Jean Goubault-Larrecq

Randomized complexity classes

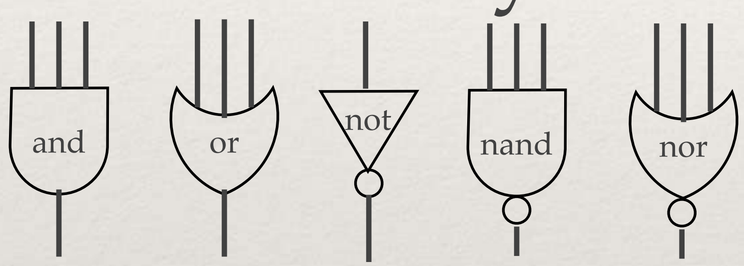
Today: **BPP** (part 2) and **P/poly**

Today

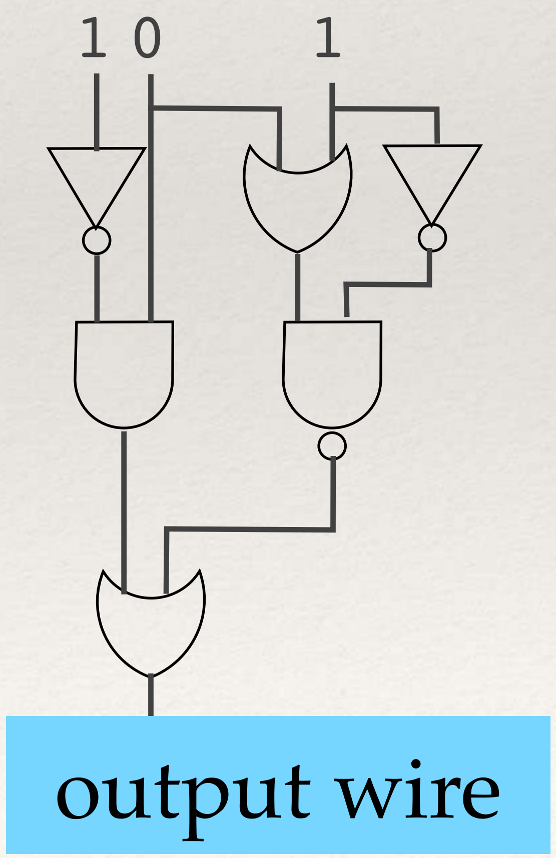
- Circuits, **P/poly**
- Adleman's theorem: $\mathbf{BPP} \subseteq \mathbf{P/poly}$
- The Karp-Lipton theorems, and consequences

Circuits

- Informally, collections of logical **gates** connected by **wires**



- Must be **acyclic**
- Wires can be **shared**
- **Fan-in** arbitrary here
  (e.g., 1=fan-in 0 and, 0=fan-in 0 or)



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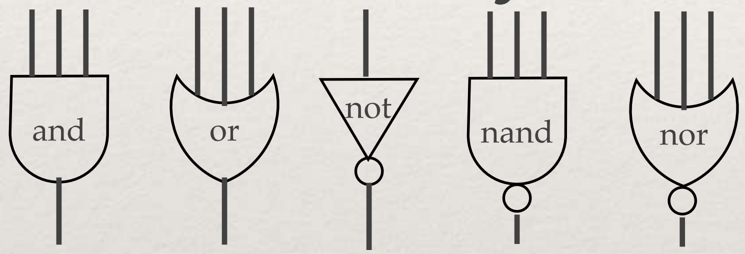

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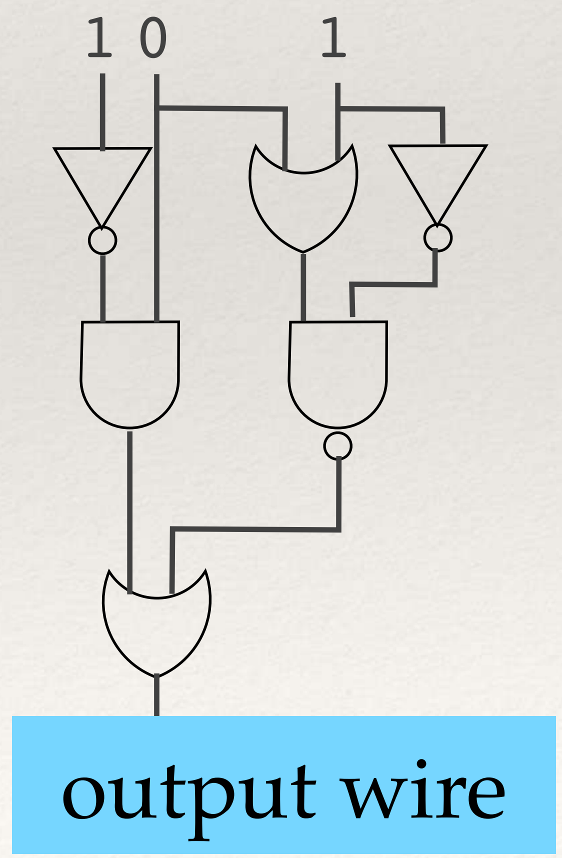
- Remember: **CIRCUIT-VALUE** is **P**-complete (for logspace reductions)



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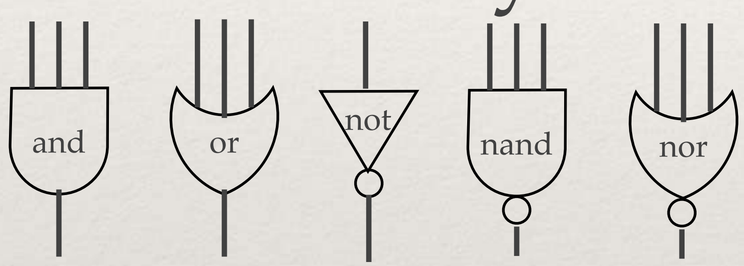

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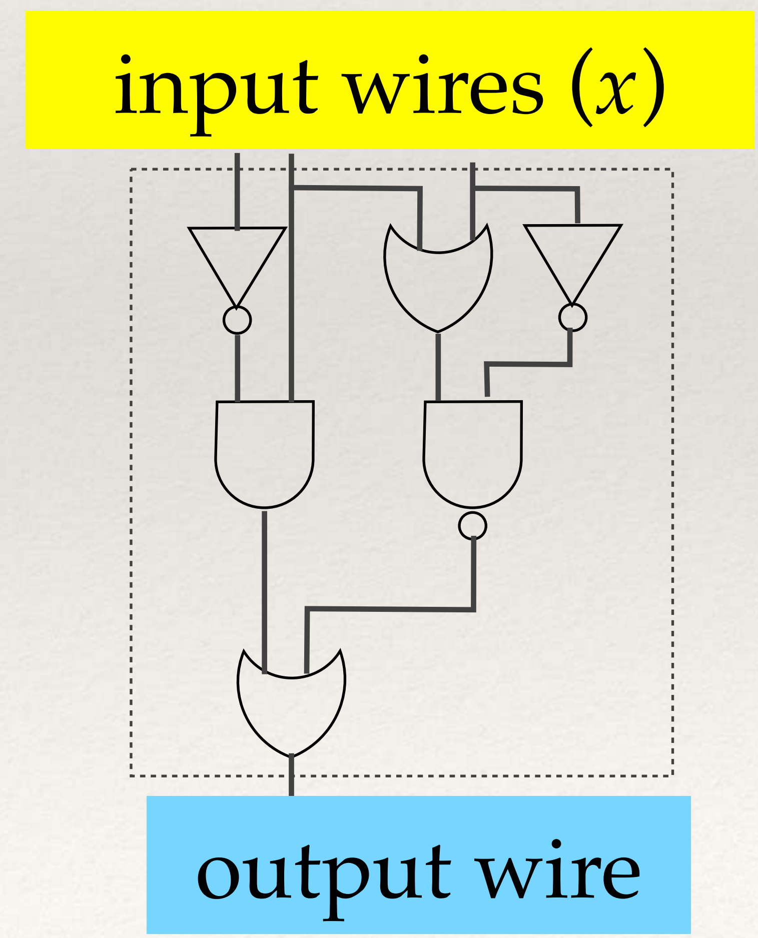

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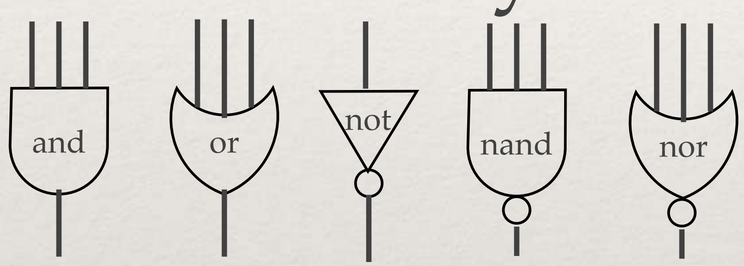

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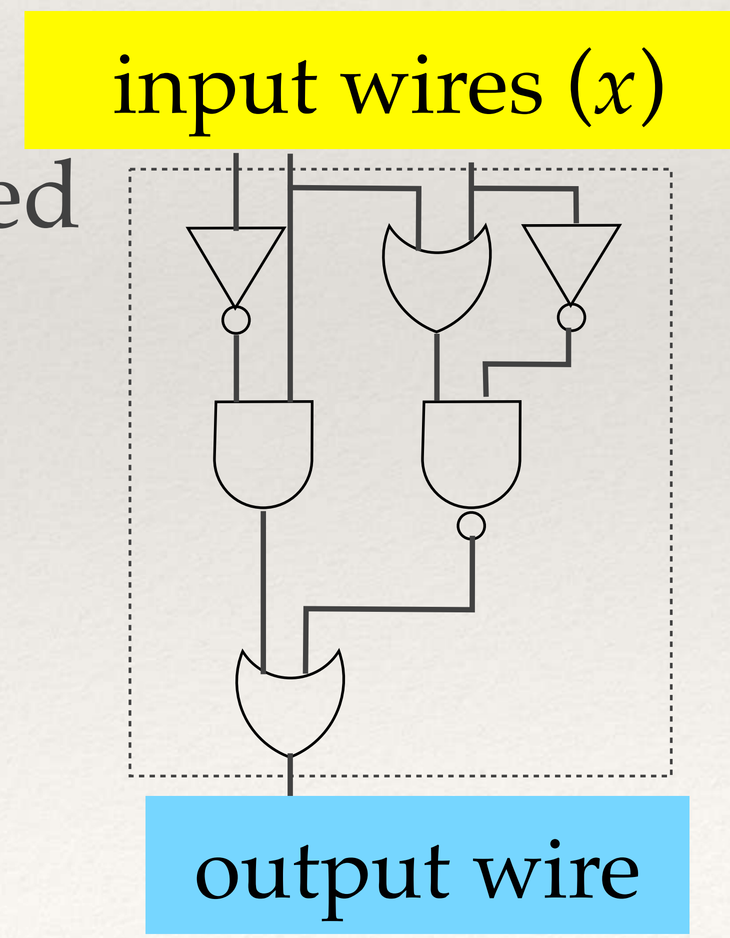
- $C[x]$ = value of C when fed input bits x



Circuits, formally: net-lists

- We encode circuits as **words (net-lists)**, e.g.:

```
3 ¬ 0
4 ∨ 1 2
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→8
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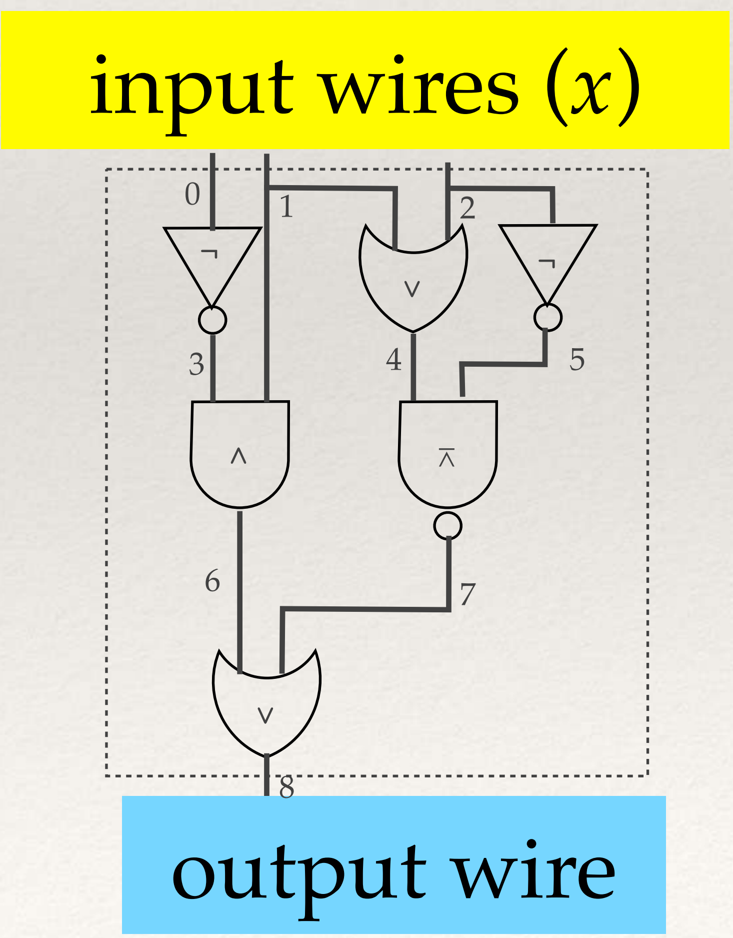

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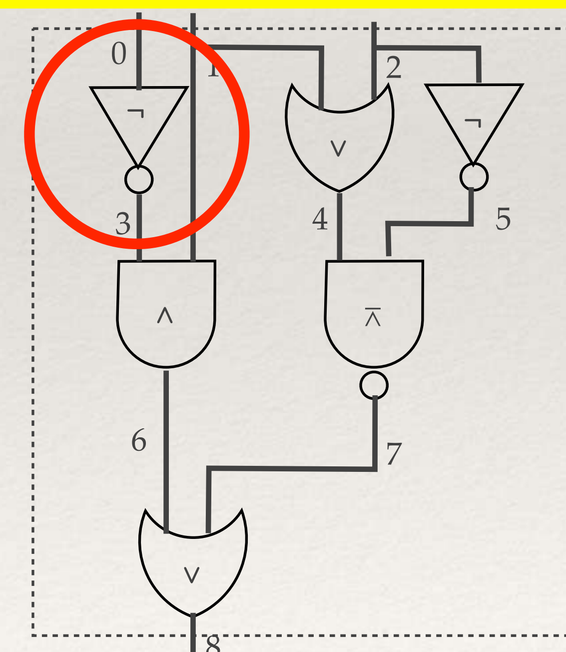

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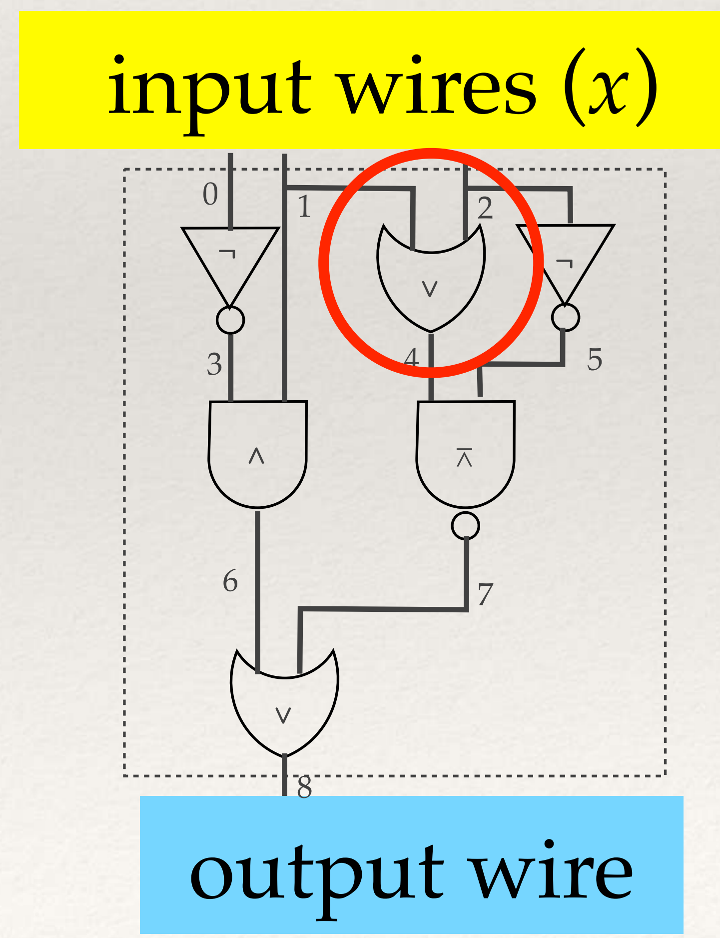

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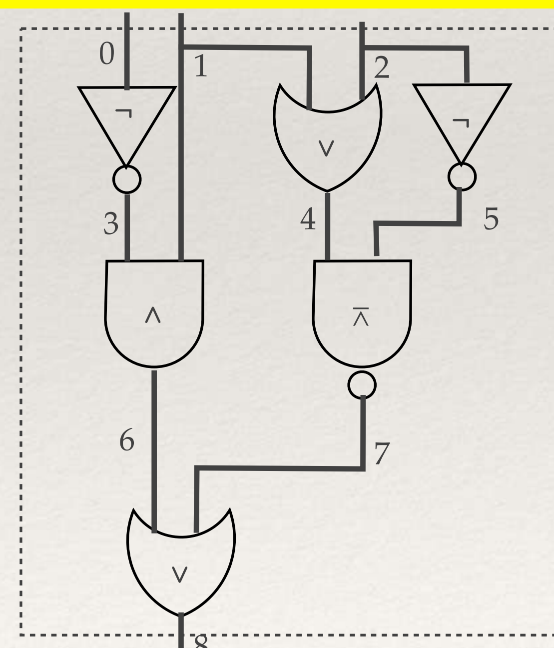

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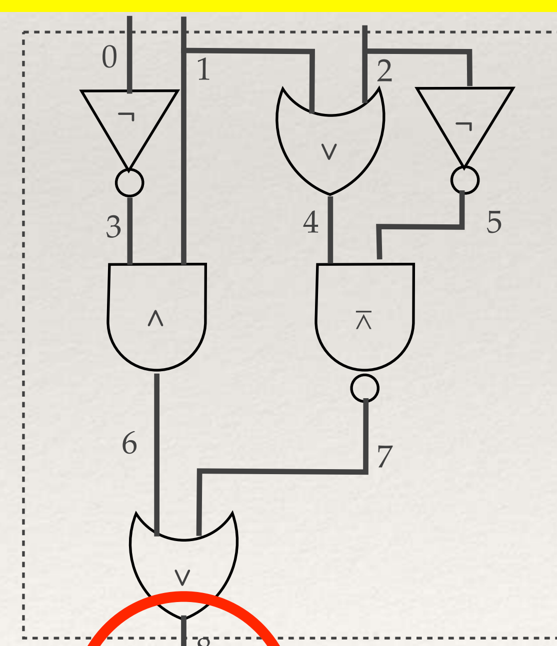

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wire numbers in binary

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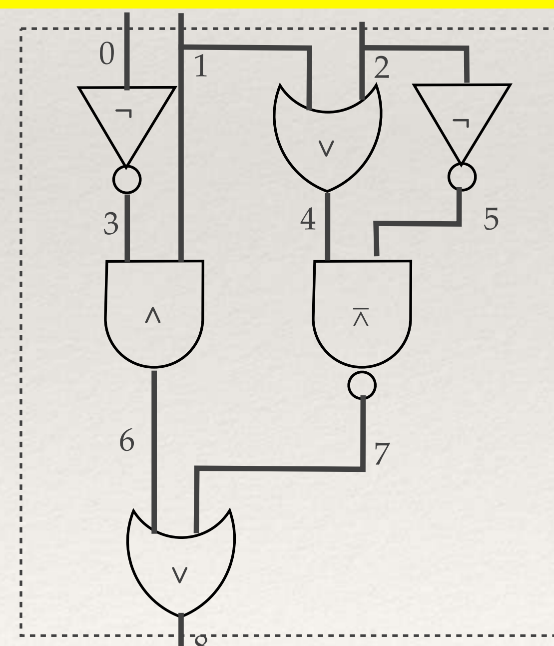

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We require wire numbers to be **sorted** (implies acyclicity)
(sortedness checkable in logspace, acyclicity is **NL**-complete)

wire numbers in binary

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$6 \;\wedge\; 1 \;\; 3$
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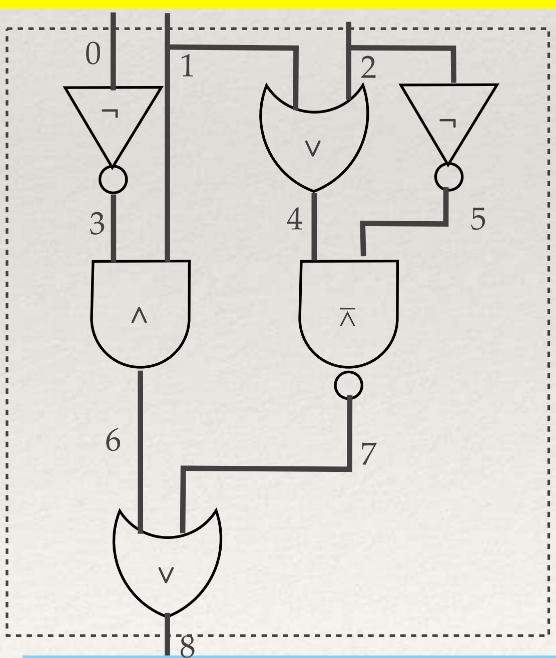

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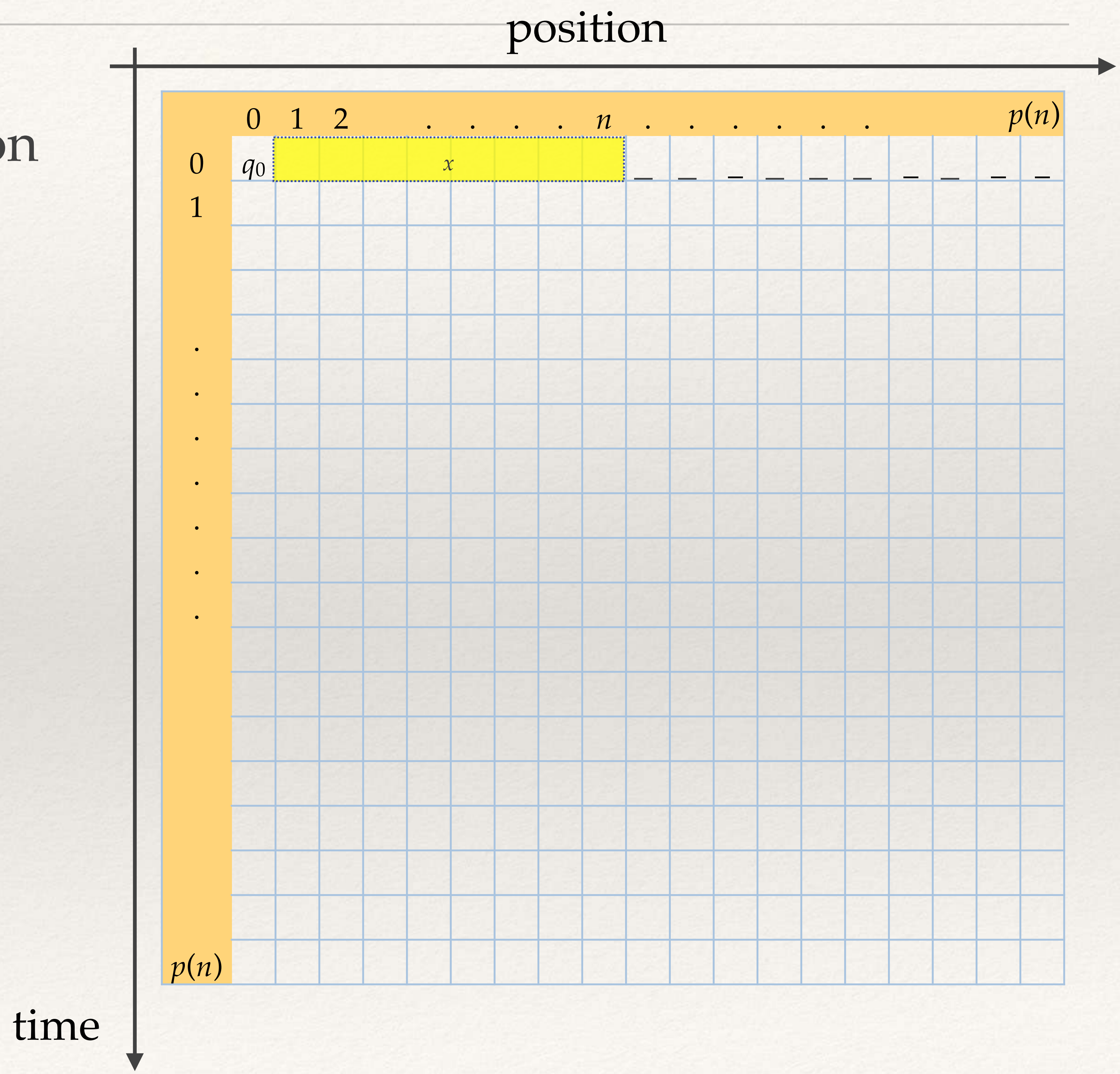
- Encode $p(n)$-time TM $\mathcal{M}$ on input x by a circuit



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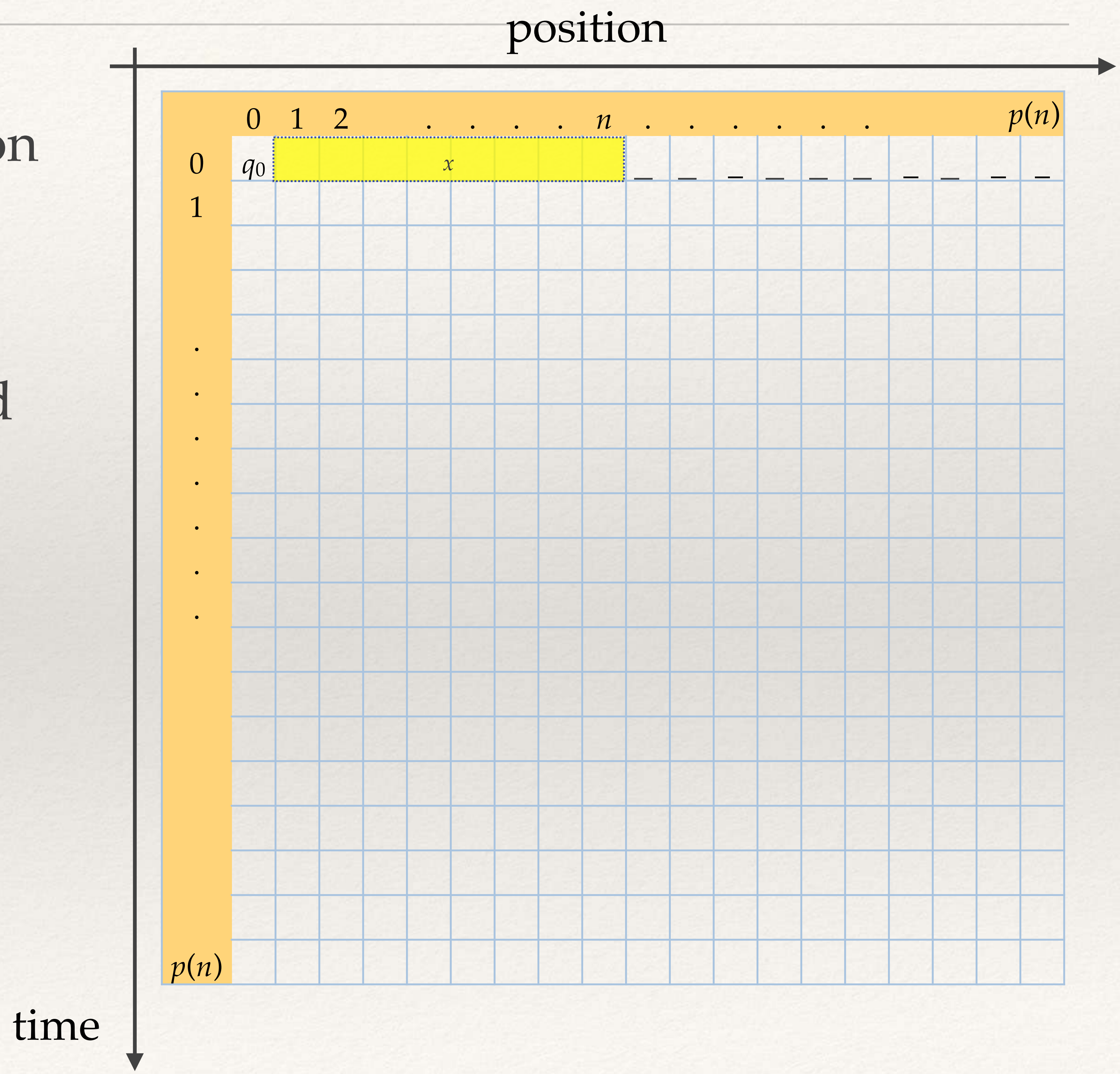

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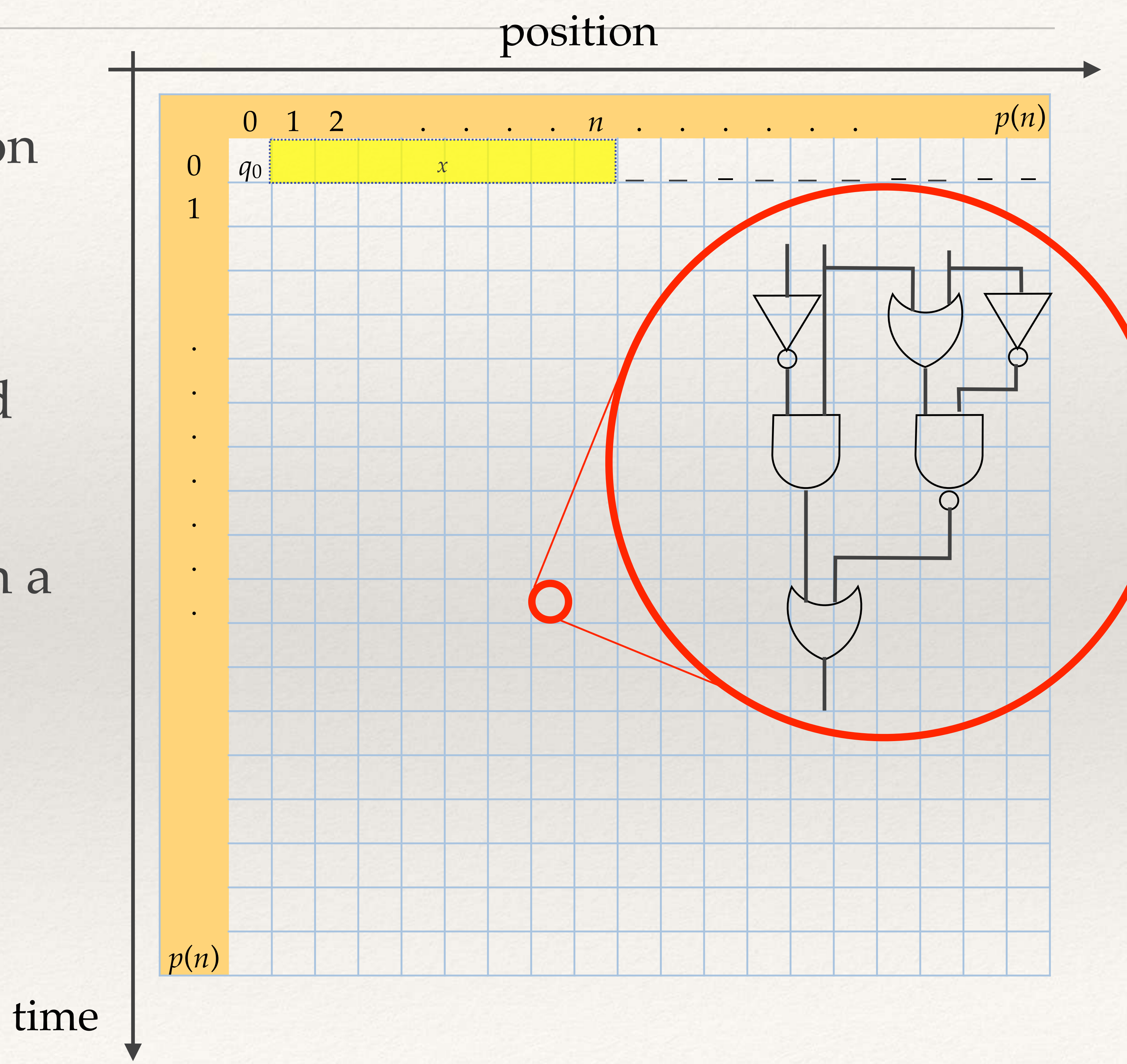

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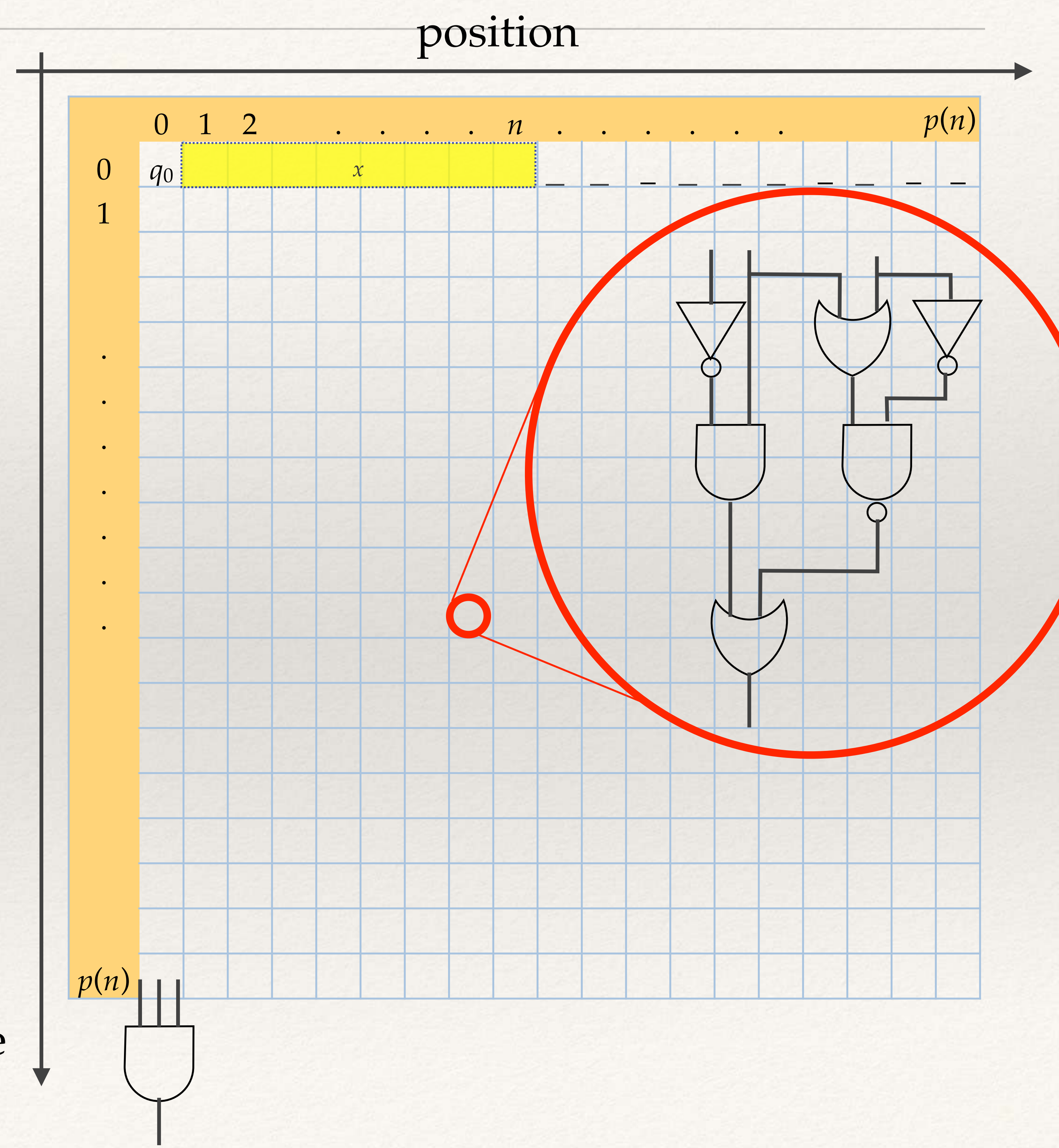

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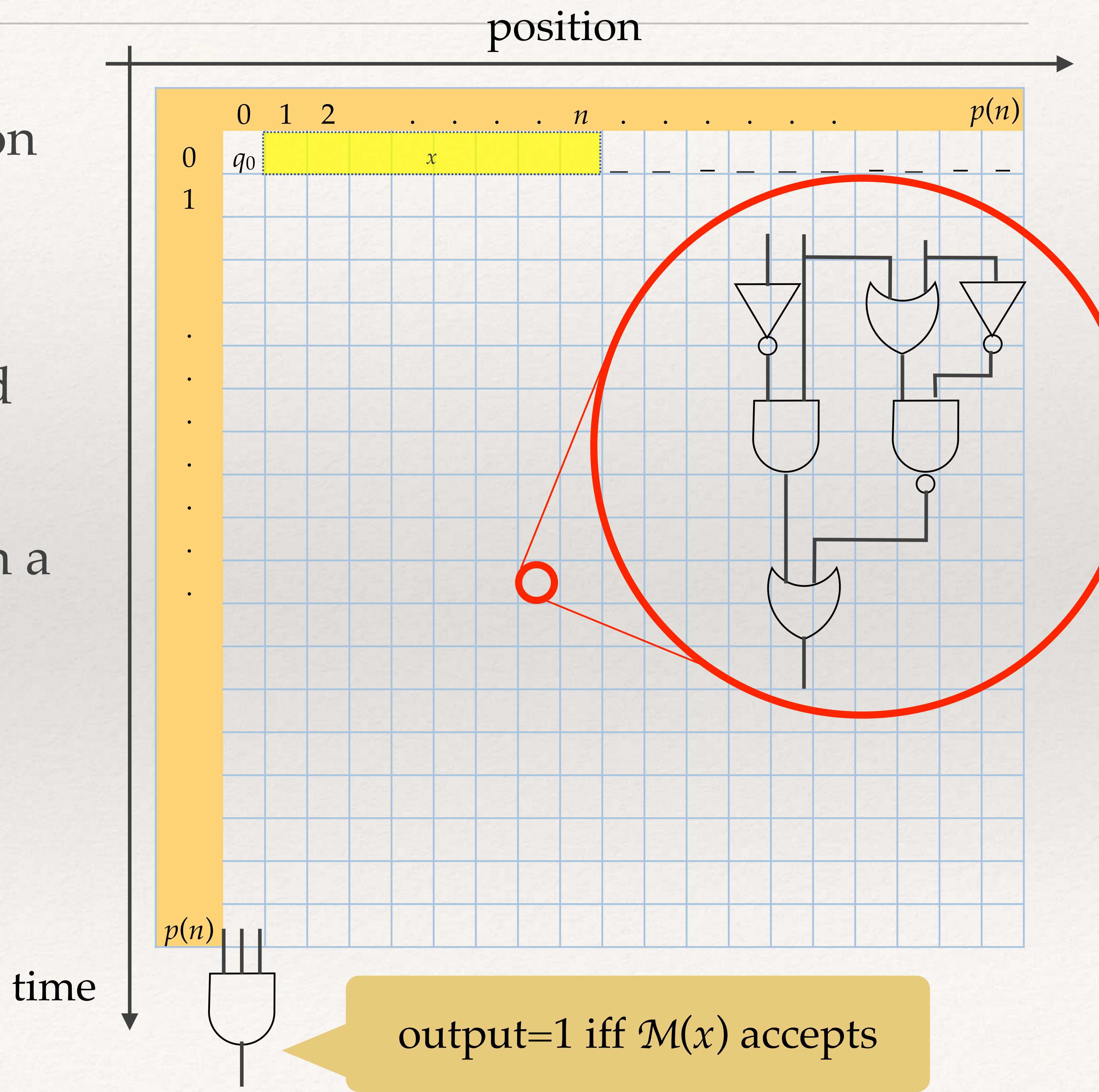
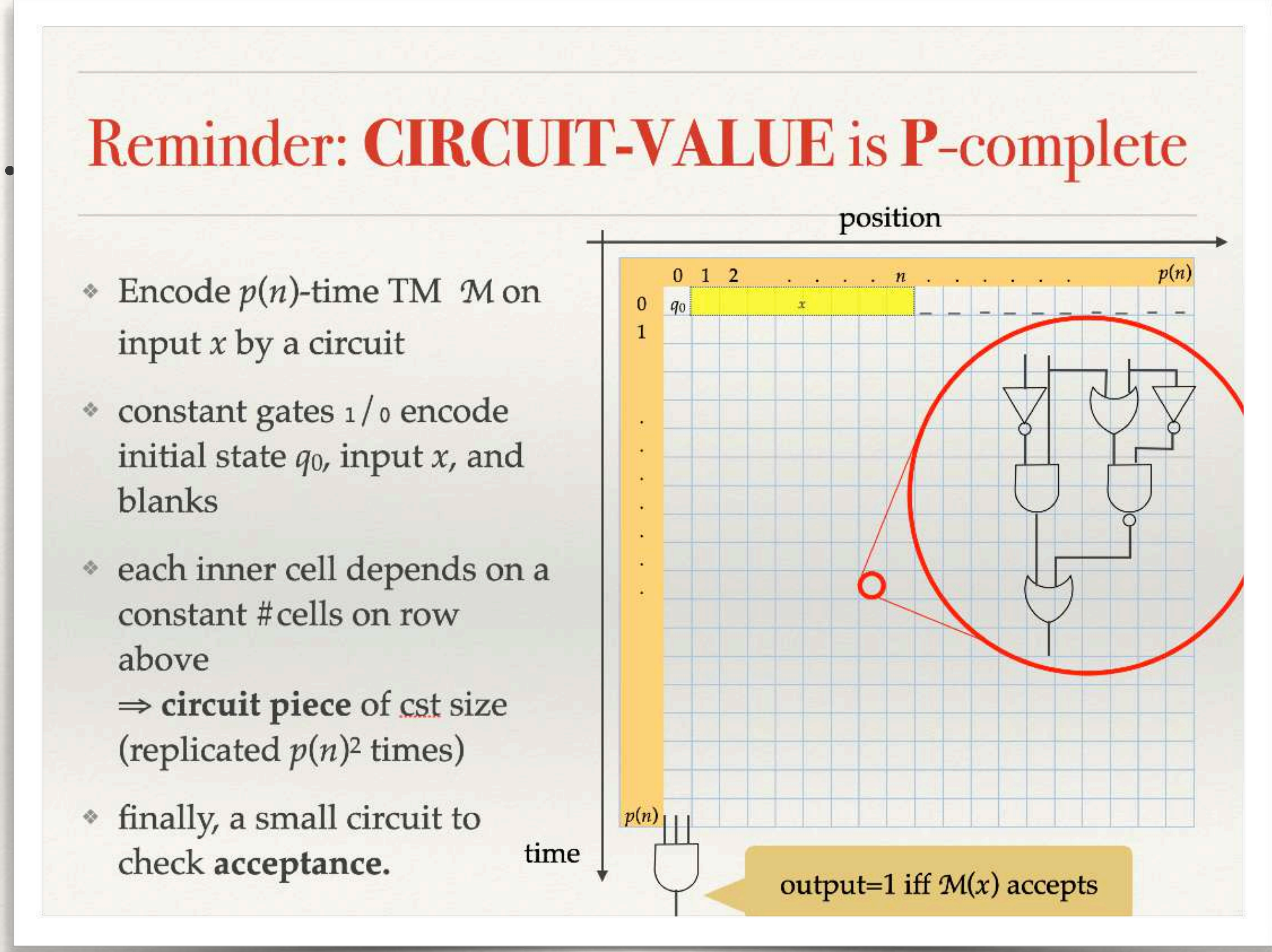


Plenty of technical details…

- Each row encodes a config. of a **one-tape** TM $\mathcal{M}$



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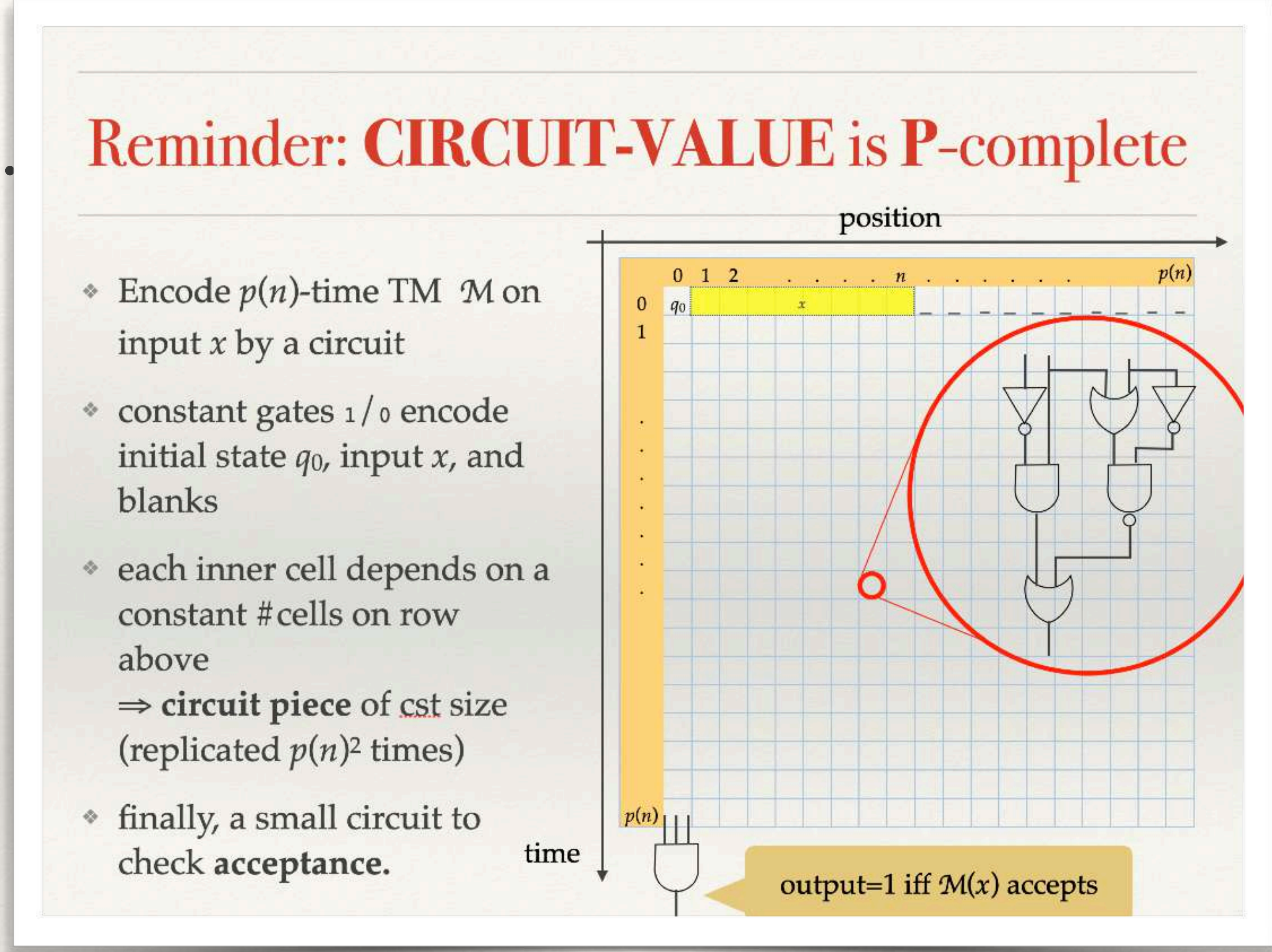

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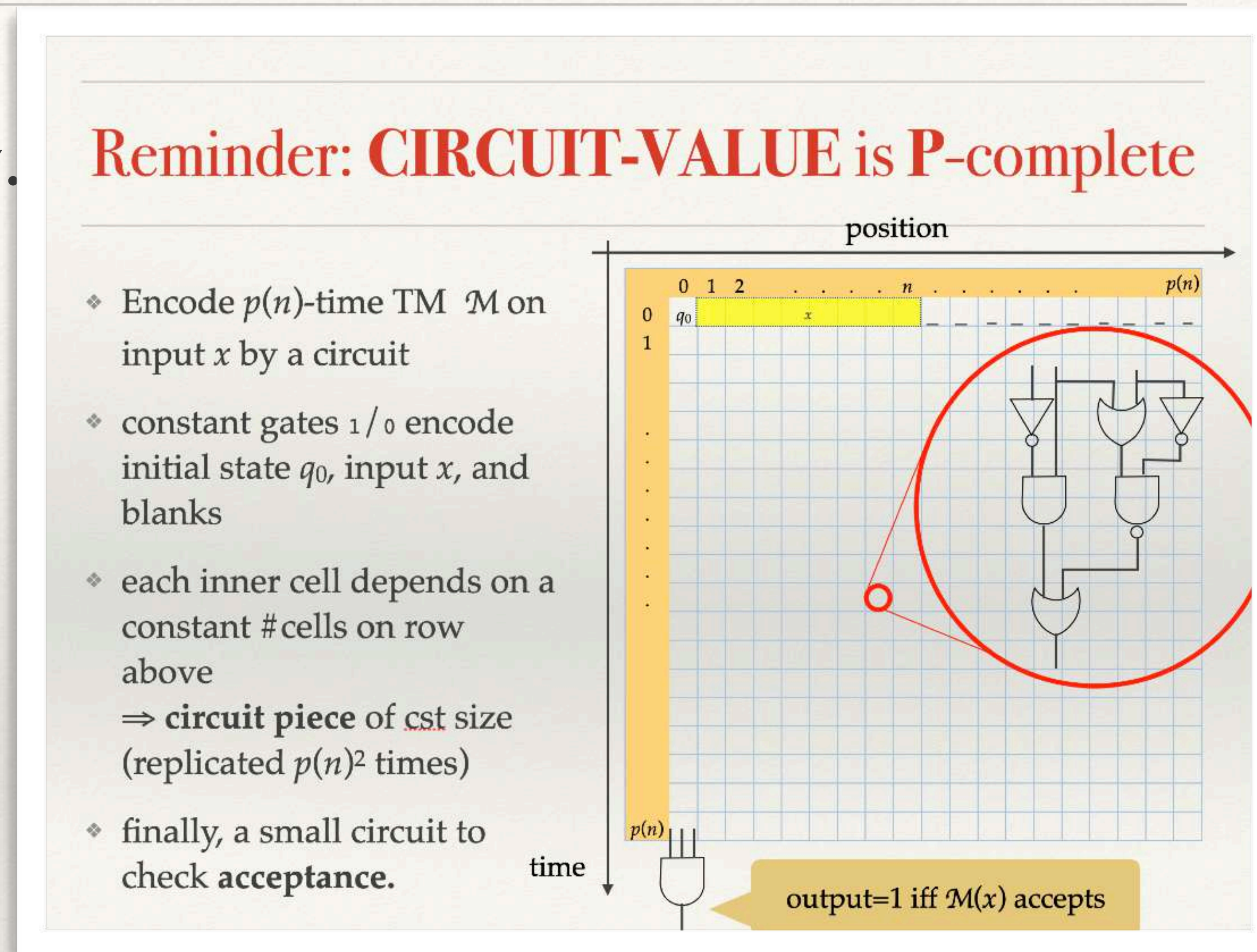

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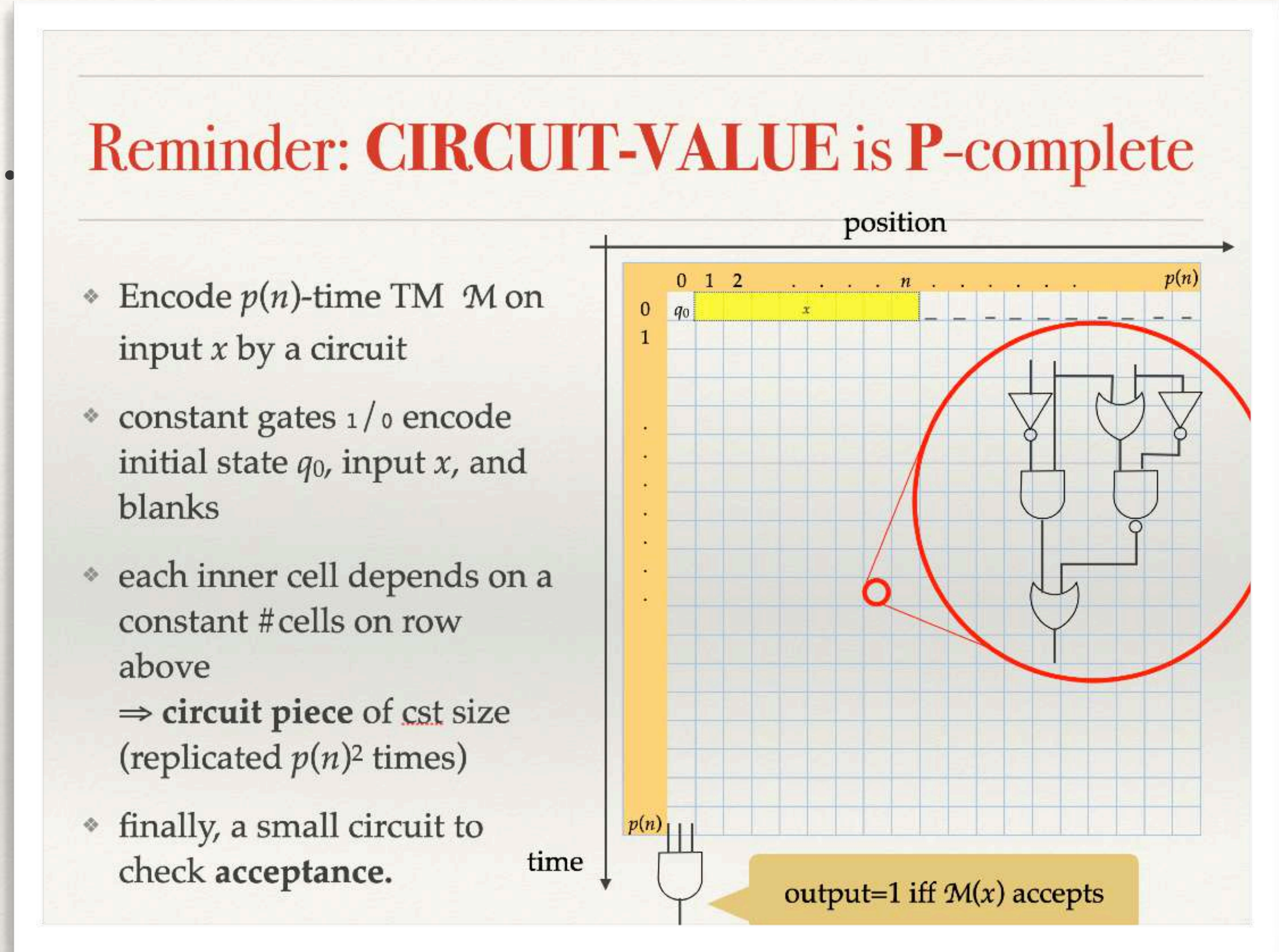

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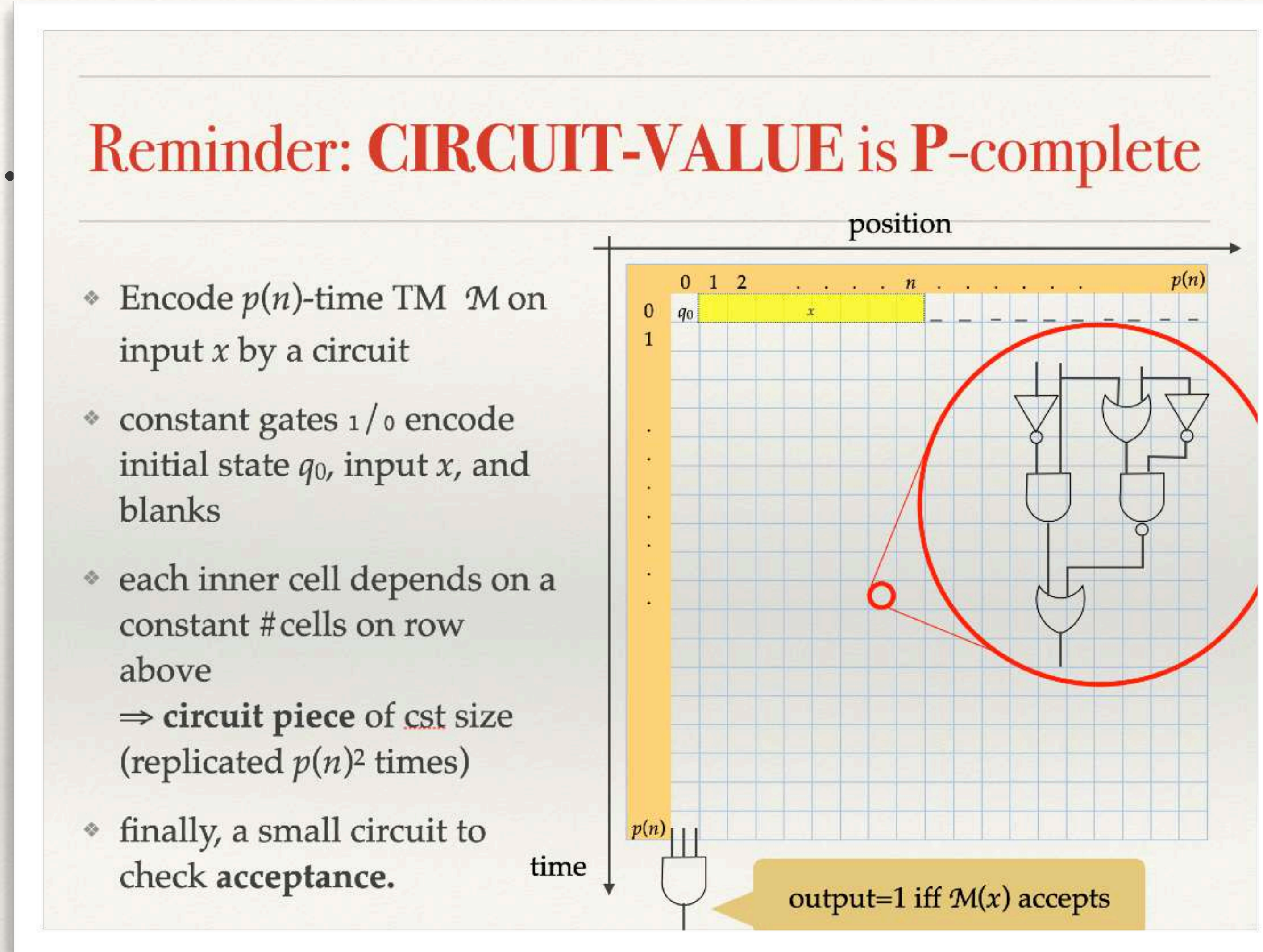
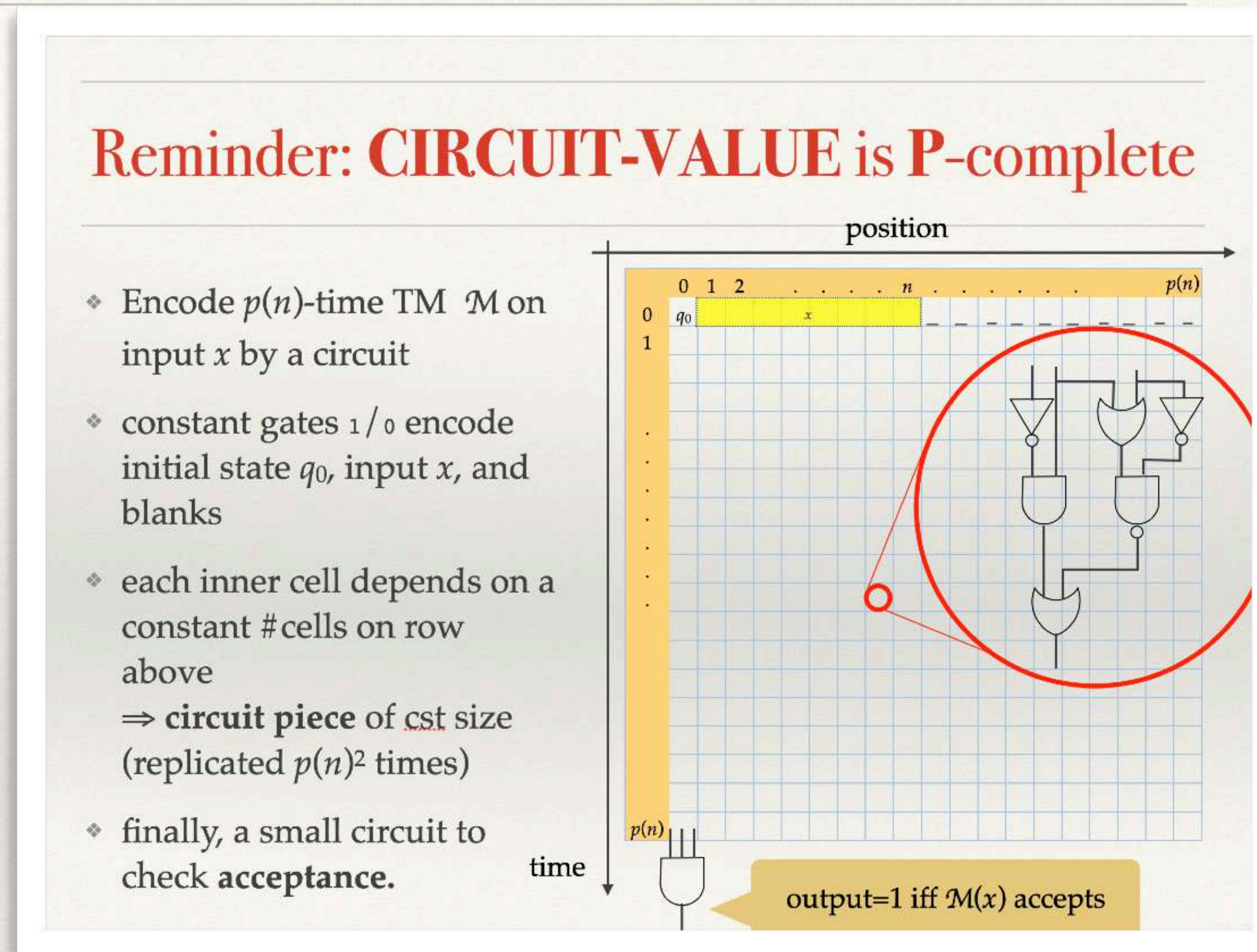


- Build the circuit in **logspace**: 2 nested loops from 0 to $p(n)$, with 2 **counters**

An important remark

- We can **precompile** a circuit C_n with n free input wires
  — without knowing x,
  — just its length n,
  — still in logspace



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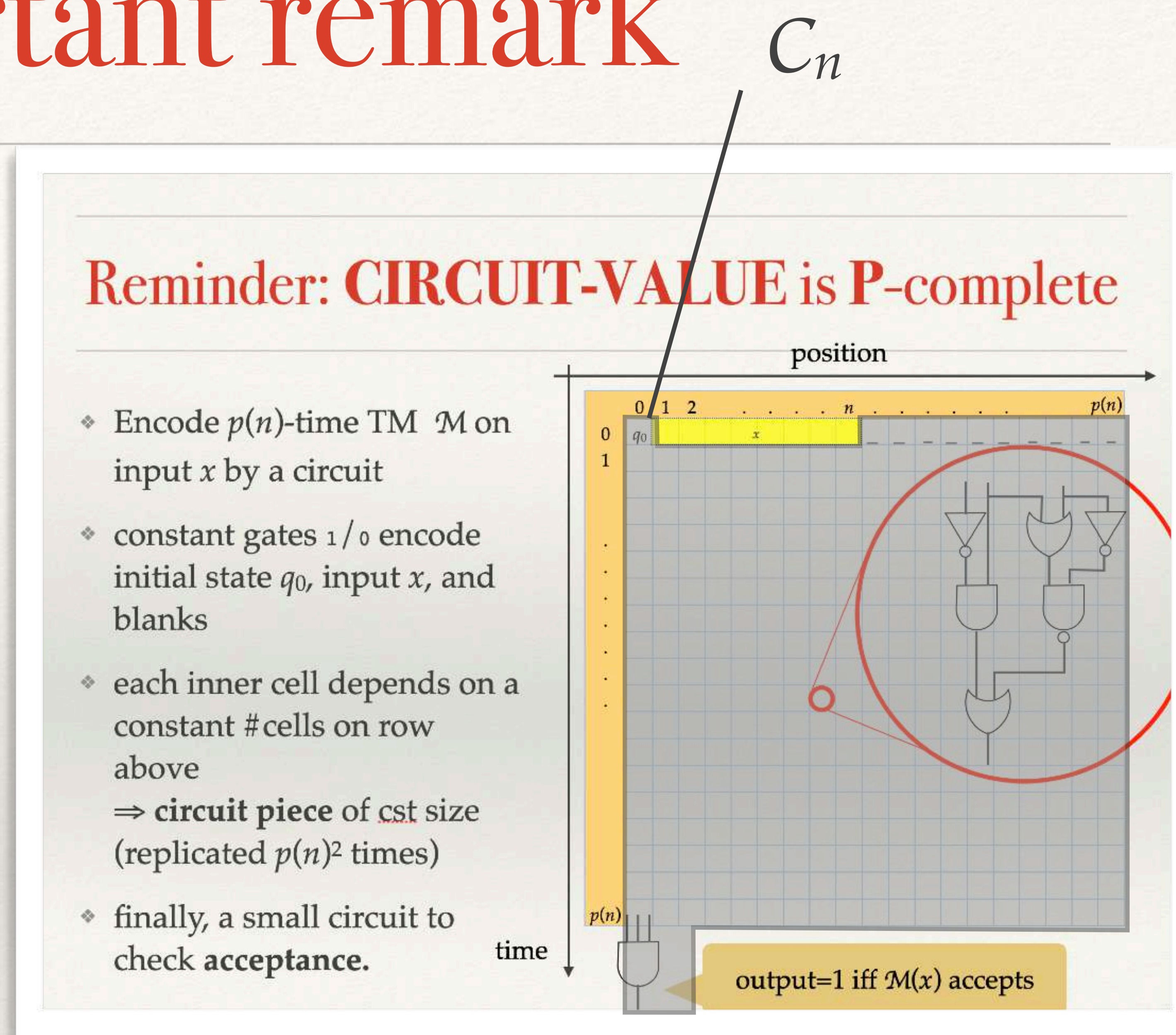

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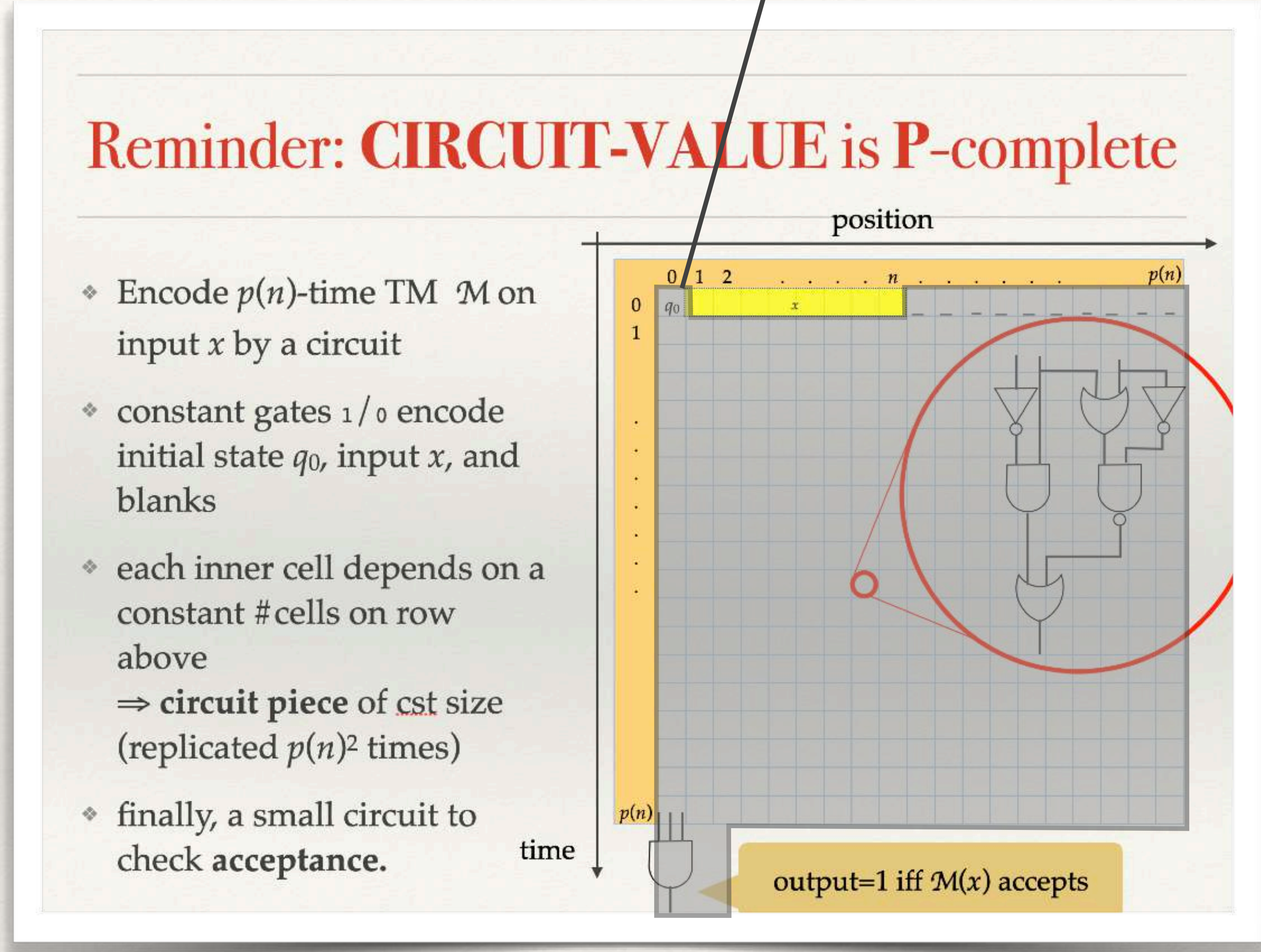
- such that for **every** x of that size n,
  $\mathcal{M}(x)$ accepts $\Leftrightarrow C_n[x]=1$

C_n



Uniform P/poly

- A language L is in **uniform P/poly** iff for every n, one can build a circuit C_n
  — in space $O(\log n)$
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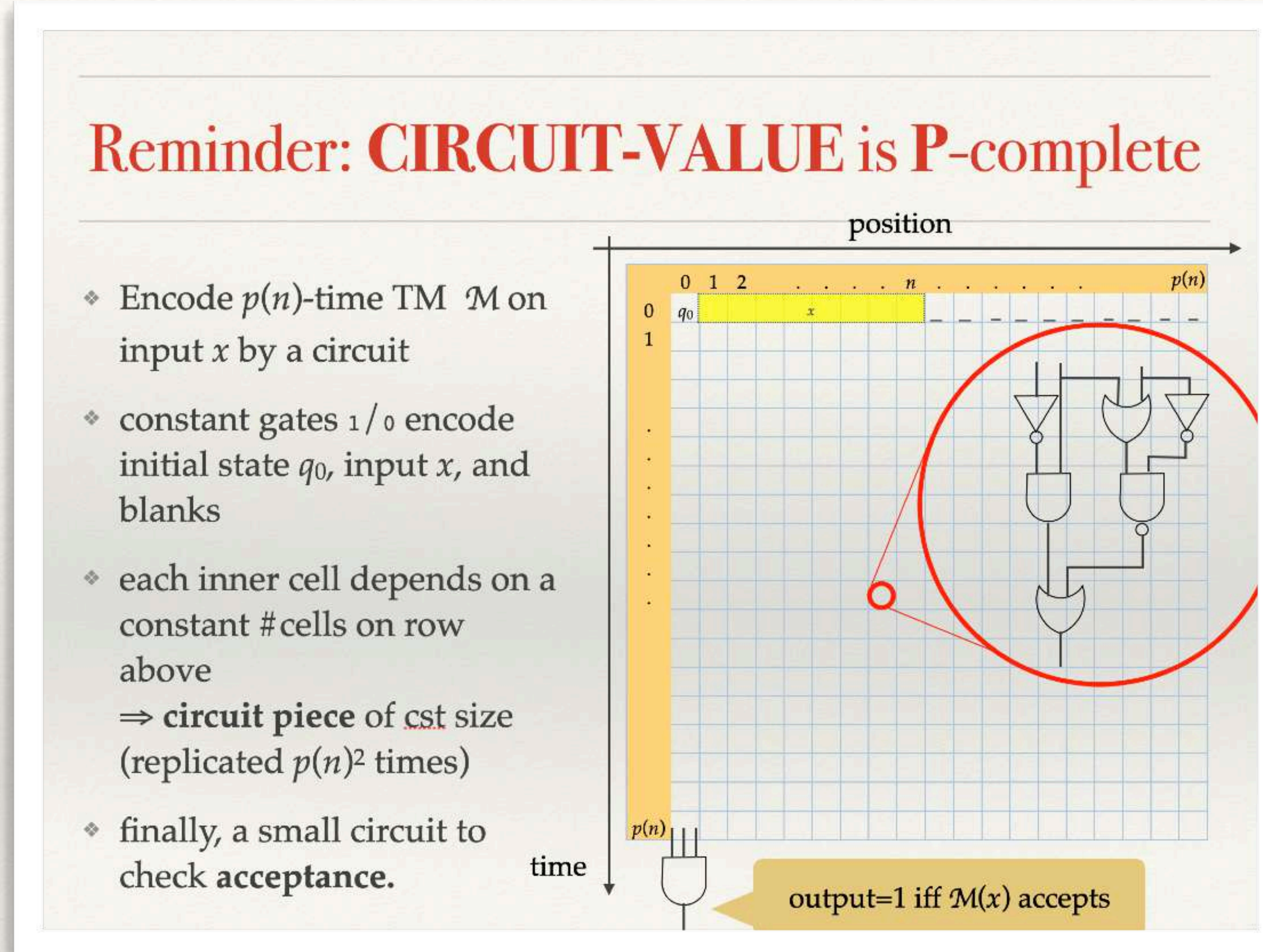

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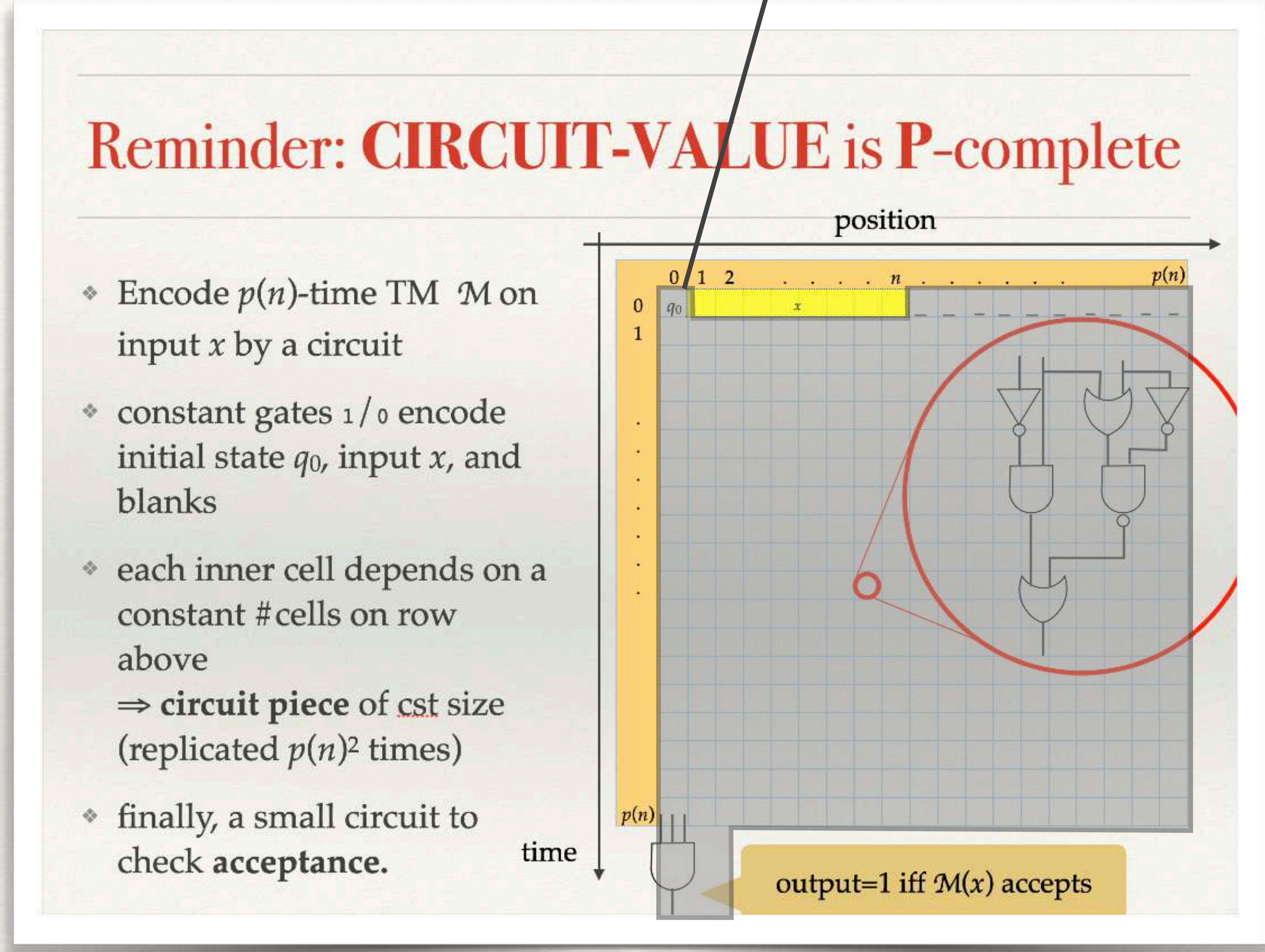

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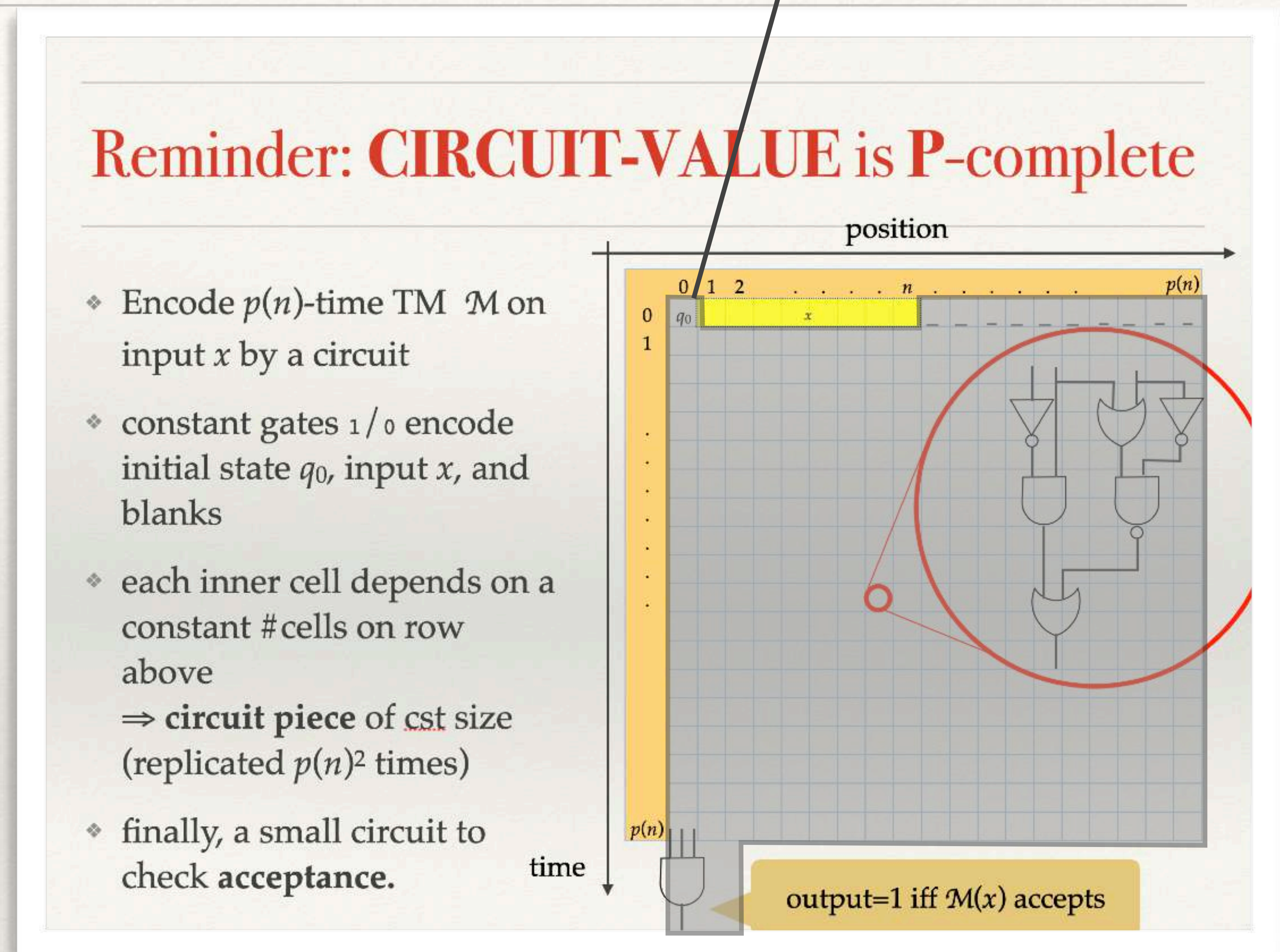

- **Prop. P $\subseteq$ uniform P/poly.** (This is what we have just proved!)

P = Uniform P/poly

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  $x \in L \Leftrightarrow C_n[x]=1$
- **Prop. P $\subseteq$ uniform P/poly**.

- In fact:
  Prop. P = uniform P/poly.
- *Proof.*
  Let $L \in$ **uniform P/poly**.
  On input x (size n),
  compute C_n in space $k \log n$,
  hence in time $O(n^k)$.
  Then evaluate $C_n[x]$
  in polytime.
  Hence $L \in$ **P**.

(Non-uniform) P/poly

- A language L is in
  ~~**uniform**~~ **P/poly** iff
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- Familiarly, we say that L **has polynomial circuits**

P/poly

- **Defn.** A language L is in **P/poly** iff

  there is a family $(C_n)_{n \in \mathbb{N}}$ of circuits:

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- It was initially hoped that we could prove that some **NP**-complete languages do **not** have polynomial circuits. That would immediately imply **P**≠**NP**, since **P** ⊆ **P/poly**.

P/poly is pretty weird

- **Prop. P/poly** contains some undecidable languages.

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- *Proof.*
  Let L be undecidable (e.g., **HALT**).
  Then $L' = \{\text{words } 1^n \mid$
  $a_1 \ldots a_k \in L,\ n = a_1 + 2a_2 + \ldots + 2^{k-1}a_k + 2^k\}$
  is undecidable, too; and C_n is…

convert from binary to unary

If $\mathrm{bin}(n) \notin L$	If $\mathrm{bin}(n) \in L$

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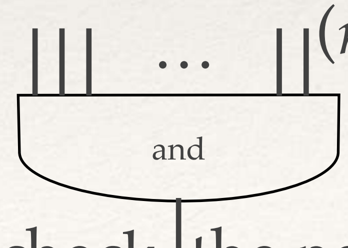
0 (ignores its input, size O(1))	 (n input bits) (size n log n: check the net-list!)

Weird, too: advice strings

- Imagine you wish to decide whether x is in L.
- … and you have a « cheat sheet » w_n depending only on n=size(x).
  How can this help?

Corpus ID: 115398060

Turing machines that take advice

R. Karp, R. J. Lipton · Published 1982 · Computer Science

https://upload.wikimedia.org/wikipedia/commons/thumb/3/3e/Karp_mg_7725-b.cr2.jpg/520px-Karp_mg_7725-b.cr2.jpg

https://cyber.gatech.edu/sites/default/files/styles/faculty_bio_pic/public/dick-lipton_1.jpg?itok=EkU43aPB

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- What if w_n is only allowed to have **polynomial size?**

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R. Karp, R. J. Lipton · Published 1982 · Computer Science

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Advice strings and **P/poly** (1/2)

Defn. A language L is in **P/poly** iff
there is a family $(C_n)_{n\in\mathbb{N}}$ of circuits:
— of size $p(n)$ (for some fixed polynomial p)
— such that for every input x (letting n be its si
$x \in L \Leftrightarrow C_n[x]=1$.

- **Prop.** $L \in$ **P/poly** iff there is a polytime TM $\mathcal{M}$ and a family $(w_n)_{n\in\mathbb{N}}$ of so-called
  advice strings:
  — of polysize $p(n)$
  — s.t. $\forall\, x$ (size n)
  $x \in L \Leftrightarrow \mathcal{M}(x,w_n)$ accepts.

Proof.

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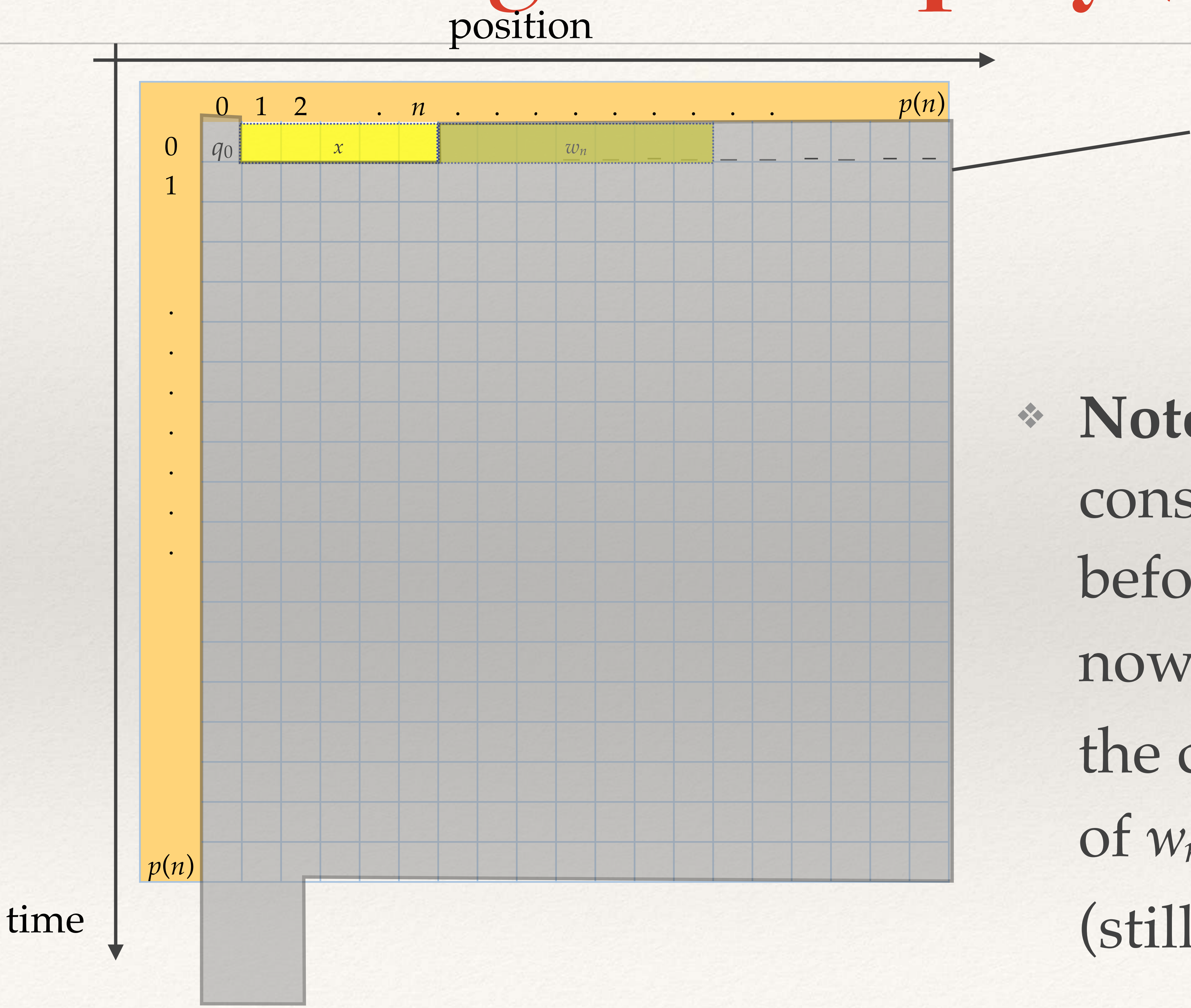
Proof.

- If $L \in$ **P/poly**, then let w_n be a net-list for C_n
- If L has advice strings w_n, then…

(see next slide)

Advice strings and **P/poly** (2/2)



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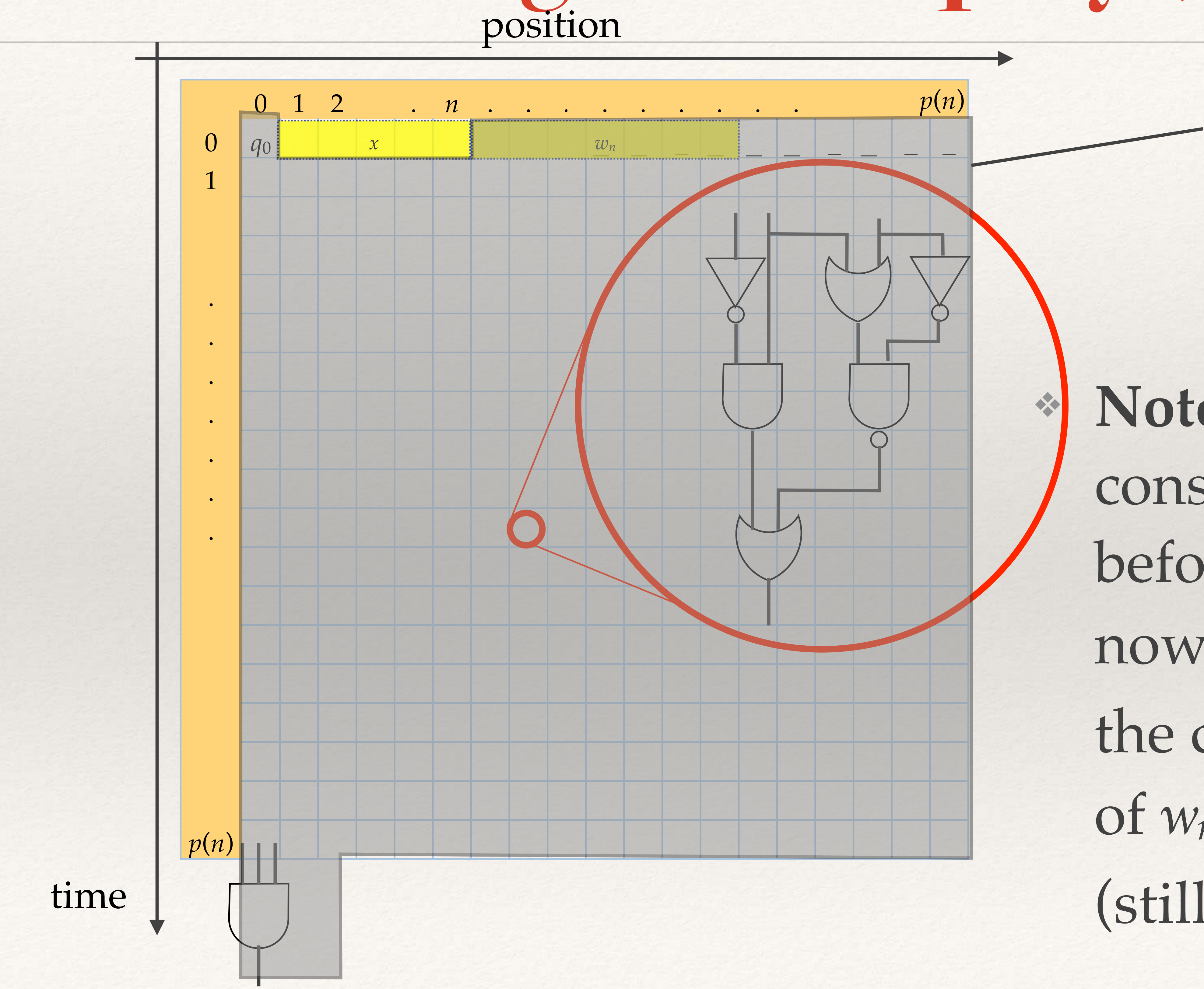

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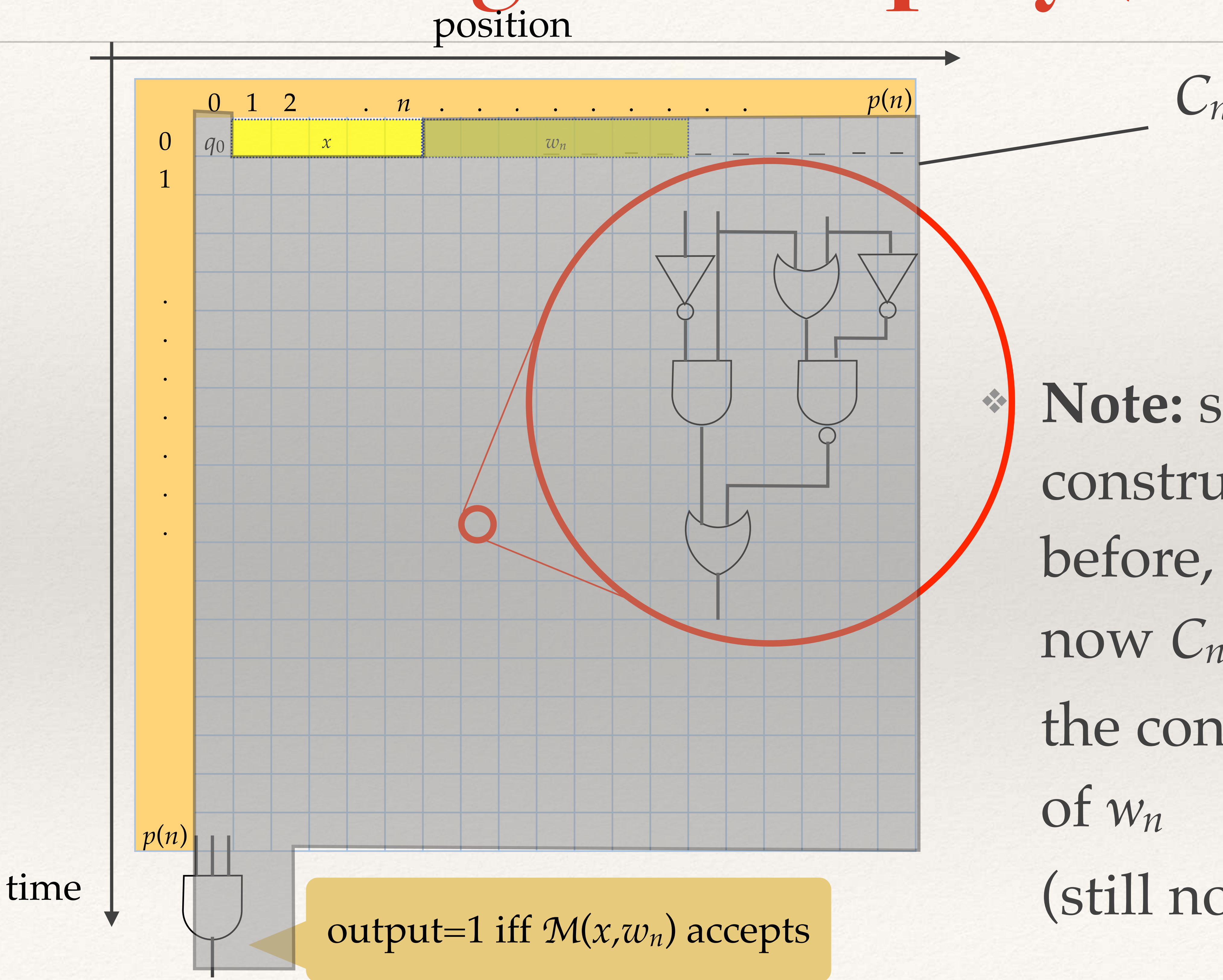

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Adleman's Theorem

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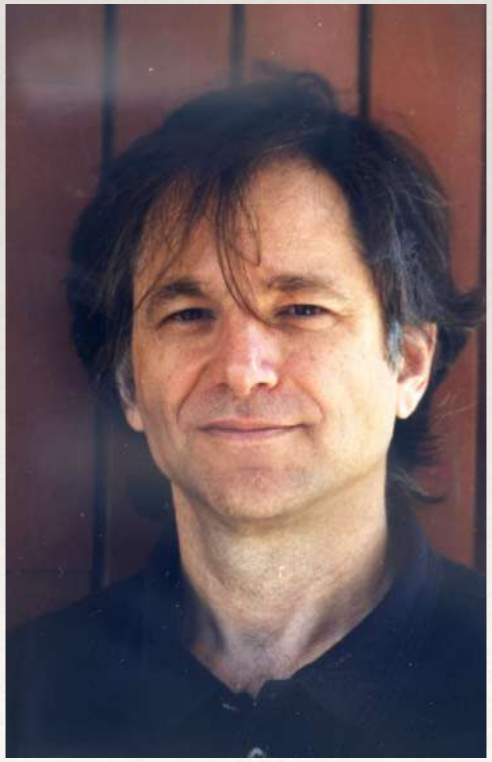
- **Theorem (Prop. 1.20). BPP $\subseteq$ P/poly.**

DOI: 10.1109/SFCS.1978.37 • Corpus ID: 15176763

Two theorems on random polynomial time

L. Adleman • Published 1978 • Computer Science •
19th Annual Symposium on Foundations of Computer Science (sfcs 1978)

The use of randomness in computation was first studied in abstraction by Gill [4]. In recent years its use in both practical and theoretical areas has become apparent. Strassen and Solovay [10]; Miller [7]; and Rabin [8] have used it to transform primality testing into a (for many purposes) tractible problem. We can see in retrospect that it was implicit in algorithms by Ber1ekamp [2], Lehmer [6], and Cippola [3] (1903!). Where the traditional method of polynomial reduction has been... CONTINUE READING



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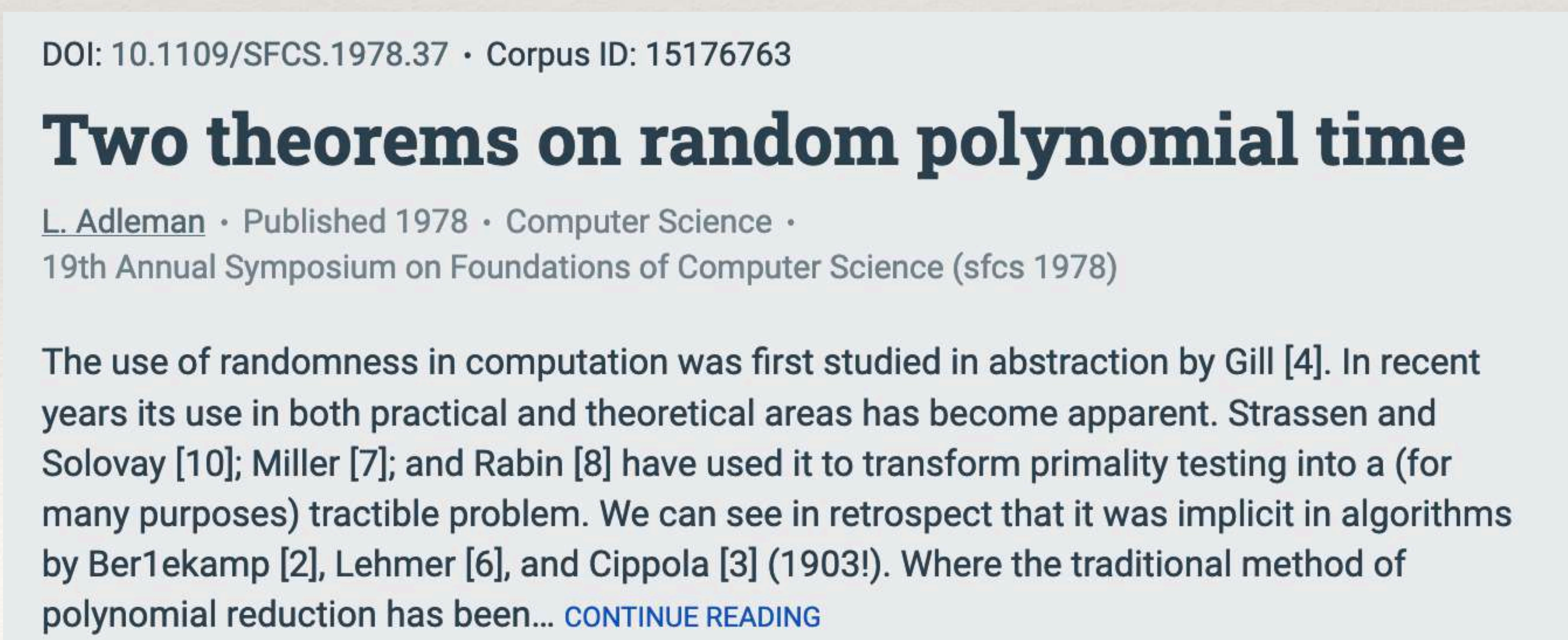
Adleman's Theorem

- **Theorem (Prop. 1.20). BPP $\subseteq$ P/poly.**
- Interestingly, we will be able to show the **existence** of the circuits C_n, (or the advice strings) but we won't be able to **compute** them (efficiently).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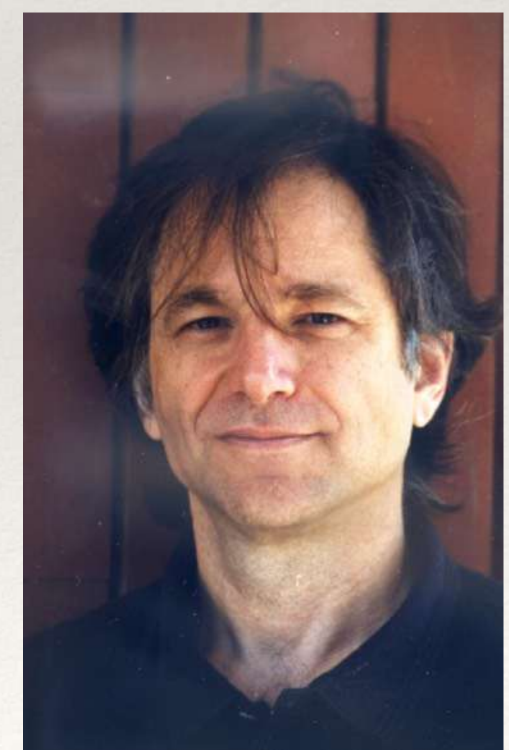

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The proof of Adleman's Theorem (1/2)

- Let L be in **BPP**.

A language L is in **BPP** if and only if there is a **polynomial-time** TM $\mathcal{M}$ such that for every input x (of size n):

$$\Pr_r(\mathcal{M}(x,r) \text{ errs}) \leq \varepsilon.$$

error $\varepsilon = 1/2^{q(n)}$

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- … < 1 if we had the good taste to pick $q(n)=n+1$, say.

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For each size n, there is a tape r_n (of size $p(n)$) such that for **every** x of size n, $\mathcal{M}(x,r_n)$ gives the correct answer, i.e.:
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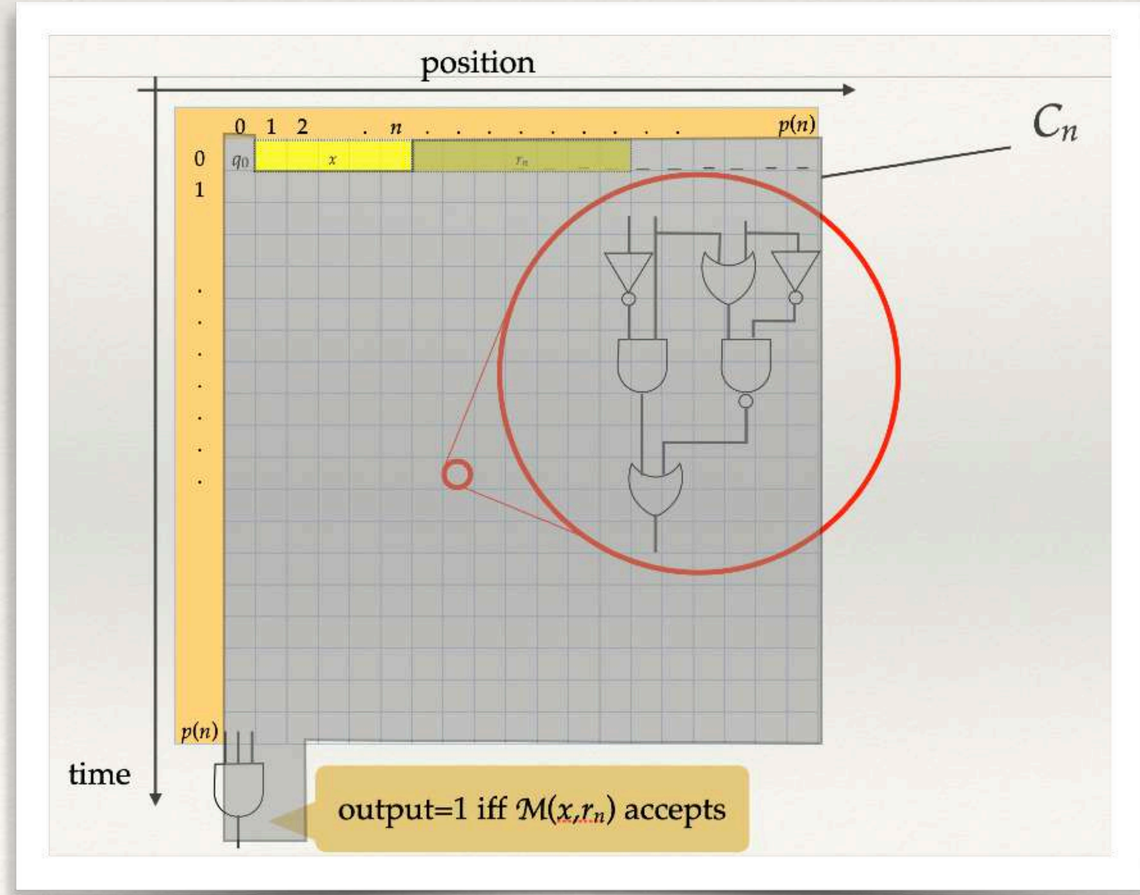
- … Just use r_n as advice string! □



The Karp-Lipton Theorems, and consequences

(Yes, them again!)

Corpus ID: 115398060

Turing machines that take advice

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$\mathbf{co}C$

- Recall that $\Pi^{\mathrm{p}}_{k}=\mathbf{co}\Sigma^{\mathrm{p}}_{k}$ for every $k\geq 1$.
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- **Fact.** **co** is monotonic: if $C \subseteq C'$, then $\mathbf{co}C \subseteq \mathbf{co}C'$.

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(**co**C is the class of complements of languages of C.)
- **Fact. co** is monotonic: if $C\subseteq C'$, then **co**$C\subseteq$**co**C'.
- (Already argued last time, as part of the Sipser-Gács-Lautemann theorem.)

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- **Claim.** For any class C, the following are equivalent:

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- $3 \Rightarrow 2$, and therefore $3 \Rightarrow 1$: similar. $1 \Rightarrow 2$: obvious. $\square$

Does PH collapse?

- We say that **PH** collapses at level 2 iff $\Sigma^p_2 = \Pi^p_2$.
  By the previous claim, equivalent to $\Pi^p_2 \subseteq \Sigma^p_2$.
- **Prop.** If $\Sigma^p_2 = \Pi^p_2$ then
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- $\Sigma^p_3 = \exists\cdot\Pi^p_2 = \exists\cdot\Sigma^p_2 = \exists\cdot\exists\cdot\mathbf{coNP} = \exists\cdot\mathbf{coNP} = \Sigma^p_2$, then
  $\Pi^p_3 = \mathbf{co}\Sigma^p_3 = \mathbf{co}\Sigma^p_2 = \Pi^p_2 = \Sigma^p_2$, etc. $\square$

The first Karp-Lipton theorem

- **Theorem (Prop. 1.21).** If **NP** $\subseteq$ **P/poly**, then the polynomial hierarchy collapses at level 2: $\Pi^P{}_2 \subseteq \Sigma^P{}_2$.
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  Where is the bug?
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Hint: this is Σ^*, not L
(just take the constant circuit 1 for C here)

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- … but what prevents it from **cheating**?
  We must **check** that the circuit C it gives us does the job.

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A thought experiment

- Imagine you want to solve **SAT**.
  You are given a clause set *S*,
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- Merlin answers: « yes »
- What can you conclude?

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- You check $\varrho \models S$, **accept** if this is true, **reject** otherwise.
- If S satisfiable, then Merlin can make you accept.
  Otherwise, you will necessarily reject.

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$S[A_1{:=}1]$ sat? (using O)

yes — no

yes	no
set $A_1{:=}1$	set $A_1{:=}0$
$S_1{:=}S[A_1{:=}1]$	$S_1{:=}S[A_1{:=}0]$

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yes: set $A_1{:=}1$, $S_1{:=}S[A_1{:=}1]$

no: set $A_1{:=}0$, $S_1{:=}S[A_1{:=}0]$

$S_1[A_2{:=}1]$ sat? (using O)

yes: set $A_2{:=}1$, $S_2{:=}S_1[A_2{:=}1]$

no: set $A_2{:=}0$, $S_2{:=}S_1[A_2{:=}0]$

Self-reducibility

- Now Merlin complains he can only decide whether S is satisfiable (using circuits C_n), **not** find a satisfying ϱ
- You retort that **SAT** is **self-reducible**:
  Given an oracle O deciding satisfiability,
  one can compute ϱ such that $\varrho \models S$ (if any).

$S[A_1{:=}1]$ sat? (using O)

yes: set $A_1{:=}1$, $S_1{:=}S[A_1{:=}1]$

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$S_1[A_2{:=}1]$ sat? (using O)

yes: set $A_2{:=}1$, $S_2{:=}S_1[A_2{:=}1]$

no: set $A_2{:=}0$, $S_2{:=}S_1[A_2{:=}0]$

$S_2[A_3{:=}1]$ sat? (using O)

yes: set $A_3{:=}1$, $S_3{:=}S_2[A_3{:=}1]$

no: set $A_3{:=}0$, $S_3{:=}S_2[A_3{:=}0]$

etc.

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Call this the « **self-reducibility machine** »

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- Instead of an oracle O, Merlin will use circuits C_m on clause sets S, S_1, S_2, …, of various sizes m.

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Self-reducibility

- Instead of an oracle O, Merlin will use circuits C_m on clause sets S, S_1, S_2, …, of various sizes m.
- m is bounded by n=size(S)
  (e.g., $S[A:=1]$ is obtained by removing clauses in which $+A$ appears, and removing $-A$ in the remaining clauses)

$S[A_1:=1]$ sat? (using O)

set $A_1:=1$
$S_1:=S[A_1:=1]$

set $A_1:=0$
$S_1:=S[A_1:=0]$

$S_1[A_2:=1]$ sat? (using O)

set $A_2:=1$
$S_2:=S_1[A_2:=1]$

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$S_2:=S_1[A_2:=0]$

$S_2[A_3:=1]$ sat? (using O)

set $A_3:=1$
$S_3:=S_2[A_3:=1]$

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A circuit for self-reducibility

- Now given (net-lists for) C_0, C_1, …, C_n as **advice** $w_{0\ldots n}$

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by the way, not quite the trick used in the lecture notes

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- **Theorem (Prop. 1.21).** If **NP** $\subseteq$ **P/poly**, then the polynomial hierarchy collapses at level 2: $\Pi^{P}_{2} \subseteq \Sigma^{P}_{2}$.
- Let $L \in \Pi^{P}_{2}$ be $\{x \mid \forall y$ of size $p(n), (x,y) \in L'\}$, $L' \in$ **NP**.

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- Hence $L = \{x \mid \forall y$ of size $p(n)$, $f(x,y) \in$ **SAT**$\}$

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- I claim that $L = \{x \mid \exists w, \forall y$ of size $p(n),\ h(f(x,y),w) \vDash f(x,y)\}$

  (huh? that was the bug, right? No, we now **check** that $h(\ldots) \vDash f(x,y)$!)

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The second Karp-Lipton theorem

- **Theorem (Prop. 1.22).** If $\mathbf{NP} \subseteq \mathbf{P/poly}$, then $\mathbf{PH} \subseteq \mathbf{P/poly}$.

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  This is in **NP**, hence has polynomial circuits C_n, too!

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size of x + cst + size of $w_{\text{size}(x,y)}$…
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- Hence L is decided by the circuits

$$C_{\text{appropriate size}}[(\_, w_{\text{size}(x,y)})] \qquad \square$$

(all sizes depending only on n=size(x), not on x itself)

Conclusion

BPP cannot be too large

- **Corollary.** If **BPP** contains **NP**, then:
  — **PH** collapses at level 2 (unlikely)
  — and is included in **P/poly**.

- *Proof.*

Adleman's Theorem

Theorem (Prop. 1.20). $\mathbf{BPP} \subseteq \mathbf{P/poly}$.

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Theorem (Prop. 1.21). If $\mathbf{NP} \subseteq \mathbf{P/poly}$, then the polynomial hierarchy collapses at level 2: $\Pi P_2 \subseteq \Sigma P_2$.

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