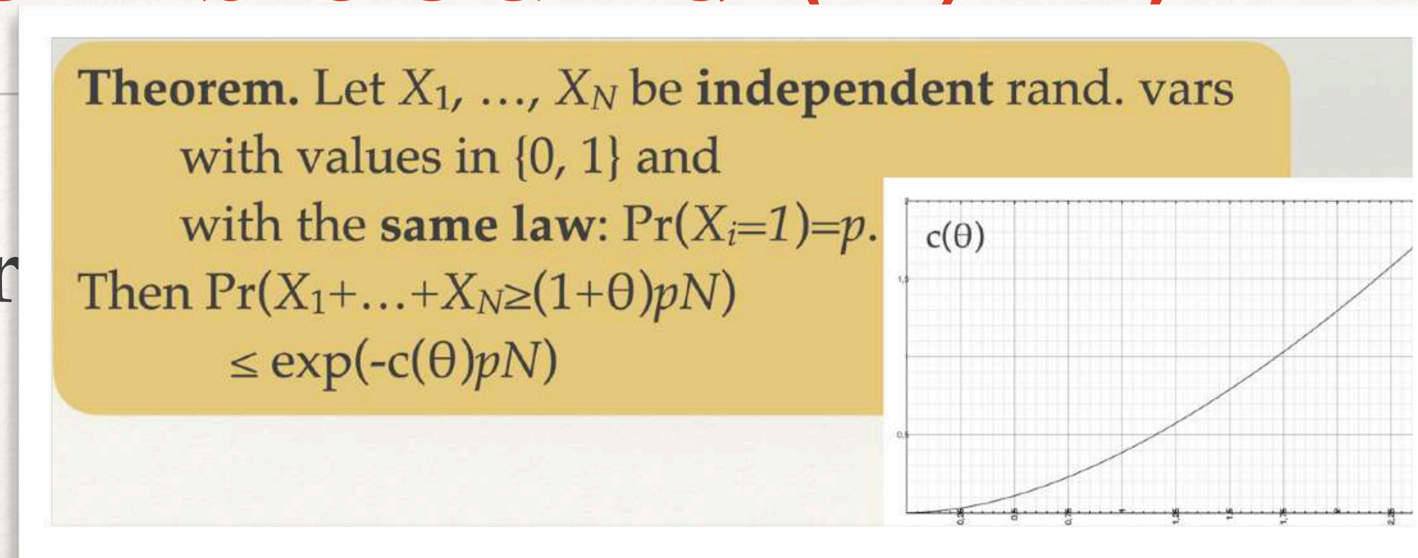


- Let t, $a > 0$ to be fixed later
- Define the rand. var
  $X = \exp(t(X_1+\ldots+X_N))$
- Note that $E(X) \leq \exp(tN) < \infty$, so we can use **Markov's inequality**:

$$\Pr(X \geq a.E(X)) \leq 1/a$$

Theorem (Markov's inequality).
Let X be a **non-negative real-valued** random variable with **finite** expectation $E(X)$. For every $a \geq 0$:
$\Pr(X \geq a.E(X)) \leq 1/a$.

Proof of Chernoff's bound (2/4)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- $\Pr(X \geq a.E(X)) \leq 1/a$ (from last slide)

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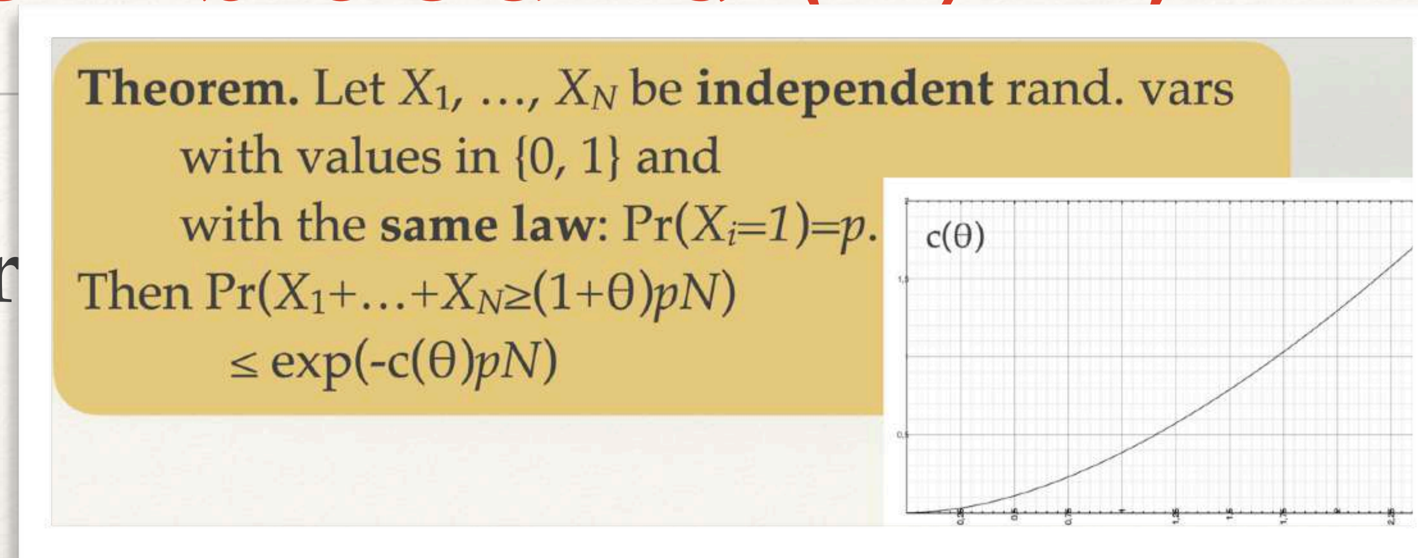

- Let t, $a > 0$ to be fixed later
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- Let us fix $a = \exp(t(1+\theta)pN) \;/\; \mathrm{E}(X)$, hence:

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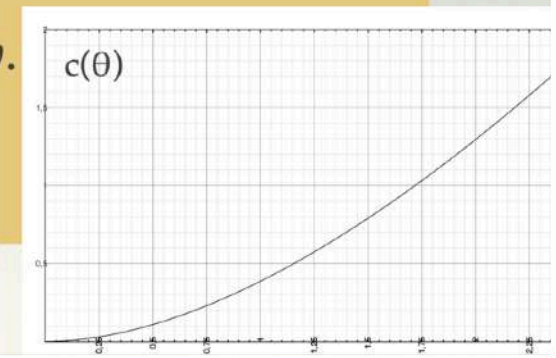

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- $$\Pr(X \geq a.\mathrm{E}(X)) \leq 1/a$$ (from last slide)
- Let us fix $a = \exp(t(1+\theta)pN)\ /\ \mathrm{E}(X)$, hence:
- $$\Pr(X \geq \exp(t(1+\theta)pN)) \leq \mathrm{E}(X)\exp(-t(1+\theta)pN))$$

This is just $\Pr(X_1+\ldots+X_N \geq (1+\theta)pN)$

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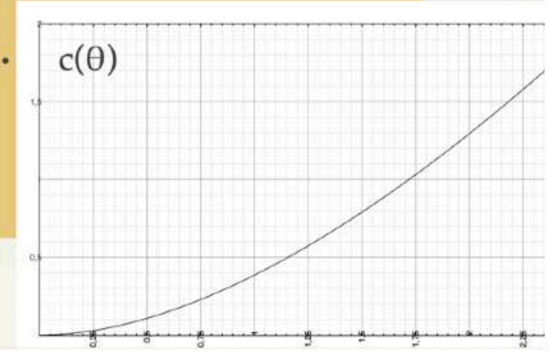

- Let $t>0$, to be fixed later
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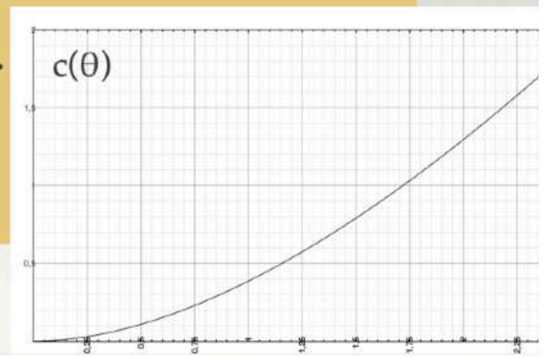

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- $\Pr(X_1+\ldots+X_N \geq (1+\theta)pN) \leq E(X) \exp(-t(1+\theta)pN))$ (from last slide)
- $E(X) = E(\Pi_{i=1}{}^N \exp(tX_i))$
  $= \Pi_{i=1}{}^N E(\exp(tX_i))$ (**independence**)
  $= \Pi_{i=1}{}^N (p \exp(t) + 1-p)$ (def. of the **law** of X_i)
  $= (p \exp(t) + 1-p)^N$
  $= (1+p(\exp(t)-1))^N \leq \exp((\exp(t)-1)pN)$

take logs:
$N \log(1+p(\exp(t)-1)) \leq Np(\exp(t)-1)$

Proof of Chernoff's bound (4/4)

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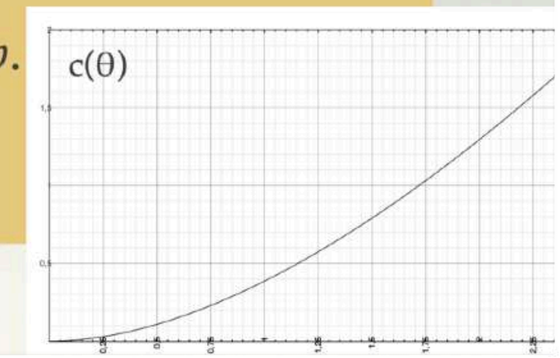

- Let $t>0$, to be fixed later
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- $\Pr(X_1+\ldots+X_N \geq (1+\theta)pN)$

$$\leq \exp((\exp(t)-1)pN)\ \exp(-t(1+\theta)pN))$$

(from last slide)

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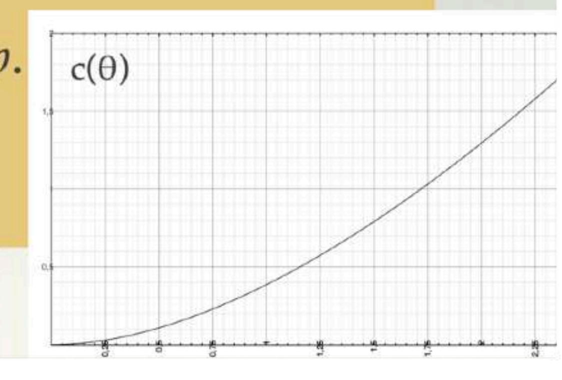

- Let $t>0$, to be fixed later
- $X = \exp(t(X_1+\ldots+X_N))$
- $\Pr(X_1+\ldots+X_N \geq (1+\theta)pN)$
  $$\leq \exp((\exp(t)-1)pN)\,\exp(-t(1+\theta)pN)$$
  (from last slide)
- Let $t = \log(1+\theta)$, so $(\exp(t)-1)pN = \theta pN$, hence

Proof of Chernoff's bound (4/4)

Theorem. Let $X_1, \ldots, X_N$ be **independent** rand. vars with values in $\{0, 1\}$ and with the **same law**: $\Pr(X_i=1)=p$.
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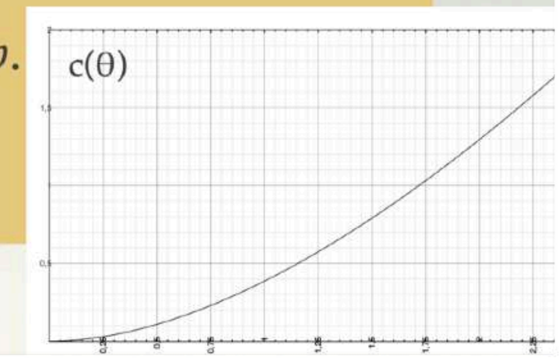

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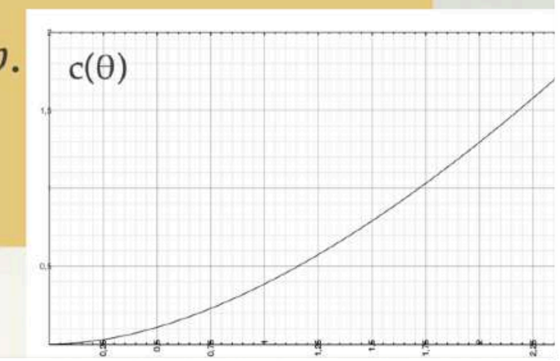

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Done! □

Call this $-c(\theta)$

A few properties of $c(\theta) = -\theta + (1+\theta)\log(1+\theta)$

- **Prop 1.** $c(\theta)$ is monotonic (for $\theta \geq 0$)
- *Proof.* $c'(\theta) = \log(1+\theta) \geq 0$

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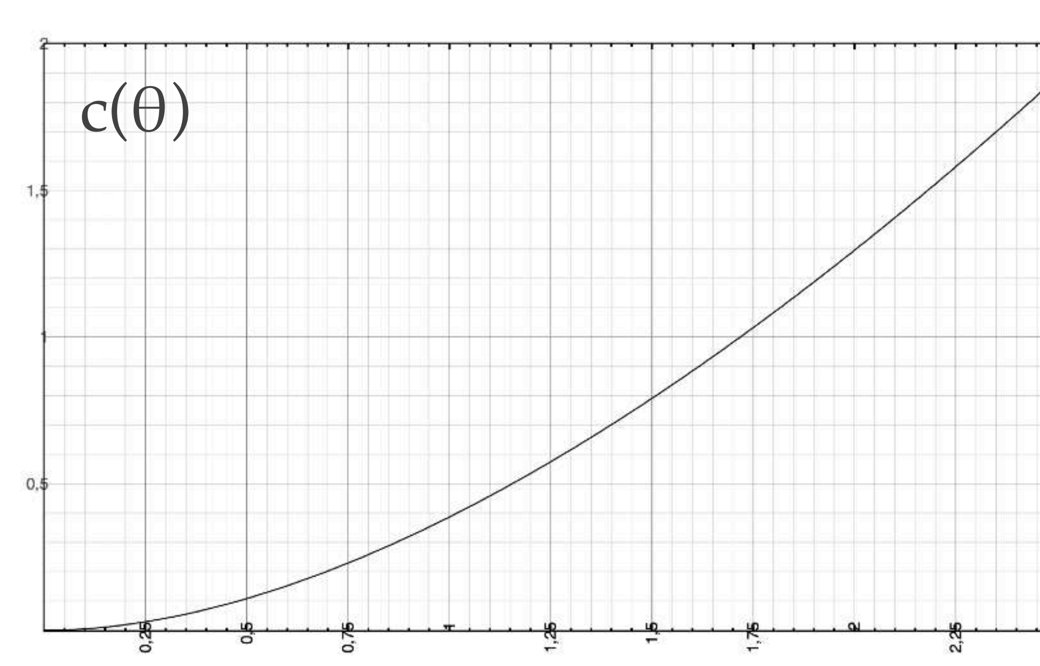

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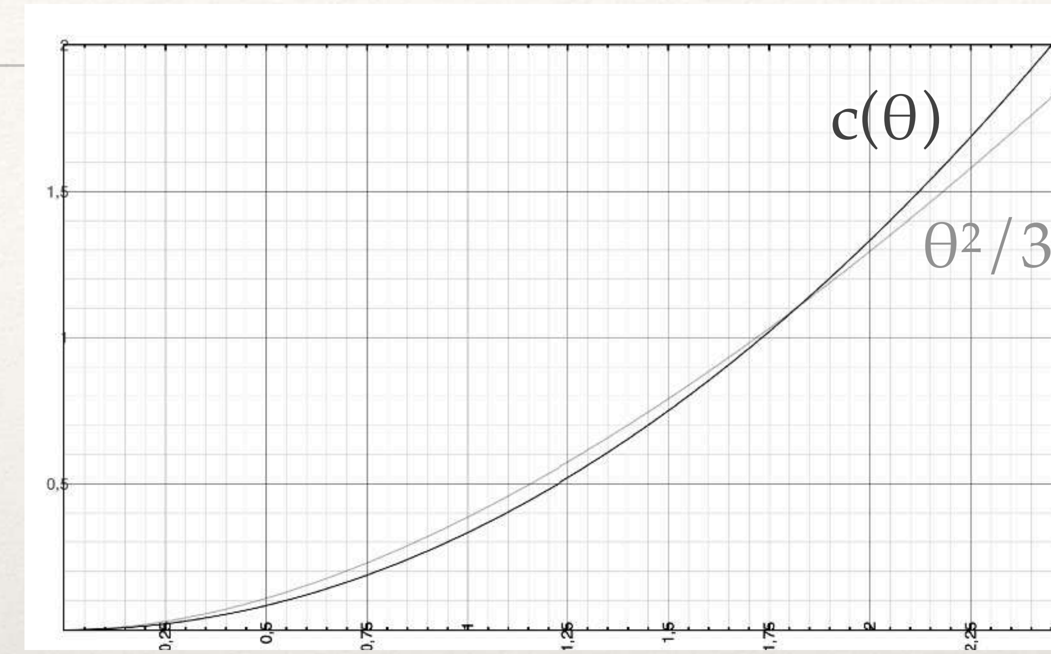

- **Prop 1.** $c(\theta)$ is monotonic (for $\theta \geq 0$)
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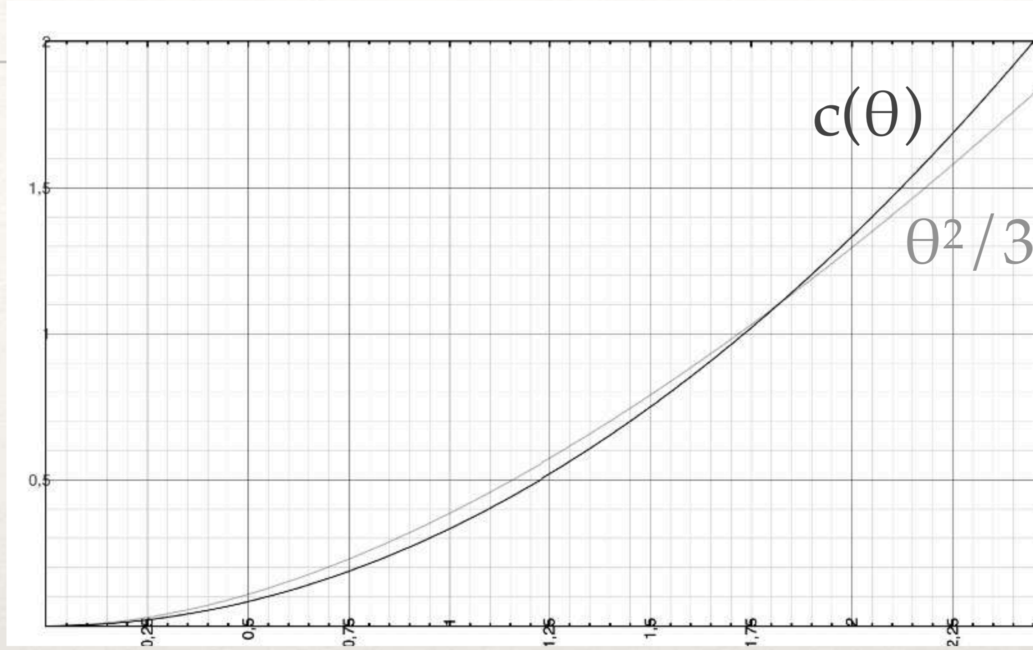

- **Prop 1.** $c(\theta)$ is monotonic (for $\theta \geq 0$)
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  $c'(0)=0$ (recall $c'(\theta) = \log(1+\theta)$)

  $c''(0)=1$ ($c''(\theta) = 1/(1+\theta)$)

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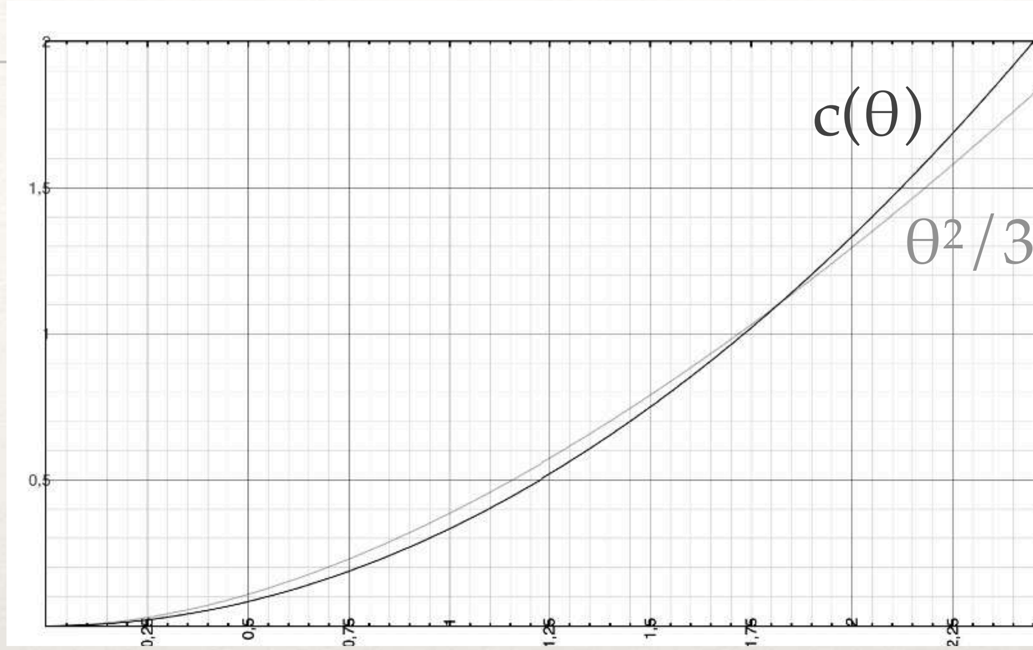

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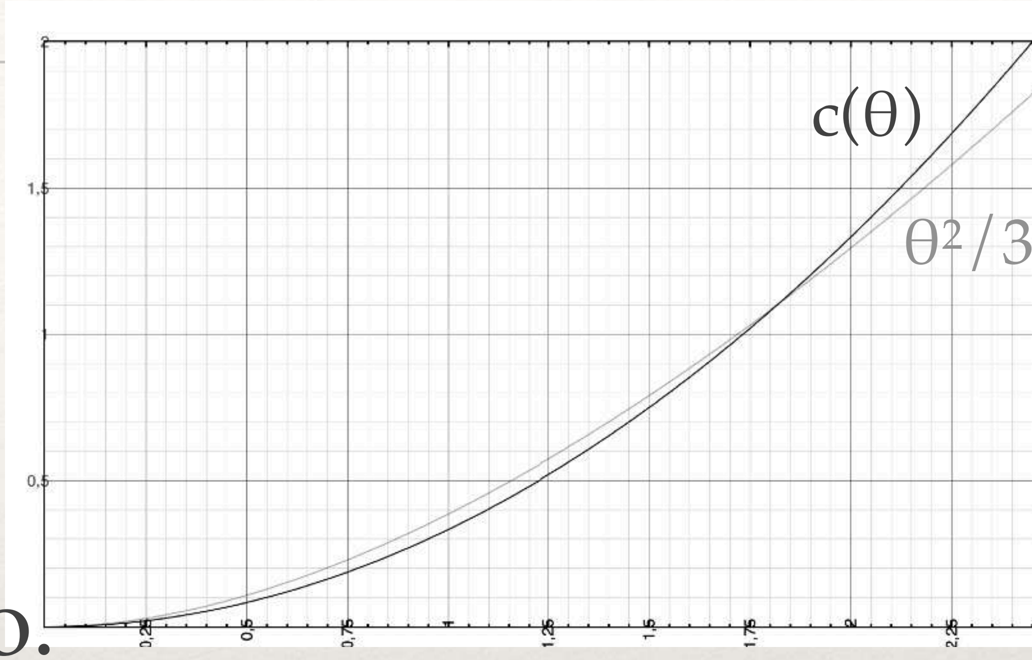
- So $c(\theta) = \theta^2/2 - \theta^3/6 + c^{(4)}(\theta_0)/24$ for some $0\leq\theta_0\leq\theta$ (Taylor)

  $\geq \theta^2/2 - \theta^3/6$ (since $c^{(4)}(\theta) = 2/(1+\theta)^3 \geq 0$)

  $\geq \theta^2/3$ (since $\theta\leq 1$) □

A few properties of $c(\theta) = -\theta + (1+\theta)\log(1+\theta)$



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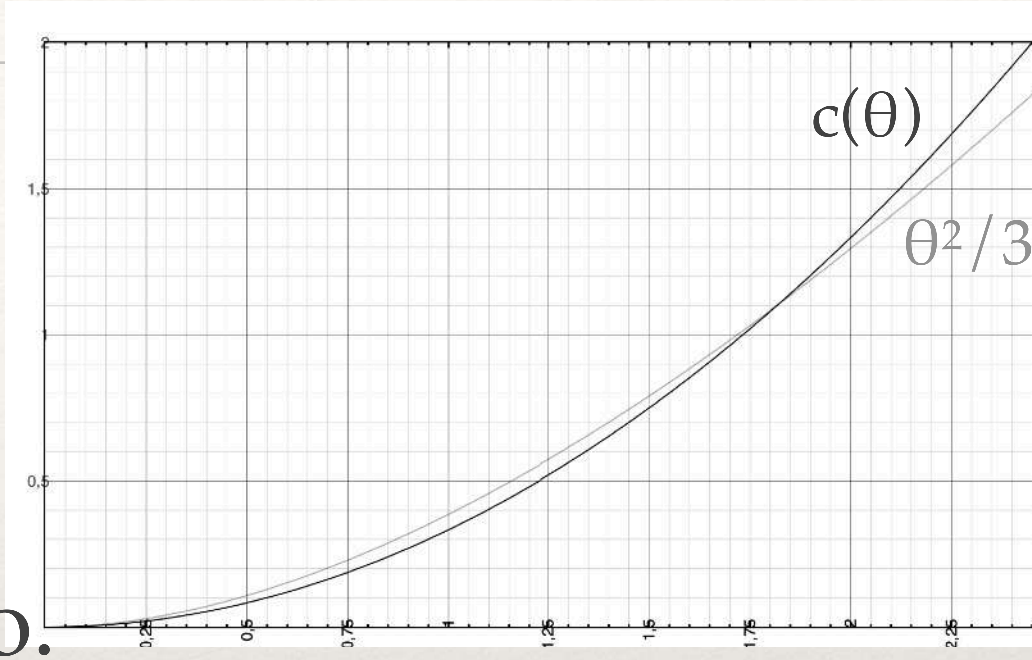

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- Proof. $c(\theta)/(1+\theta) = -\theta/(1+\theta) + \log(1+\theta)$

  Derivative: $-1/(1+\theta)^2 + 1/(1+\theta) = \theta/(1+\theta)^2 \geq 0$ □

Application to voting (1/4)

- Assume that $\Pr_r(\mathcal{M}(x,r) \text{ errs}) \leq 1/3$,
  what is the probability P that more than 1/2 of N votes
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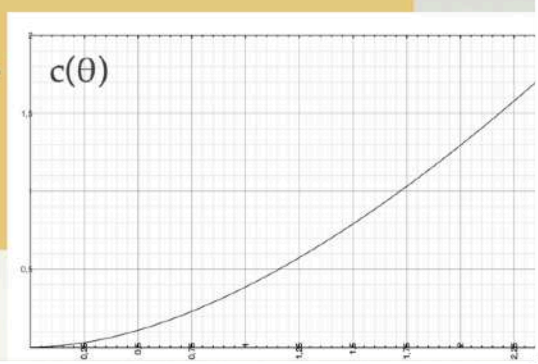

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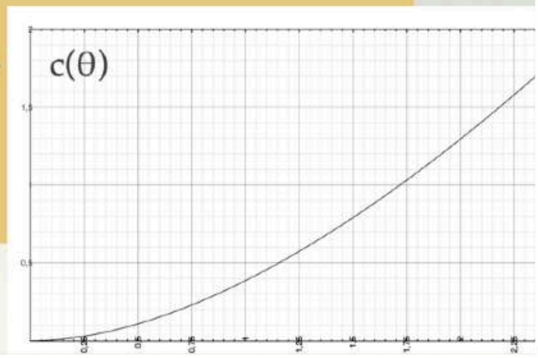

- Let $X_i = 1$ iff $\mathcal{M}(x,r_i)$ errs:
  all assumptions satisfied
  with $p{\leq}1/3$
- Take $\theta{=}1/(2p){-}1$, so $(1{+}\theta)p = 1/2$: $\quad$ $\mathrm{P} \leq \exp(-c(\theta)pN)$

Application to voting (2/4)

- Assume that $\Pr_r(\mathcal{M}(x,r)\text{ errs}) \leq 1/3$,
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- Take $\theta=1/(2p)-1$, so $(1+\theta)p = 1/2$: $\quad$ $P \leq \exp(-c(\theta)pN)$
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Application to voting (2/4)

- Assume that $\Pr_r(\mathcal{M}(x,r) \text{ errs}) \leq 1/3$, what is the probability P that more than 1/2 of N votes $\mathcal{M}(x,r_1)$, …, $\mathcal{M}(x,r_N)$ err?
- Take $\theta=1/(2p)-1$, so $(1+\theta)p = 1/2$: $\quad$ $P \leq \exp(-c(\theta)pN)$
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- I.e., $P \leq \exp(-c(\theta)/(1+\theta) \,.\, 1/2\, N)$

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  (since $p \leq 1/3$, so $\theta \geq 1/2$; plus Prop 3)

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- $\leq \exp(-c(1/2)/(3/2) \,.\, 1/2\, N)$
  (since $p \leq 1/3$, so $\theta \geq 1/2$; plus Prop 3)
- $\leq \exp(-(1/2)^2/3/(3/2) \,.\, 1/2\, N)$ (Prop 2)
  $= \exp(-N/36)$

Prop 1. $c(\theta)$ is monotonic (for $\theta \geq 0$)

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Application to voting (3/4)

- Assume that $\Pr_r(\mathcal{M}(x,r)\text{ errs}) \leq 1/3$, what is the probability P that more than 1/2 of N votes $\mathcal{M}(x,r_1), \ldots, \mathcal{M}(x,r_N)$ err?
- Answer: at most $\exp(-N/36)$

Error reduction for **BPP**

- First, a useful trick.
  Let us say that $\mathcal{M}(x,r)$ **errs**
  iff ($x \in L$ and $\mathcal{M}(x,r)$ rejects)
  or ($x \notin L$ and $\mathcal{M}(x,r)$ accepts)
- (That used to be implicit.)
- Then:

A language L is in **BPP** if and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):

if $x \in L$ then $\Pr_r$ [$\mathcal{M}(x,r)$ accepts] $\geq 2/3$

if $x \notin L$ then $\Pr_r$ [$\mathcal{M}(x,r)$ accepts] $\leq 1/3$.

A language L is in **BPP**(ε) nd only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):

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if $x \notin L$ then $\Pr_r$ [$\mathcal{M}(x,r)$ accepts] $\leq \varepsilon$

error = ε

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$$\Pr_r [\mathcal{M}(x,r) \text{ errs}] \leq \varepsilon.$$

error = ε

Error reduction for **BPP**

- Let L be in **BPP**, as here →
- Build new rand. TM $\mathcal{M}'$ by:
- ```
  yeas := 0
  for i=1 to N:
      draw r at random
      if M(x,r) accepts:
          yeas++
  ```
  accept if yeas$\geq N/2$, else **reject**

A language L is in **BPP** if and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):

$$\Pr_r\ (\mathcal{M}(x,r) \text{ errs}) \leq 1/3.$$

A language L is in **BPP(ε)** and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):

$$\Pr_r\ (\mathcal{M}(x,r) \text{ errs}) \leq \varepsilon.$$

error = ε
  ```

Error reduction for **BPP**

- Let L be in **BPP**
- Build new rand. TM $\mathcal{M}$' by:
- `yeas` := 0
  `for` i=1 to N:
      draw r at random
      if $\mathcal{M}(x,r)$ accepts:
          `yeas`++
  accept if `yeas`≥$N/2$, else **reject**
- $\mathcal{M}$' errs on input x iff at least half of the calls to $\mathcal{M}(x,r)$ err
- That happens with probability
  $\leq \exp(-N/36)$
- … ≤ ε provided that
  we pick $N \geq -36 \log \varepsilon$

Error reduction for **BPP**

- Let L be in **BPP**
- Build new rand. TM $\mathcal{M}'$ by:
- `yeas` := 0
  `for` i=1 to N:
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  if $\mathcal{M}(x,r)$ accepts:
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- $\mathcal{M}'$ errs on input x iff at least half of the calls to $\mathcal{M}(x,r)$ err
- That happens with probability $\leq \exp(-N/36)$
- … ≤ ε provided that we pick $N \geq -36 \log \varepsilon$

Note: if $\mathcal{M}$ runs in polytime $p(n)$,
then $\mathcal{M}'$ runs in **polytime** = -36 log ε $p(n)$ + cst.

Error reduction for **BPP**

- Hence **BPP**$(=$ **BPP**$(1/3)) \subseteq$ **BPP**(ε) for ε arbitrarily close to 0

Error reduction for **BPP**

- Hence **BPP**(= **BPP**(1/3)) ⊆ **BPP**(ε)
  for ε arbitrarily close to 0
- By a similar argument, we can replace 1/3 by any η, 0<η<1/2, so **BPP**(η) ⊆ **BPP**(ε)
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- Recalling that **BPP**(ε) $\subseteq$ **BPP**(η) if ≤ η, we obtain:

Error reduction for **BPP**

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- **Theorem.** For every ε, 0< ε<1/2, **BPP**=**BPP**(ε).

Error reduction for **BPP**

- Hence **BPP**(= **BPP**(1/3)) $\subseteq$ **BPP**(ε)
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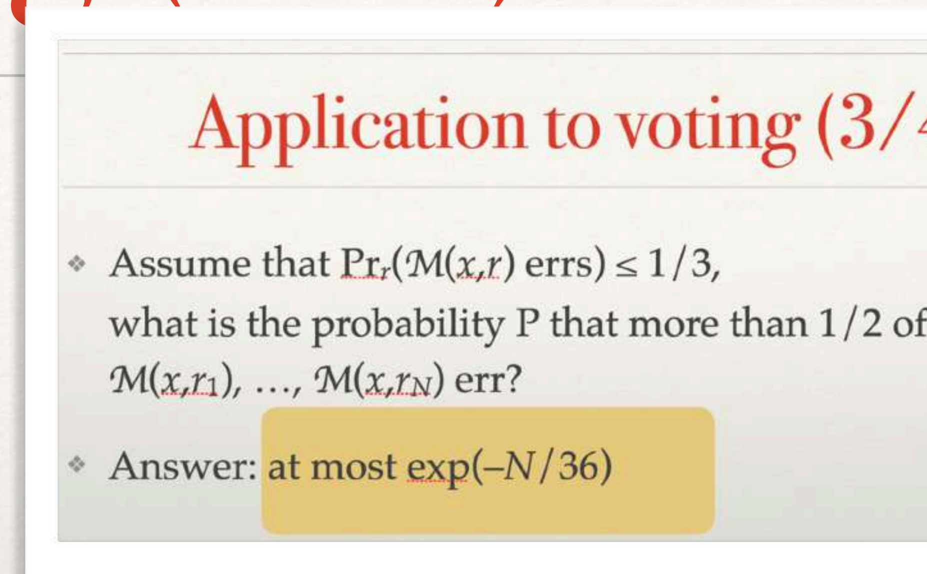
- Recalling that **BPP**(ε) $\subseteq$ **BPP**(η) if ≤ η, we obtain:
- **Theorem.** For every ε, 0< ε<1/2, **BPP**=**BPP**(ε).
- … but can we do better?

Application to voting (4/4)

- Assume that $\Pr_r(\mathcal{M}(x,r)\text{ errs}) \leq 1/3$,
  how large should N be so that
  the probability P that more than $1/2$
  of N votes $\mathcal{M}(x,r_1), \ldots, \mathcal{M}(x,r_N)$ err
  is $\leq 1/2^{q(n)}$?


Application to voting (3/4

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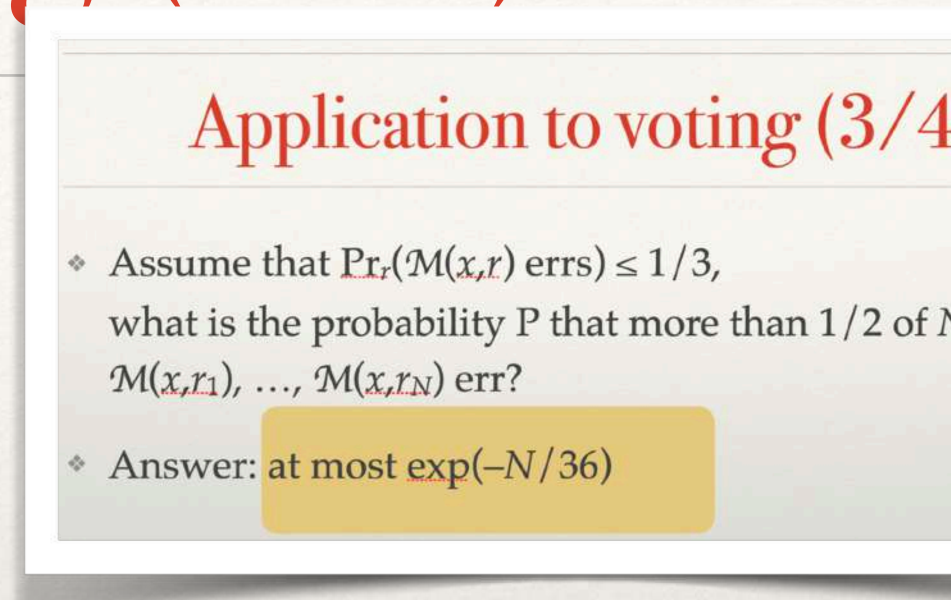
Answer: at most $\exp(-N/36)$

Application to voting (4/4)



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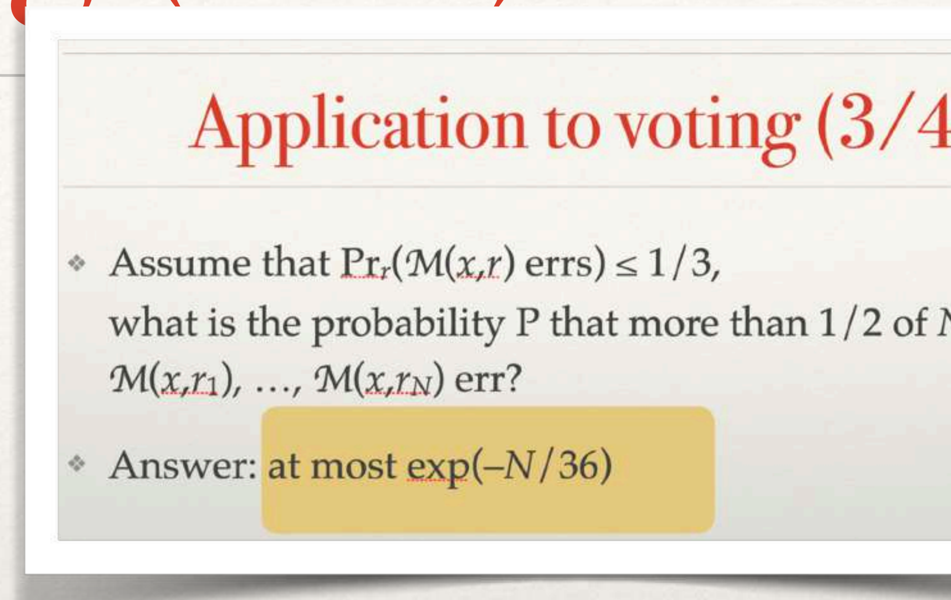
- Assume that $\Pr_r(\mathcal{M}(x,r)$ errs$) \le 1/3$, **how large** should N be so that the probability P that more than $1/2$ of N votes $\mathcal{M}(x,r_1), \ldots, \mathcal{M}(x,r_N)$ err is $\le 1/2^{q(n)}$?
- Answer: at least $36\, q(n) \log 2$

Application to voting (4/4)



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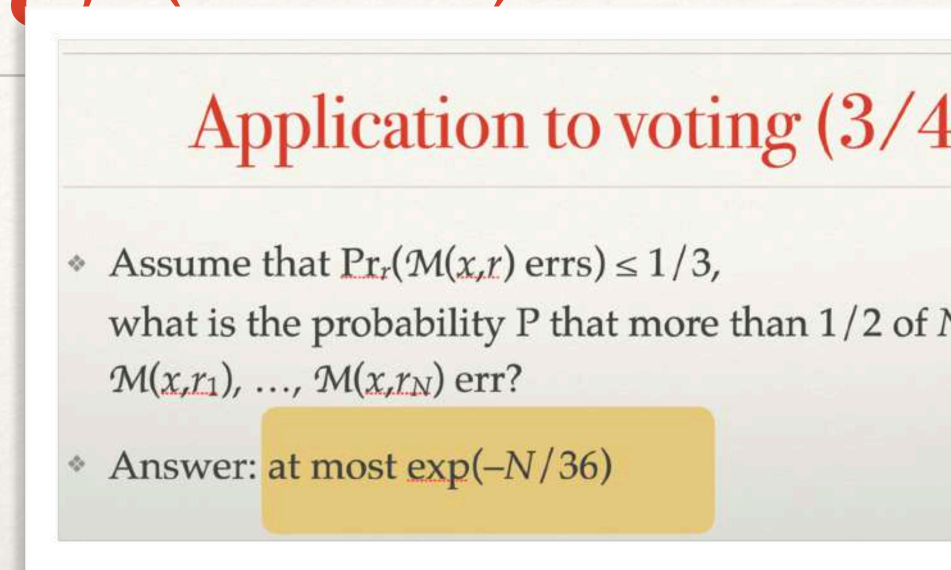
- Answer: at least $36\, q(n) \log 2$
- *Proof.* $\exp(-N/36) \leq 1/2^{q(n)}$ iff
  $-N/36 \leq -q(n) \log 2$

Application to voting (4/4)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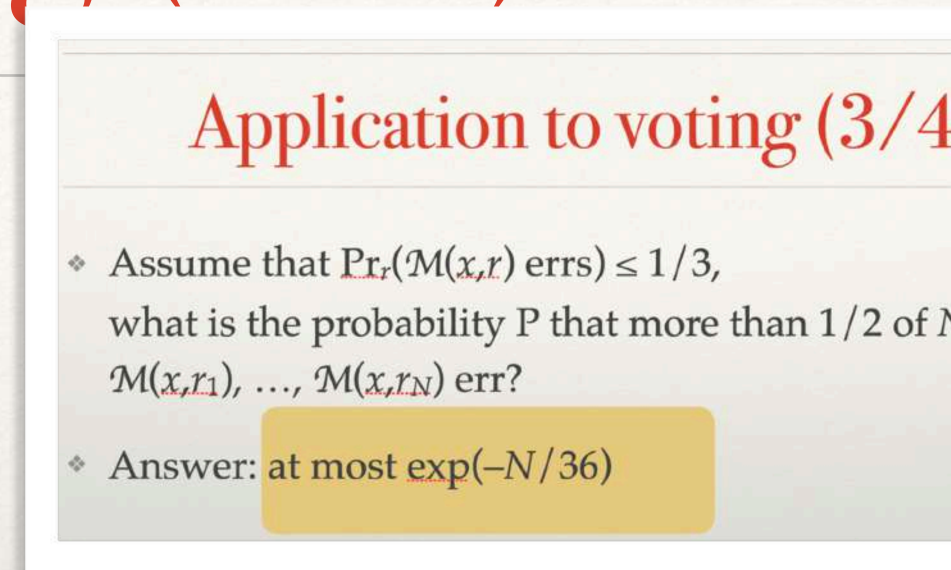
The only magical formula
you'll need to remember
for error reduction by majority voting

- *Proof.* $\exp(-N/36) \le 1/2^{q(n)}$ iff

$$-N/36 \le -q(n) \log 2$$

Application to voting (4/4)



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- *Proof.* $\exp(-N/36) \le 1/2^{q(n)}$ iff
  $-N/36 \le -q(n) \log 2$

The only magical formula
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Note: if $q(n)$ is polynomial,
this is polynomial, too

Error reduction for **BPP** revisited

- Let L be in **BPP**
- Build new rand. TM $\mathcal{M}$' by:
- `yeas` := 0
  `for` i=1 to $N := 36\, q(n) \log 2$:
  draw r at random
  if $\mathcal{M}(x,r)$ accepts:
  `yeas++`
  accept if `yeas`$\geq N/2$, else **reject**

- $\mathcal{M}$' errs on input x iff at least half of the calls to $\mathcal{M}(x,r)$ err
- That happens with probability
  $\leq 1/2^{q(n)}$

Error reduction for **BPP** revisited

- Let L be in **BPP**
- Build new rand. TM $\mathcal{M}$' by:
- `yeas` := 0
  `for` i=1 to $N := 36\, q(n) \log 2$:
  draw r at random
  if $\mathcal{M}(x,r)$ accepts:
  `yeas`++
  accept if `yeas`$\geq N/2$, else **reject**

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Note: if $\mathcal{M}$ runs in polytime $p(n)$, and $q(n)$ is polynomial
then $\mathcal{M}$' runs in **polytime** = $O(q(n)\, p(n) \log n)$ [log n for operations on the counter i]

Error reduction for **BPP** revisited

A language L is in **BPP**(ε) nd only if
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The new landscape

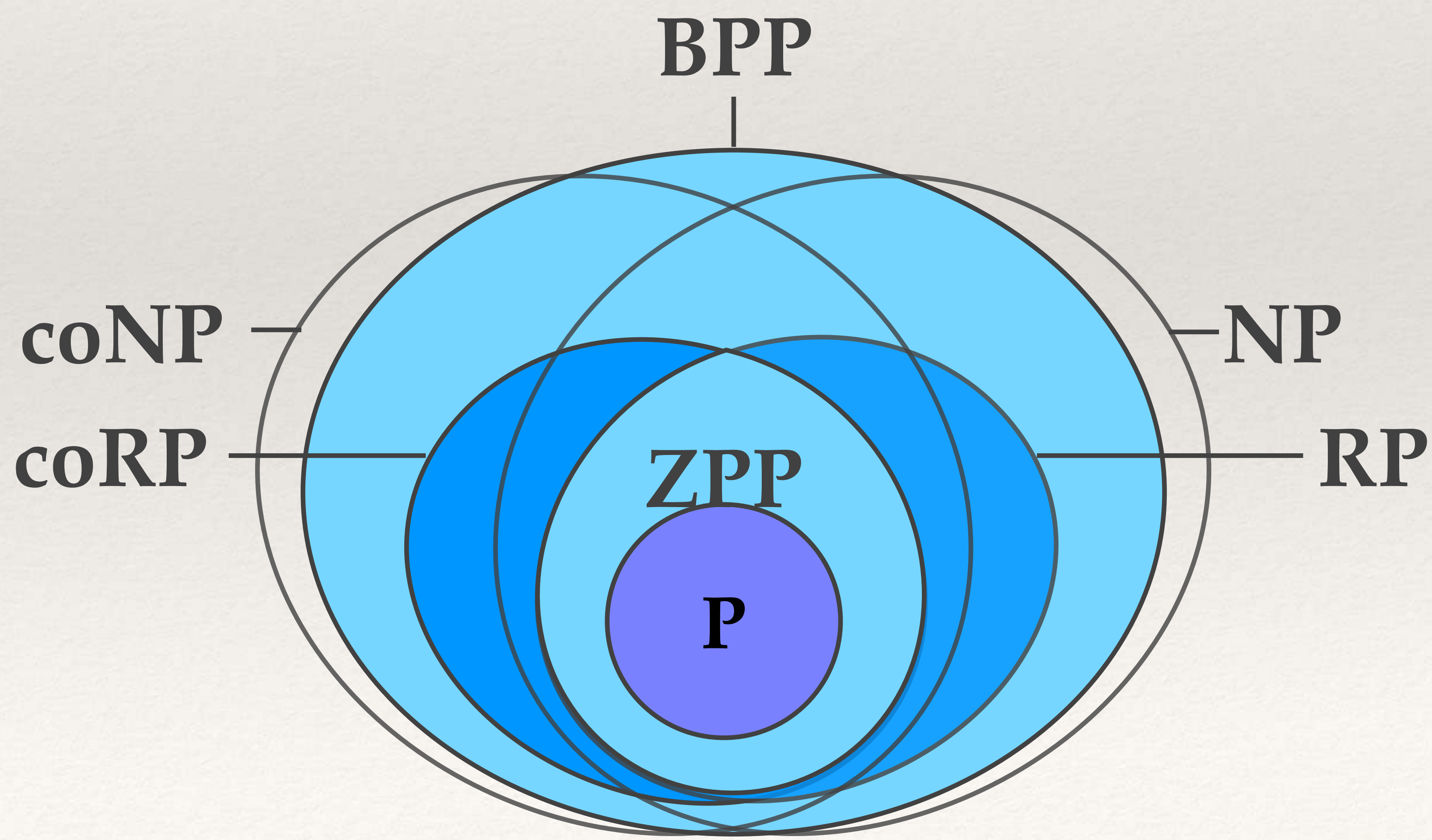
BPP vs. other complexity classes

- Both **RP** and **coRP** are included in **BPP**

  (if you make a mistake with prob. 0, then this prob. is ≤1/3!)

- **BPP** is closed under complements: **BPP=coBPP**

  (easy)



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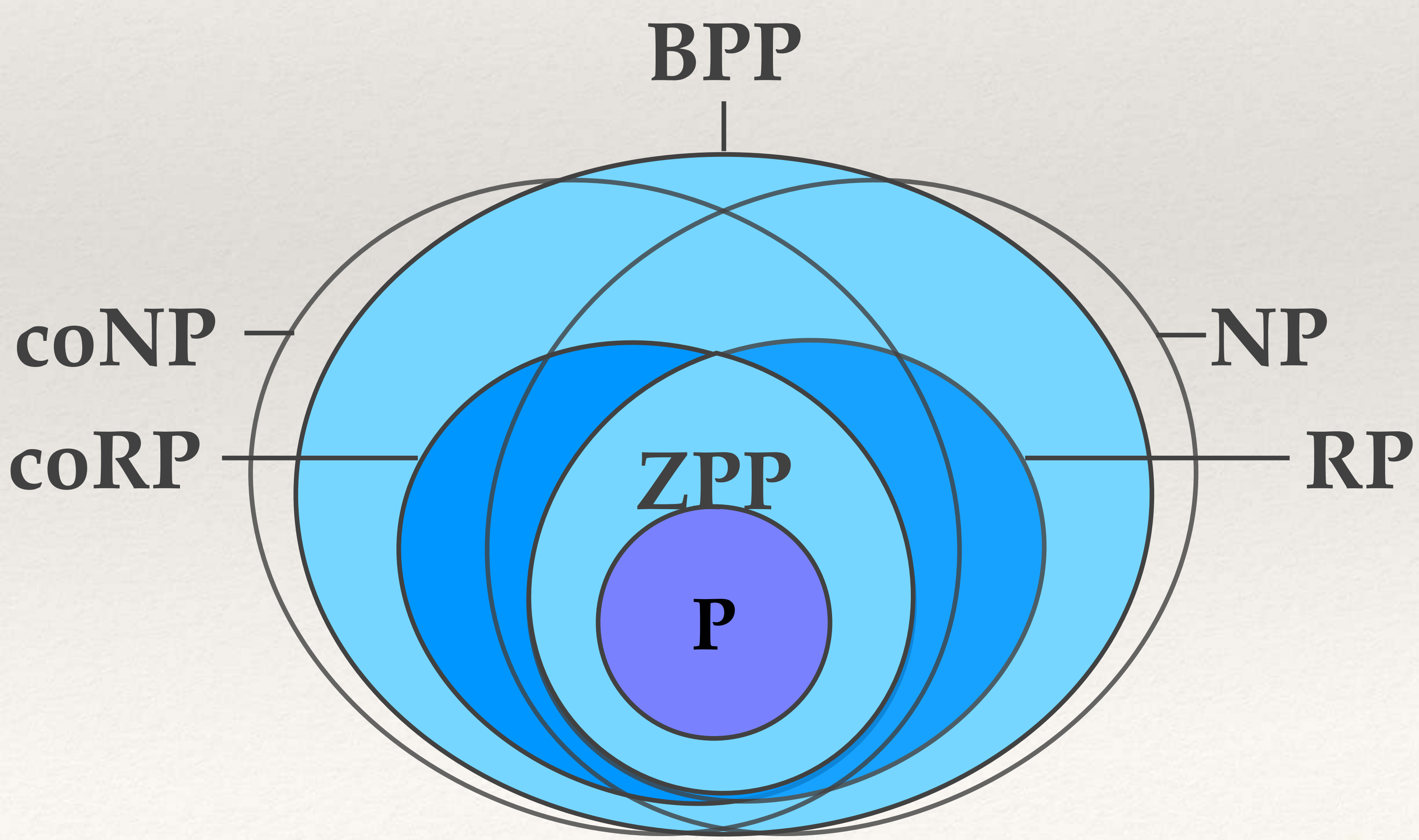
- … but what is the relation between **BPP**, **NP**, **coNP**, etc.?



BPP cannot be too large

- It is unknown whether **BPP** $\subseteq / \supseteq$ **NP** (eqv., **coNP**)
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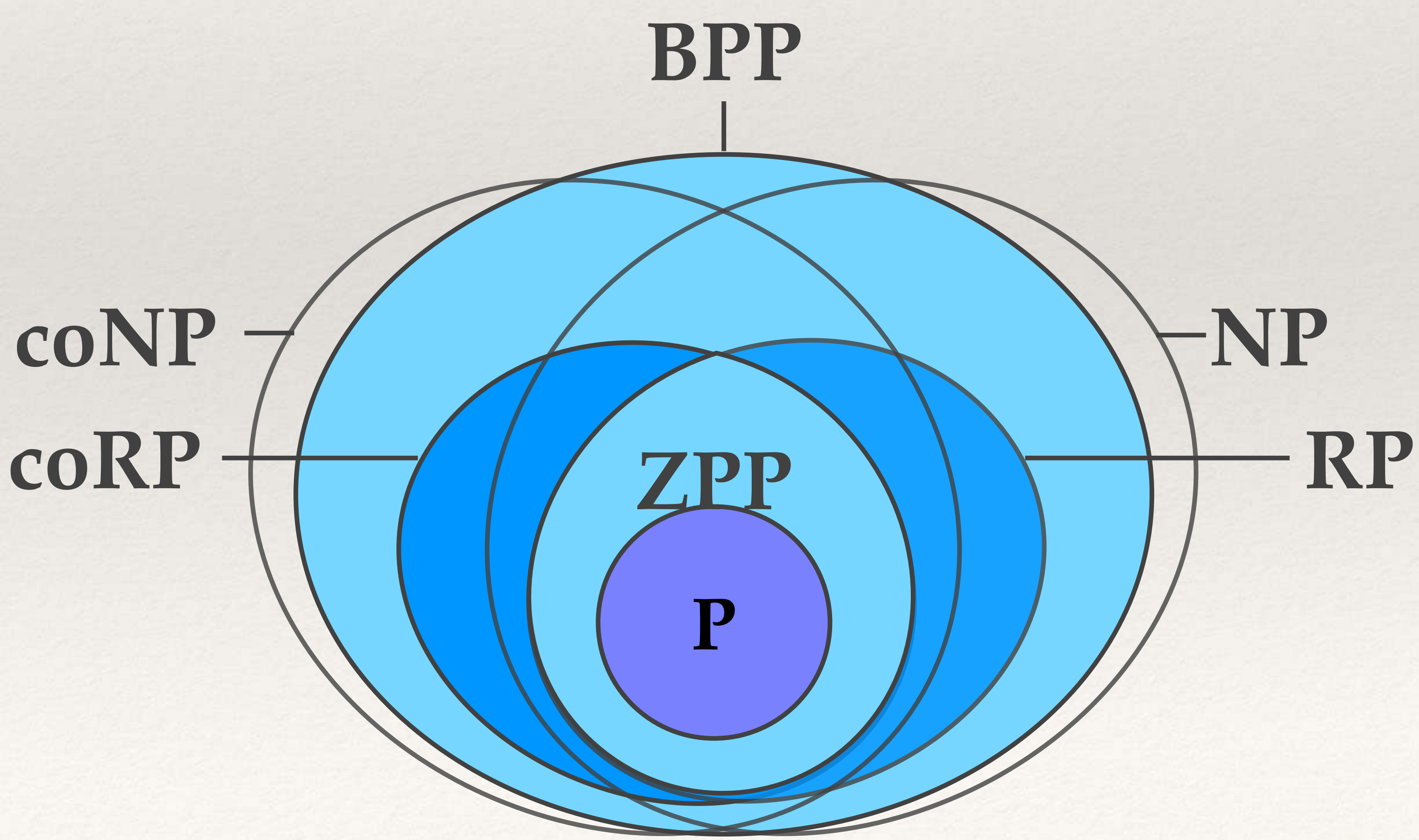

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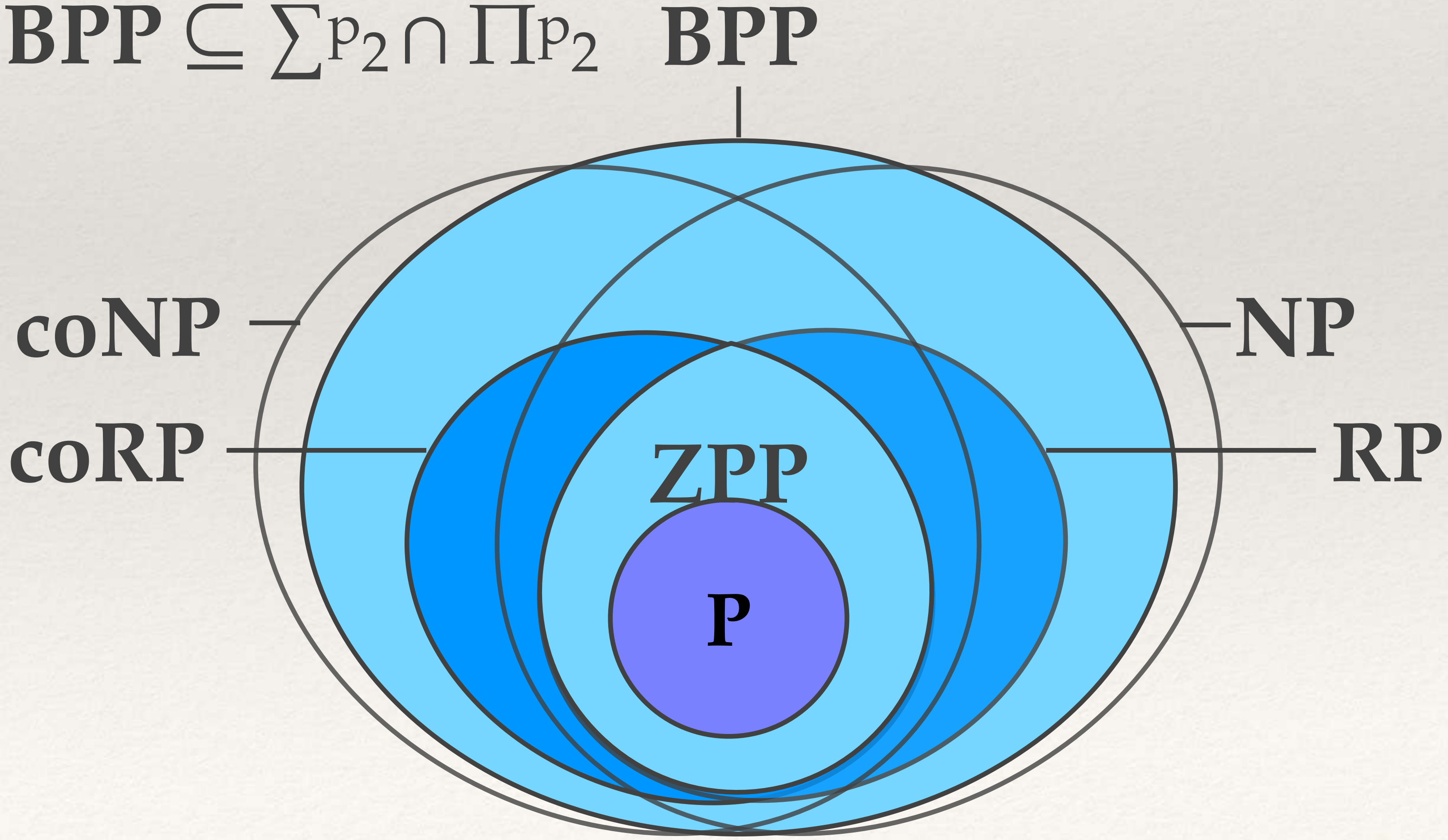

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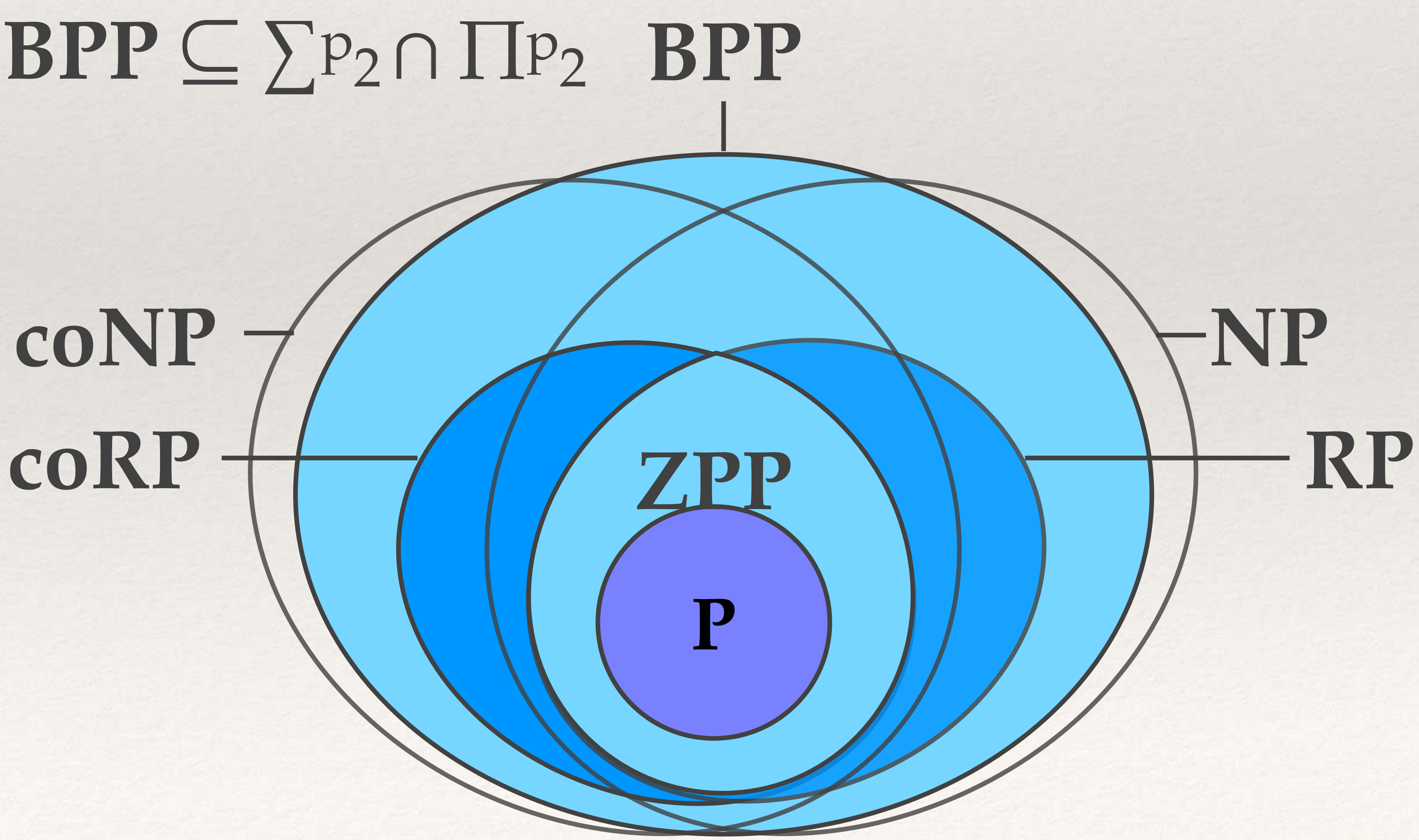

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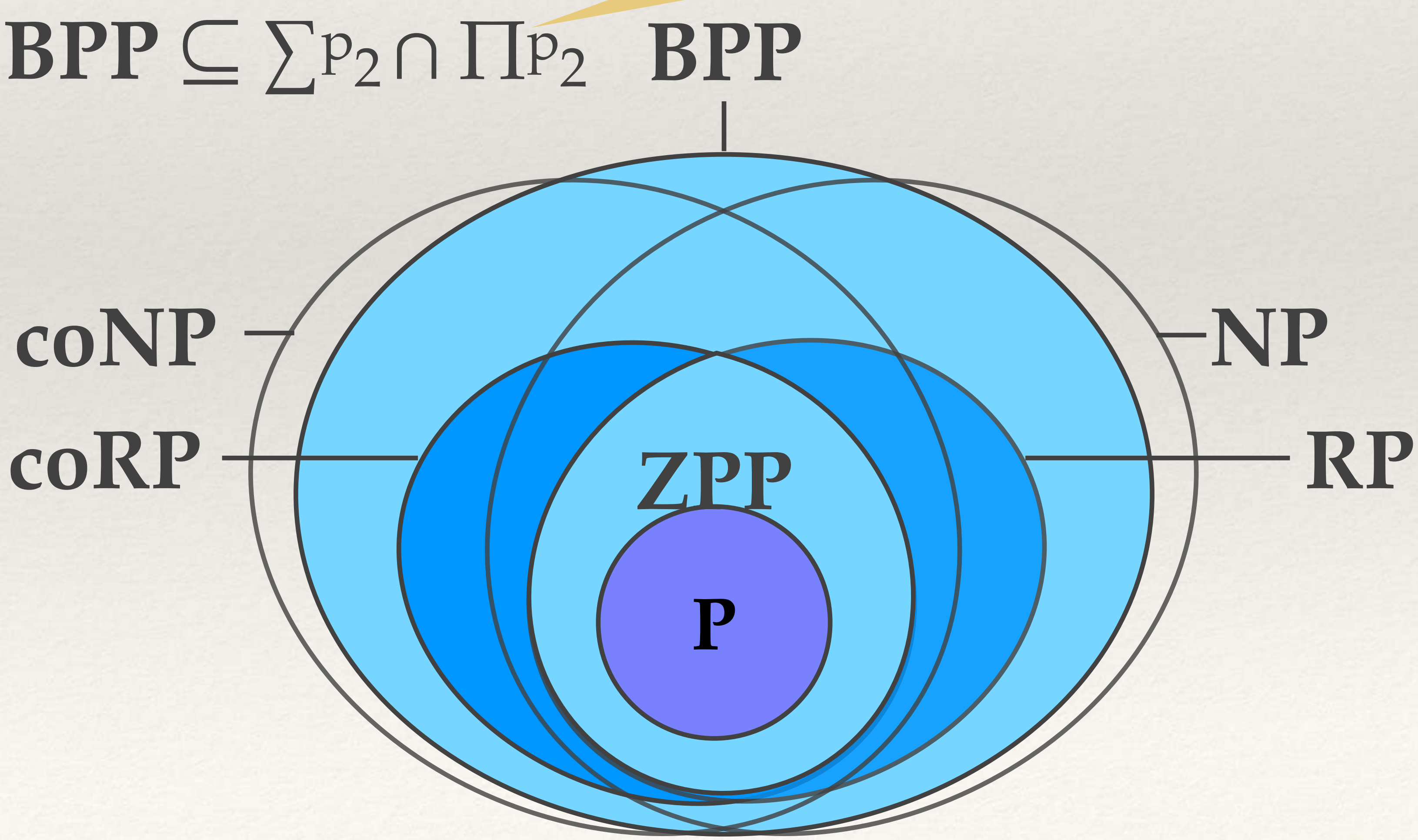

The Sipser-Gács-Lautemann Theorem

http://lpcs.math.msu.su/~ver/photo_album/Collegues/lautemann+allender+wagn

https://gravatar.com/avatar/dc36e666740ff9480eb738e556c887a4?s=200

https://upload.wikimedia.org/wikipedia/commons/thumb/3/34/MIT-Science_Sipser_Michael.jpg/440px-MIT-Science_Sipser_Michael.jpg

The Sipser-Gács-Lautemann theorem

- **Theorem (Sipser-Gács-Lautemann, Prop. 1.24.)**
  BPP $\subseteq \Sigma^p_2 \cap \Pi^p_2$.
- *Proof sketch.*
  It is enough to prove **BPP** $\subseteq \Sigma^p_2$.
  Proceeds by **derandomization**.
  In order to do so, we will to prove
  the **existence** of something
  Funnily, this will involve Erdös' **probabilistic method.**

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$\Sigma^{P}_{2} = \exists \cdot$ **coNP**
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To prove that $\exists\, t, P(t)$,
just show that $\Pr_t(P(t)) \neq 0$, or
equivalently that $\Pr_t(\neg P(t)) < 1$

Lautemann's trick

- Let $L \in \mathbf{BPP}$, decided with error
  $\varepsilon = 1/2^n$ (not $1/3$)
  in polytime $p(n)$

A language L is in $\mathbf{BPP}(\varepsilon)$ nd only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):

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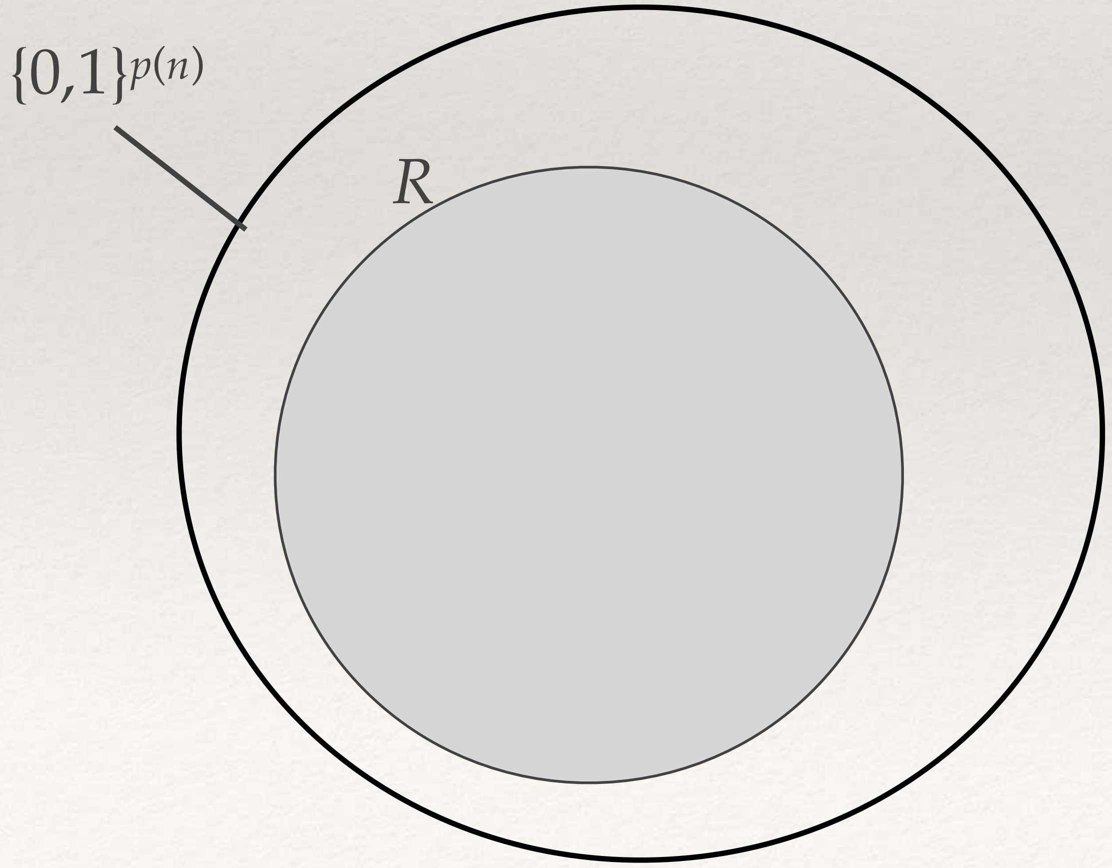

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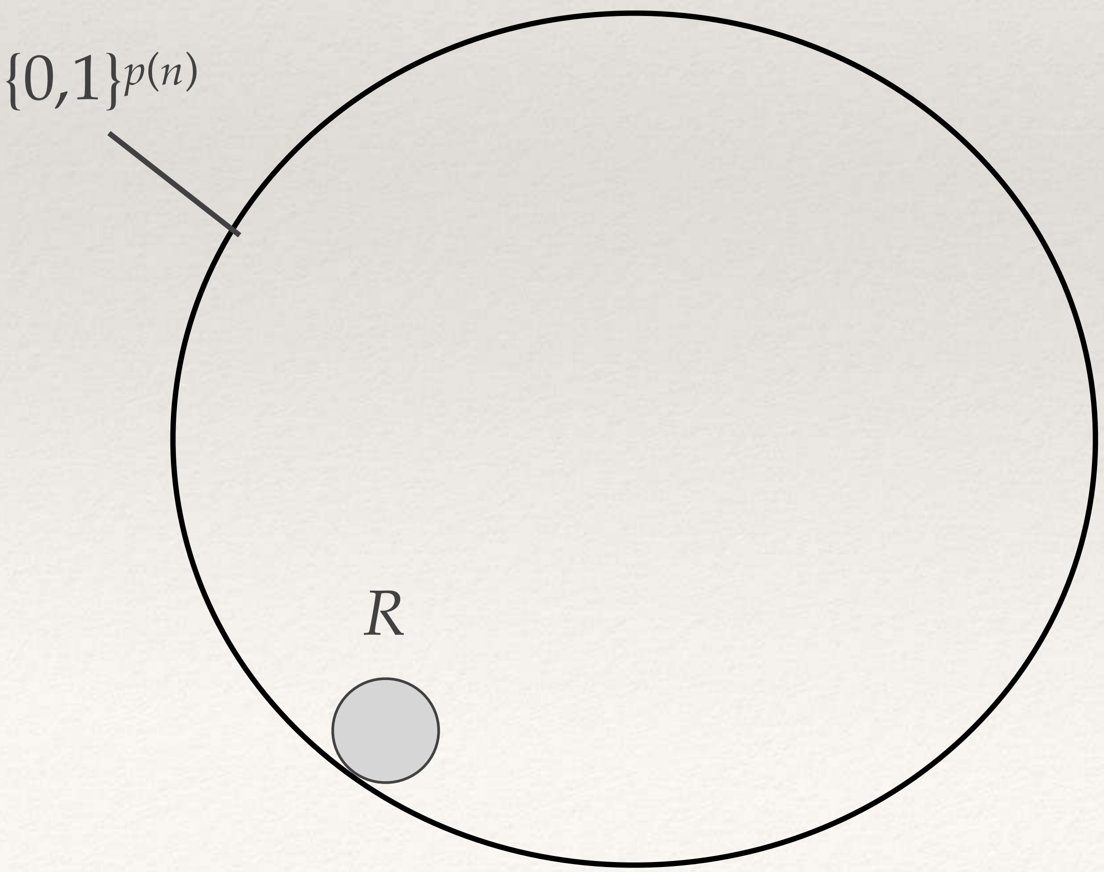
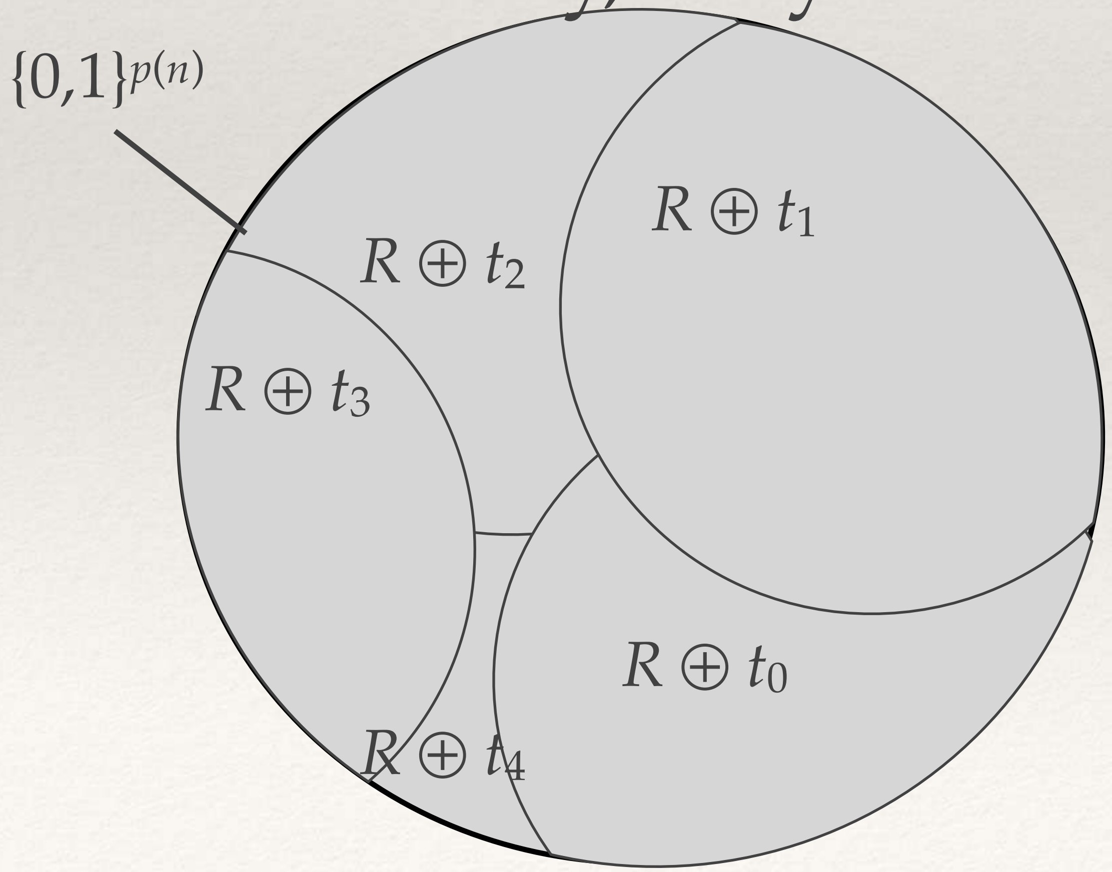


or **tiny**, if $x \notin L$
(covers a proportion $\leq 1/2^n$ of the whole space)

Lautemann's trick

- $R = \{r \in \{0{,}1\}^{p(n)} \mid \mathcal{M}(x,r) \text{ accepts}\}$ is **huge** or **tiny**
- We claim there are **translations** $R \oplus t_i$ of R such that:
— if R huge, then the translations cover the whole space
— if R tiny, they do not.



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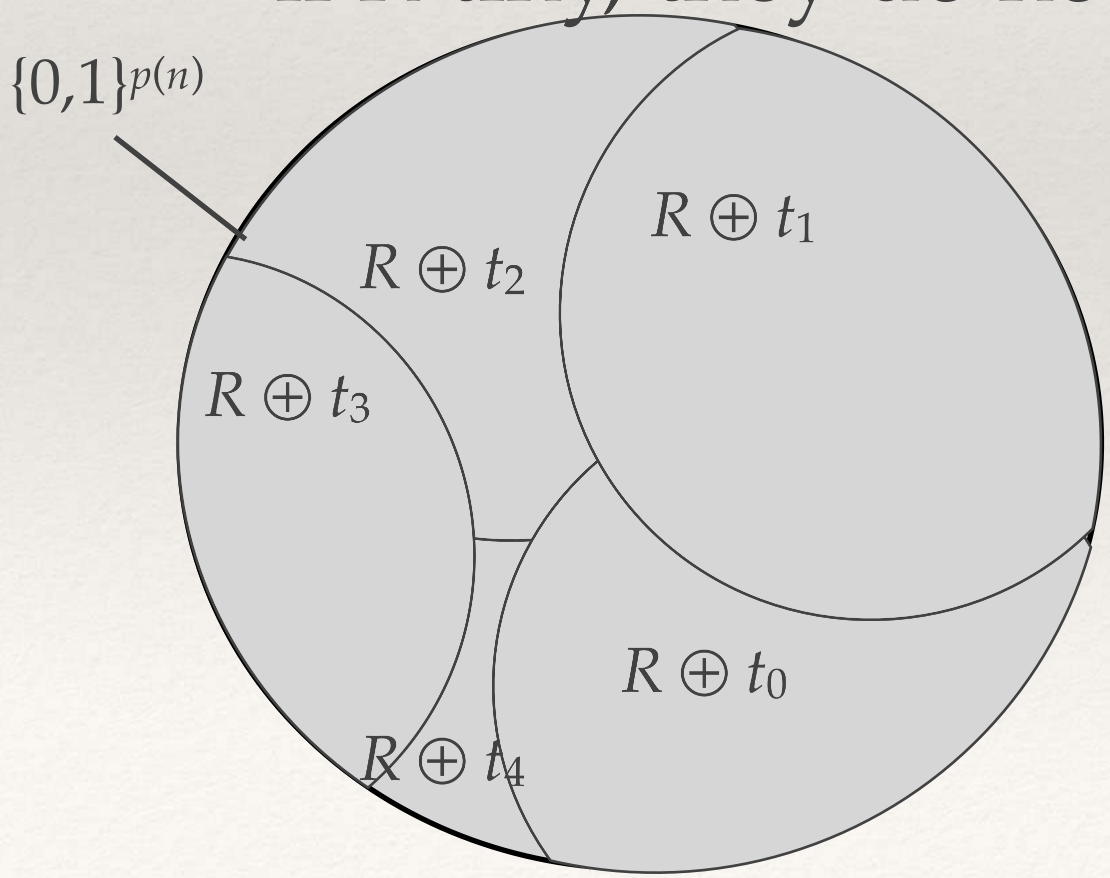

←**huge**

$(x \in L)$

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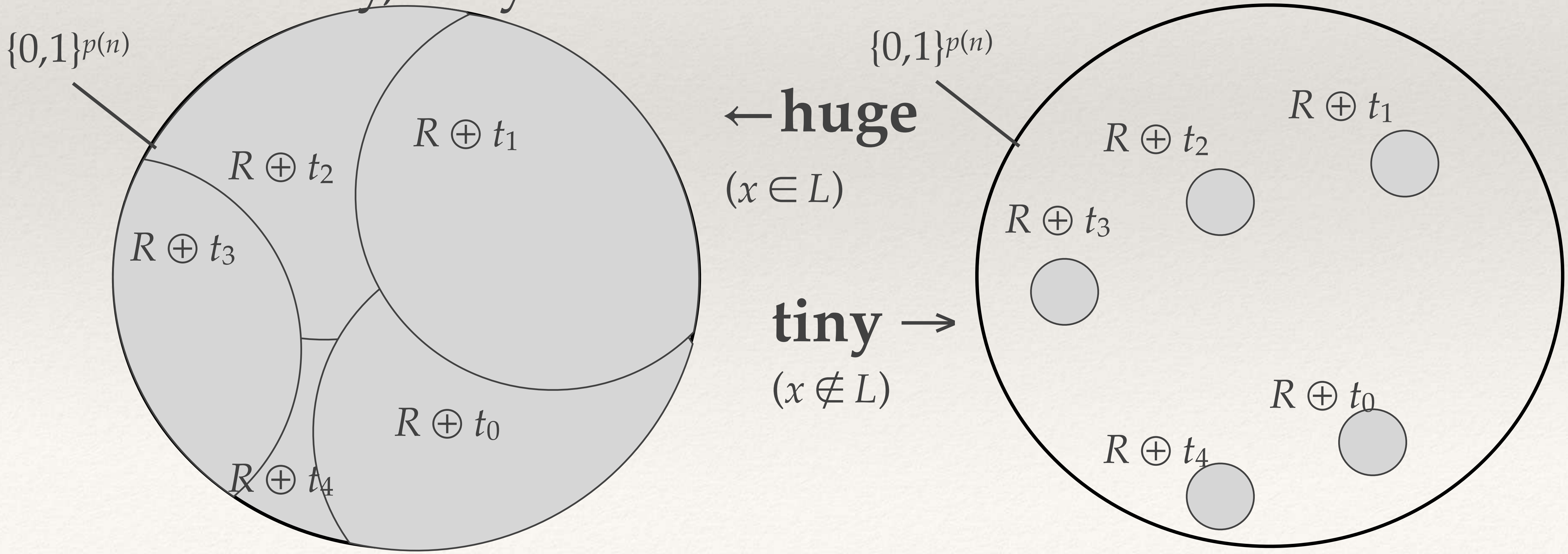

Translations?

- The computer science view:

$\oplus$ is **bitwise exclusive-or**

r	0 0 0 0 1 0 1 0 0 0 0 1 0 1 0 0 1 0 0 0 0 1 1
t	1 0 1 0 1 1 0 1 0 1 0 1 1 0 1 0 0 1 0 1 0 1 1
$r \oplus t$	1 0 1 0 0 1 1 1 0 1 0 0 1 1 1 0 1 1 0 1 0 0 0

- $$R \oplus t = \{r \oplus t \mid r \in R\}$$

Translations?

- The algebraist's view: $\{0,1\}$ is the **field** $\mathbb{Z}/2\mathbb{Z}$,
  — exclusive or $\oplus$ is **addition (mod 2)**
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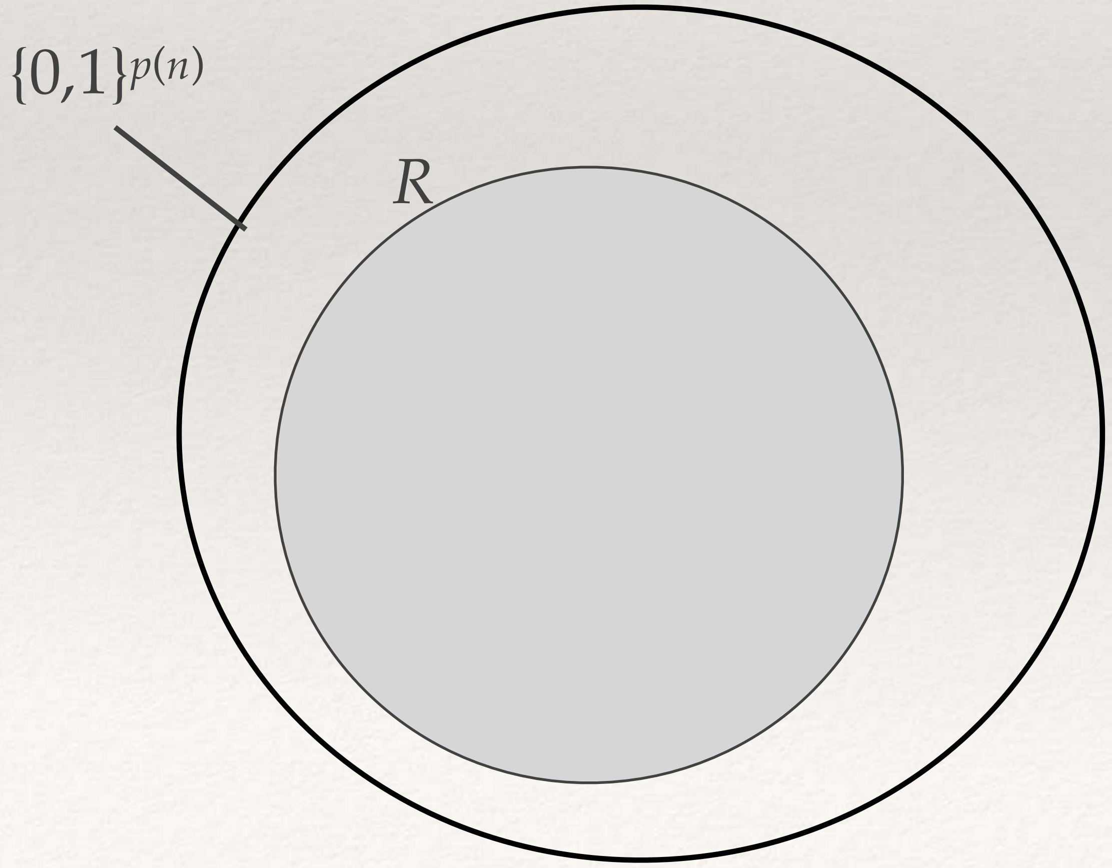

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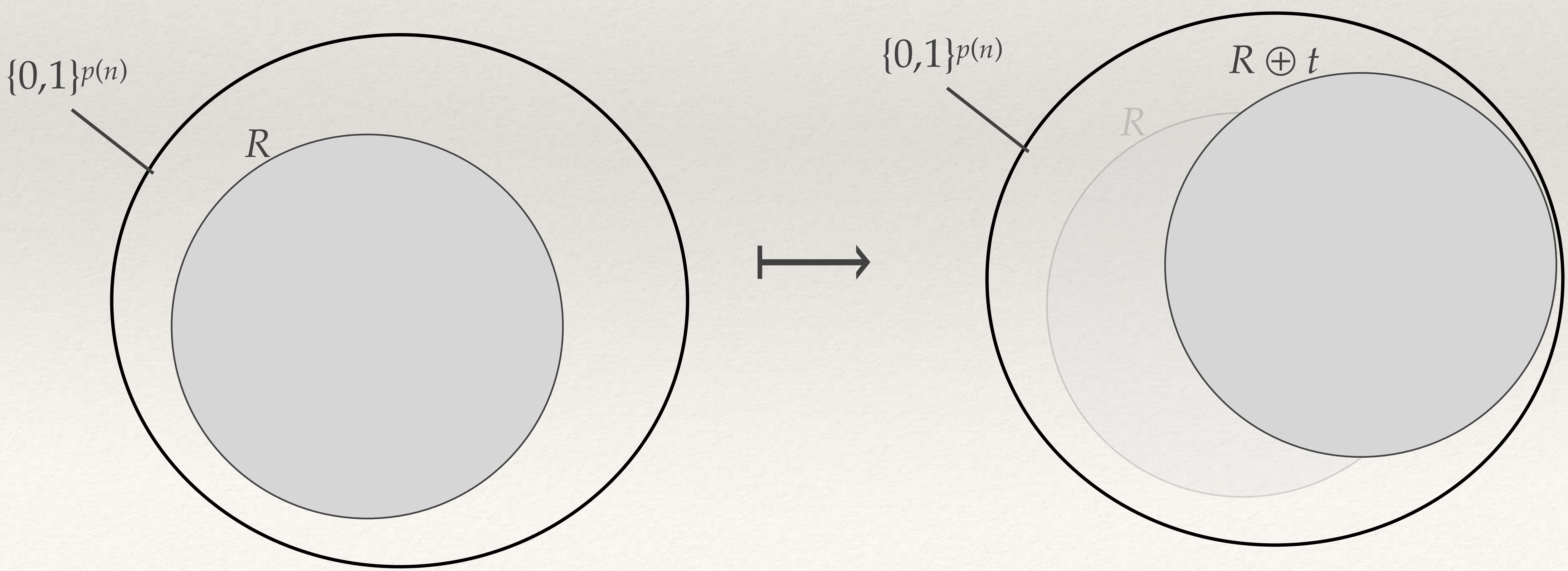

The huge case (1/3)

- Assume card $R \geq (1-1/2^n)2^{p(n)}$ (« R is huge »)
- **Claim.** $\exists\ t_0, \ldots, t_{\ulcorner m/n \urcorner}$ $(m=p(n))$ such that
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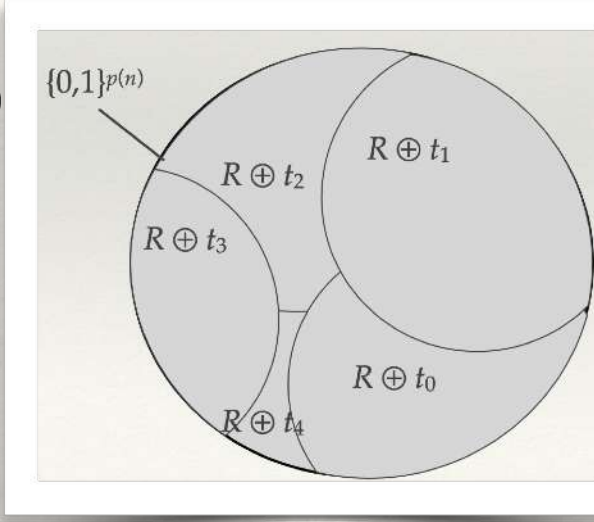

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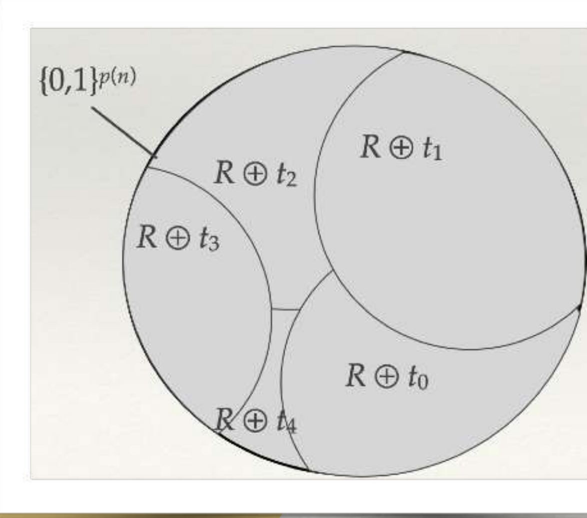

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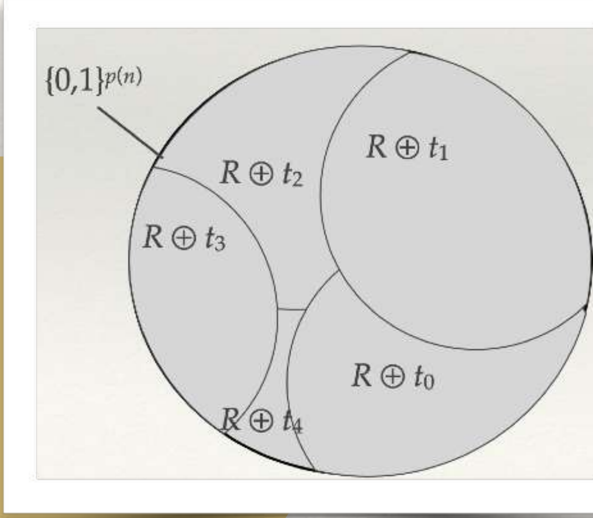

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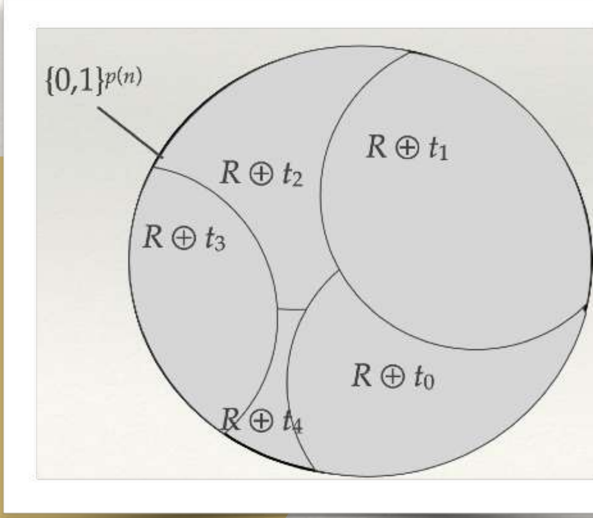

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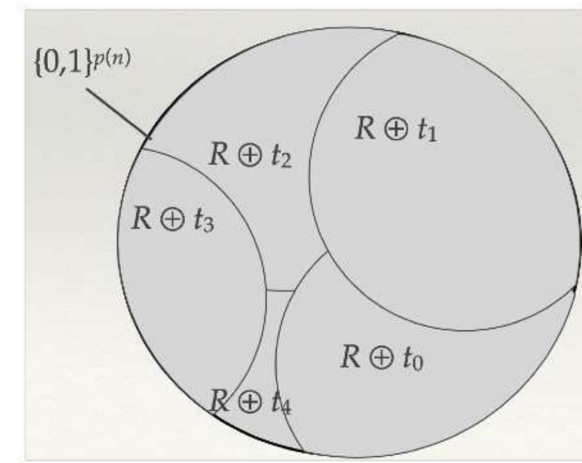

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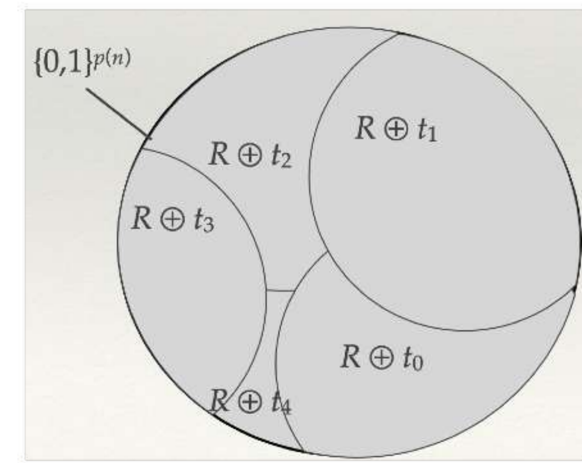

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Sum bound: $\Pr(\exists\ldots) \leq \sum \Pr(\ldots)$

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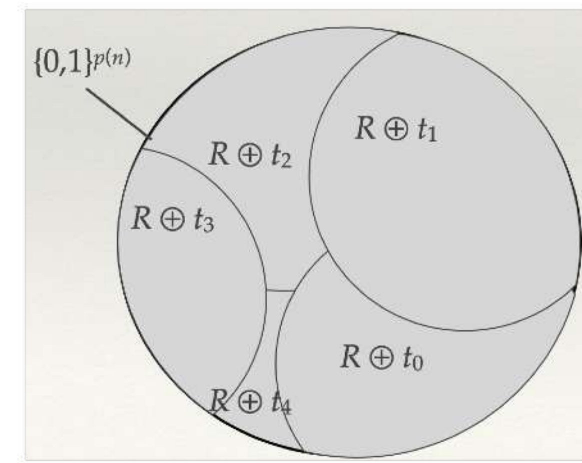

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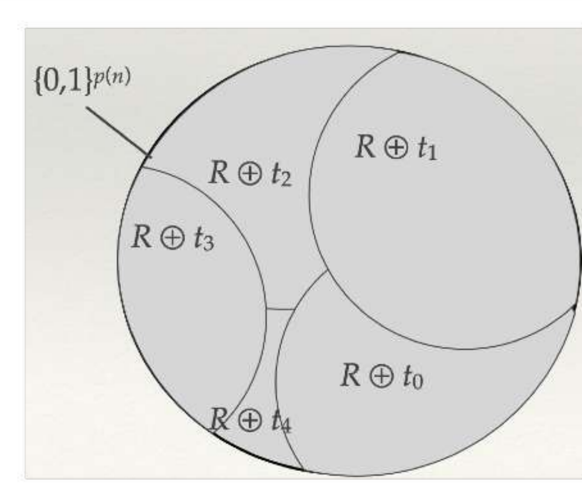
Sum bound: Pr(∃…)≤∑ Pr(…)

Oh yes, that is a sum of $2^{p(n)}$ terms here!

The huge case (2/3)



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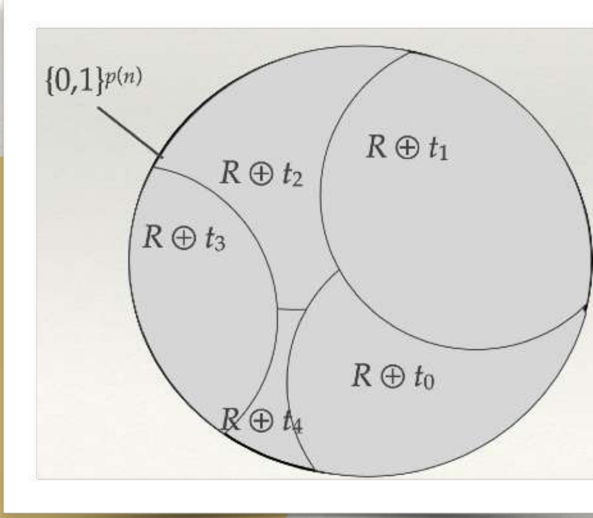

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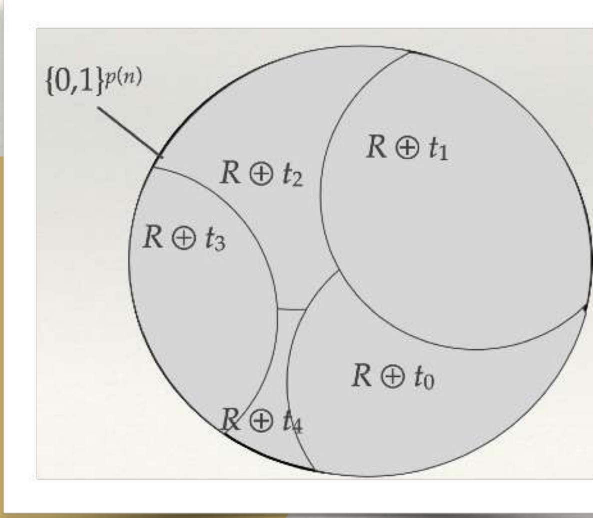

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(independence)

The huge case (2/3)

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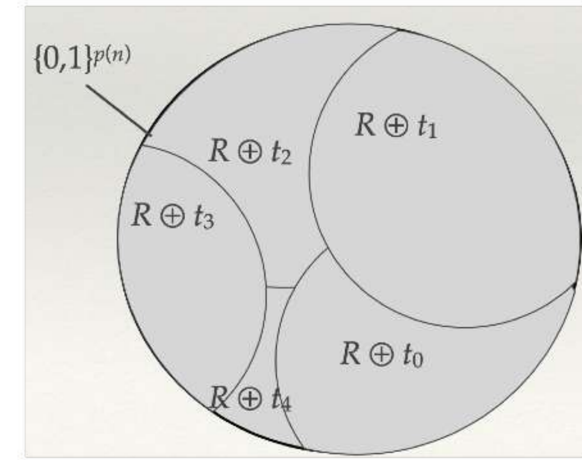

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- $= \sum_r \prod_{i=0}^{\ulcorner m/n \urcorner} \Pr_{\underline{t}} (r \notin R \oplus t_i)$ **(independence)**
- $= \sum_r \prod_{i=0}^{\ulcorner m/n \urcorner} \Pr_{\underline{t}} (r \oplus t_i \notin R)$ ($r{\in}R{\oplus}t$ iff $r{\ominus}t{\in}R$… but $\oplus{=}\ominus$ mod 2)

The huge case (3/3)

- Assume card $R \geq (1-1/2^n)2^{p(n)}$ (« R is huge »)
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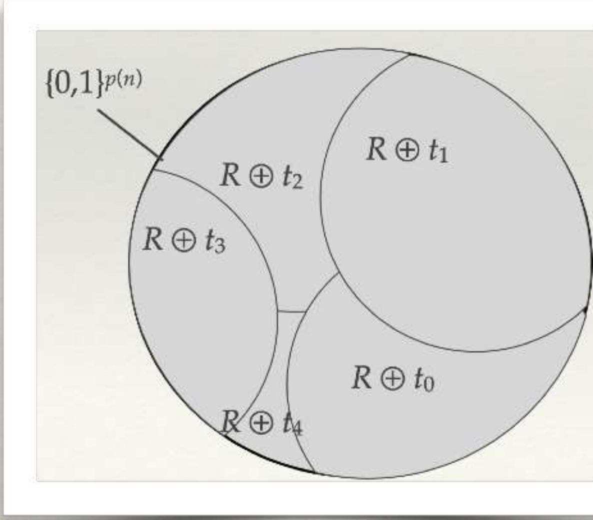

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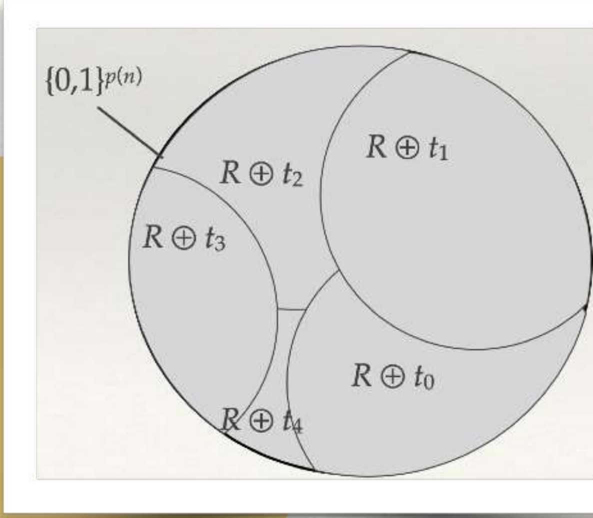

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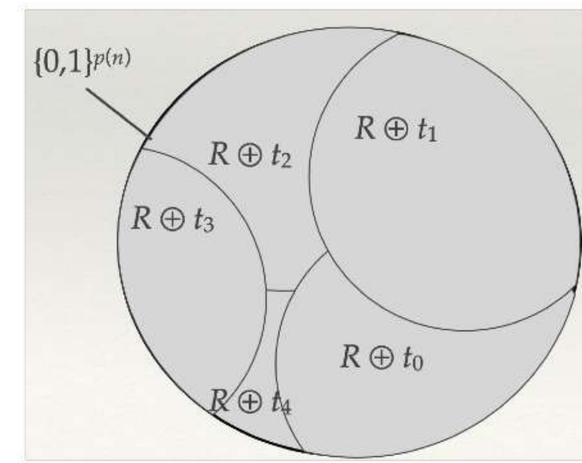

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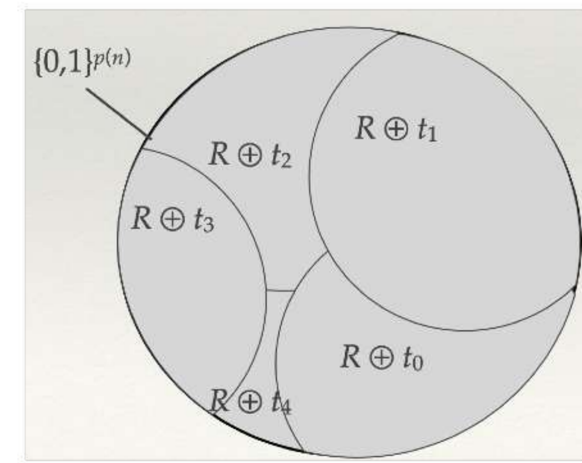

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- $\leq 2^m (1/2^n)^{\lceil m/n \rceil + 1} \leq 1/2^n < 1$ (at least if $n \neq 0$). Done! □

The tiny case

- Assume card $R \leq (1/2^n)2^{p(n)}$ (« R is tiny »)
- **Claim.** $\forall\, t_0, \ldots, t_{\ulcorner m/n \urcorner}$ $(m{=}p(n))$,
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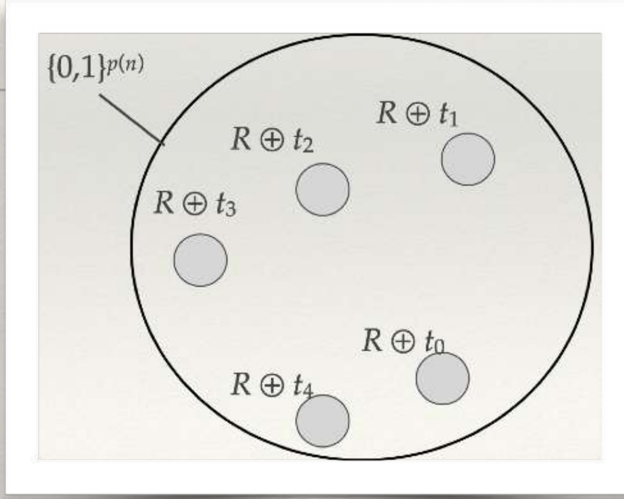

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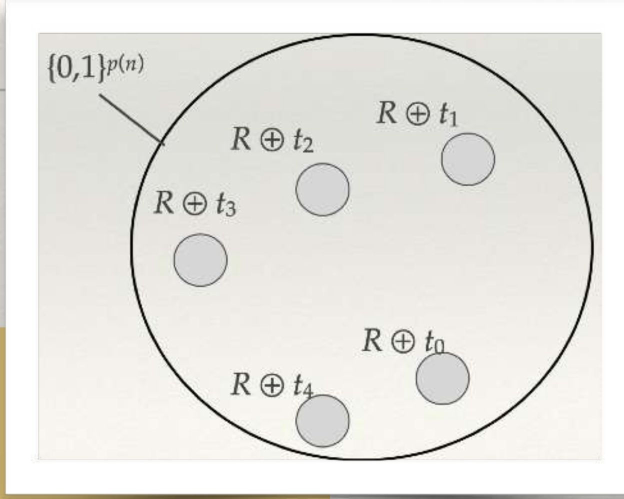

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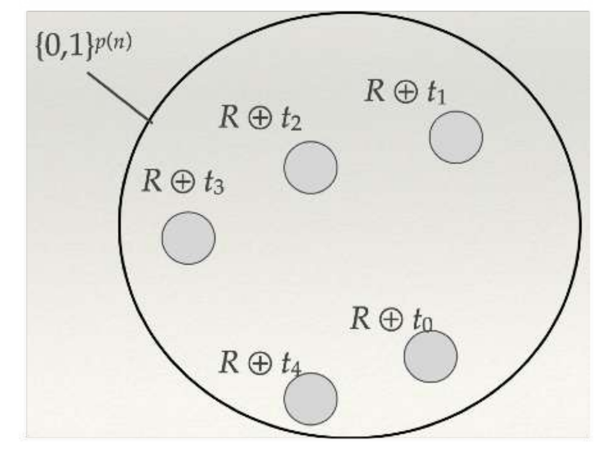

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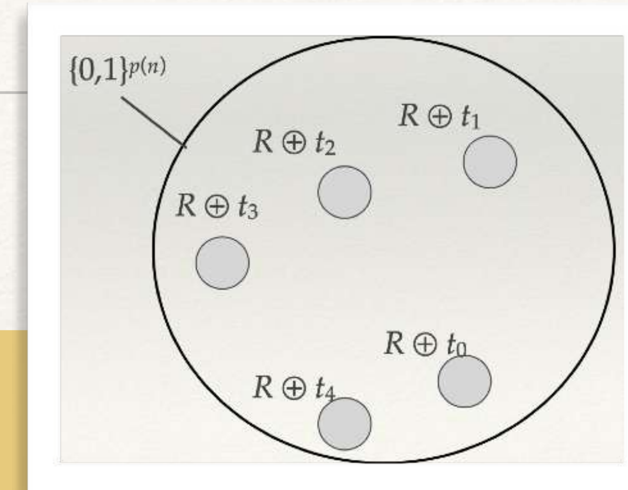
The tiny case

if $n \geq n_0$



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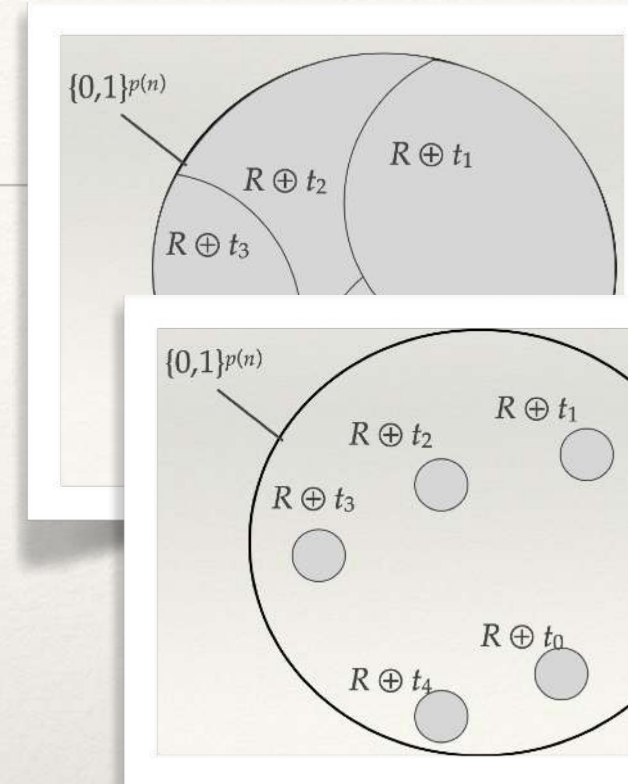
  $= O(\text{poly}(n)/2^n)\; 2^{p(n)}$
- **strictly smaller** than card $\{0,1\}^m = 2^{p(n)}$
  … if n large enough (say $n \geq n_0$). ☐

Testing huge vs. tiny



- $R \oplus t_0, \ldots, R \oplus t_{\lceil m/n \rceil}$ covers $\{0,1\}^m$ iff:

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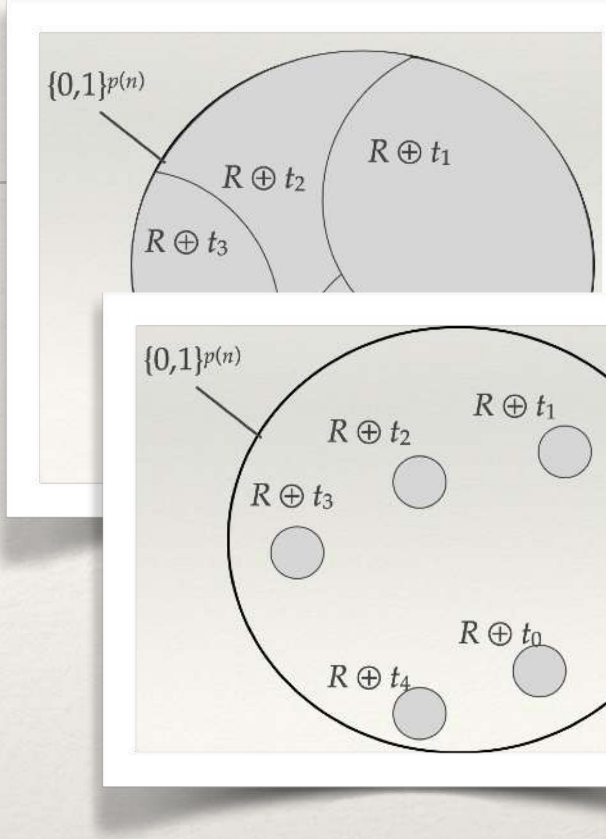

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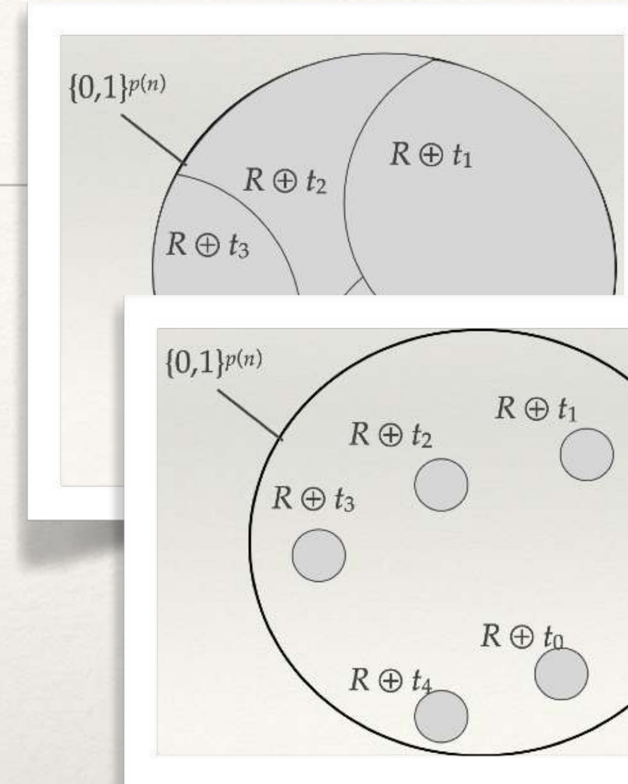

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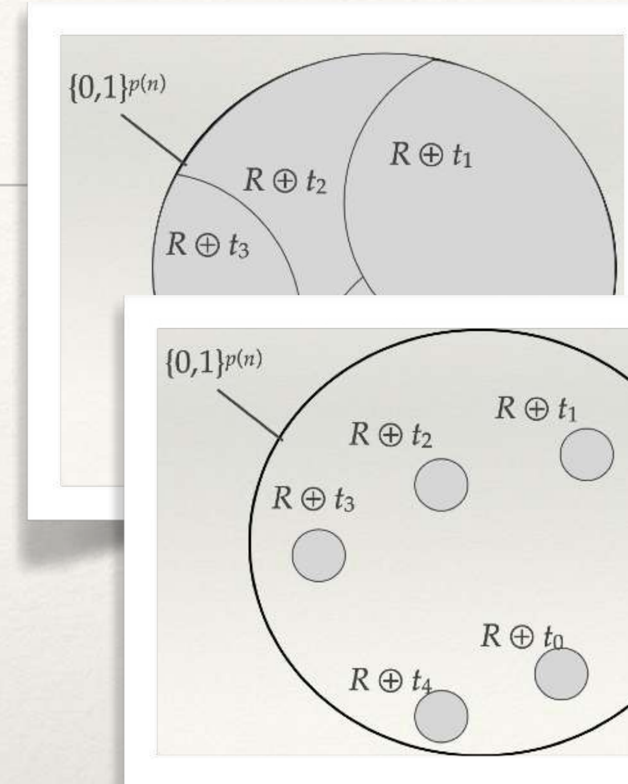

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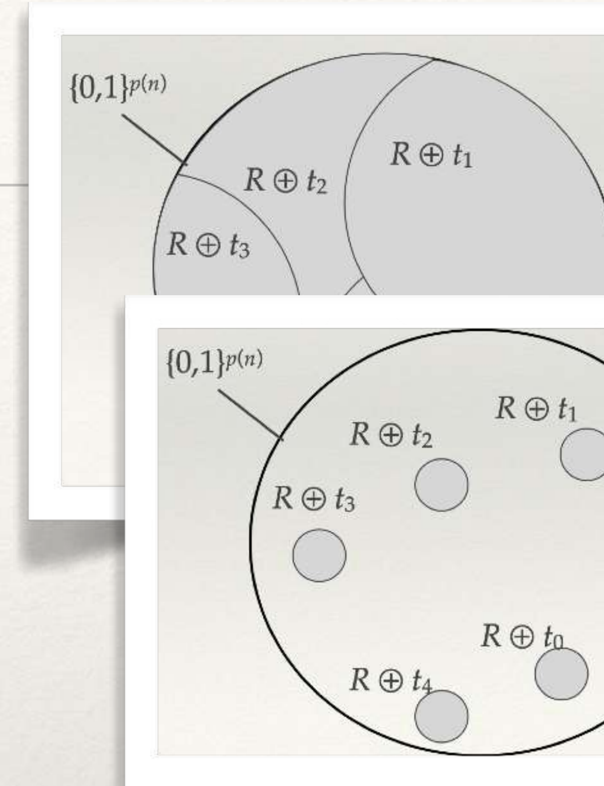

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- If $x \notin L$, such $t_0, \ldots, t_{\lceil m/n \rceil}$ do not exist (for $n \geq n_0$).

The algorithm

- Hence, for every x of size $n \geq n_0$,

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- For $n < n_0$, **tabulate** the answers.
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- Since L is arbitrary in **BPP**, **BPP** $\subseteq \Sigma^p_2$. □

The Sipser-Gács-Lautemann theorem

- **Theorem (Sipser-Gács-Lautemann, Prop. 1.24.)**
  BPP $\subseteq \Sigma^{p}_{2} \cap \Pi^{p}_{2}$.
- *End of proof.*
  We have shown **BPP** $\subseteq \Sigma^{p}_{2}$.
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Useful **Lemma.** Given two classes C_1, C_2,
if $C_1 \subseteq C_2$ then **co**$C_1 \subseteq$ **co**C_2.
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No, **co**C is **not** the complement of C.
It is the class of complements of languages in C.

Next time…

P/poly

- We will introduce a strange complexity class defined by **families of circuits**:

  P/poly

- Studying it, we will eventually show that **BPP** probably does **not** contain **NP**
  … otherwise **PH** would collapse at level 2!