Jean Goubault-Larrecq

Randomized complexity classes

Today: the Arthur vs. Merlin hierarchy **collapses**

Today

- Babai's theorem: **MA** $\subseteq$ **AM**
- Also, **MAM** $\subseteq$ **AM**
- A detour through promise problems
- The Arthur vs. Merlin hierarchy collapses.

MA is included in **AM**

Converting **MA** to **AM**

Converting MA to AM

Why not just use « skolemization »?
(Hint: does not work.)

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  (x of size n, r of poly size $q(n)$, y of poly size $p(n)$) such that $\forall x$,
  — either (1): $(\exists y, \mathrm{E}r, F(x,y,r)) \geq 1\text{–}1/2^n$ (« huge »)
  — or (2): $(\exists y, \mathrm{E}r, F(x,y,r)) \leq 1/2^n$ (« tiny »)

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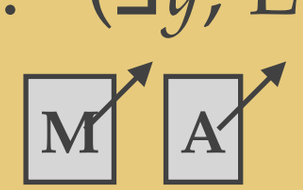

- Then for every poly $g(n)$, and for n large enough,
  — in case (1), $F'(x) \geq 1\text{–}1/2^{g(n)}$
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M A

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Permuting E over ∃, à la Lautemann

- In case (1) (« **huge** »),
  $(\mathrm{E}r,\ F(x,y,r)) \geq 1-1/2^n$ for some y.
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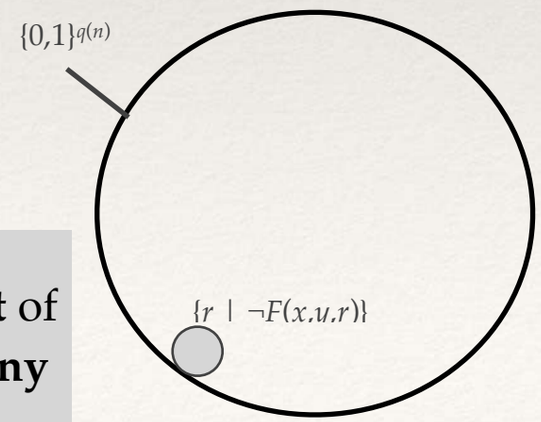
The **complement** of $\{r \mid F(x,y,r)\}$ is **tiny**



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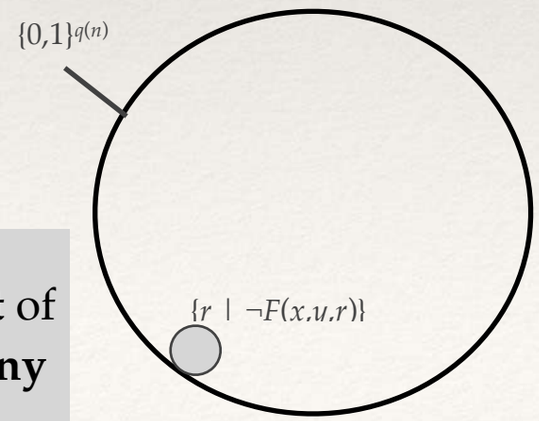

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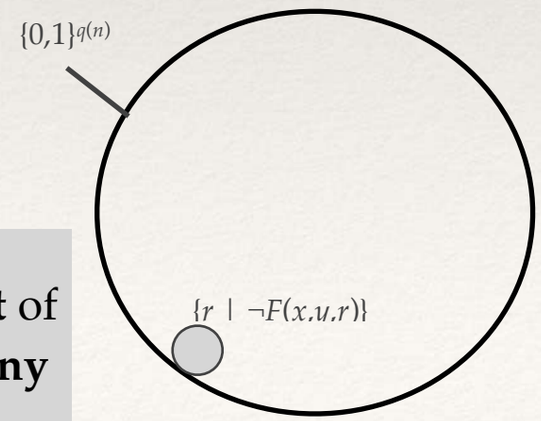

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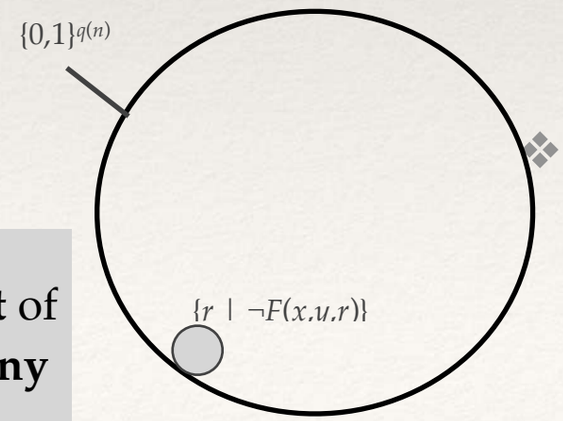

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- Since k=poly(n), this is
  < 1 for n large enough.

Permuting E over ∃, à la Lautemann

- In case (2) (« **tiny** »),
  $(\mathrm{E}r, F(x,y,r)) \leq 1/2^n$ for **every** y.

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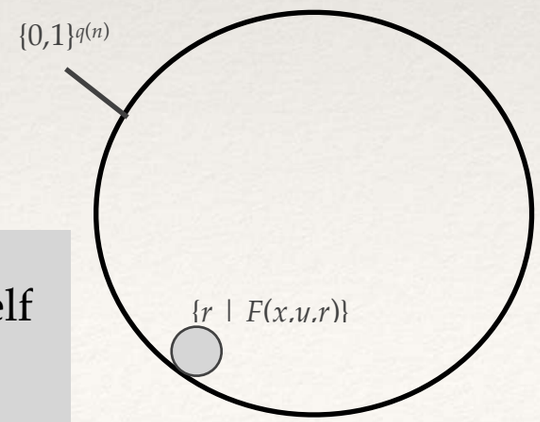
$\{r \mid F(x,y,r)\}$ itself is **tiny**



Permuting E over ∃, à la Lautemann

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- $\Pr_r(F(x,y,r)) \leq 1/2^n$ for **every** y.
  ($F(x,y,r)$ is a **predicate**!)
- $\Pr_{r1,\ldots,rk}(\exists y,r',\ \bigwedge_{i=1}^{k} F(x,y,r' \oplus r_i))$
  $\leq \sum_{y,r'} \Pr_{r1,\ldots,rk}(\bigwedge_{i=1}^{k} F(x,y,r' \oplus r_i))$
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  (**independence**)

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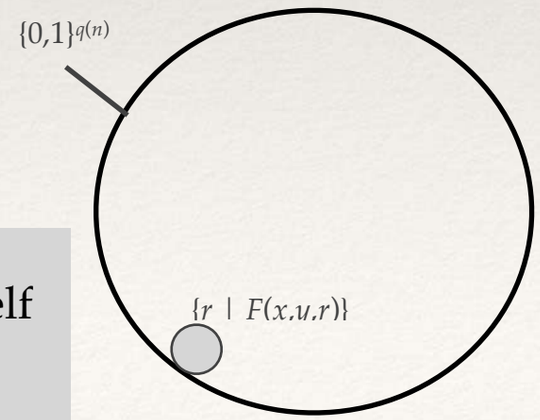

Permuting E over ∃, à la Lautemann

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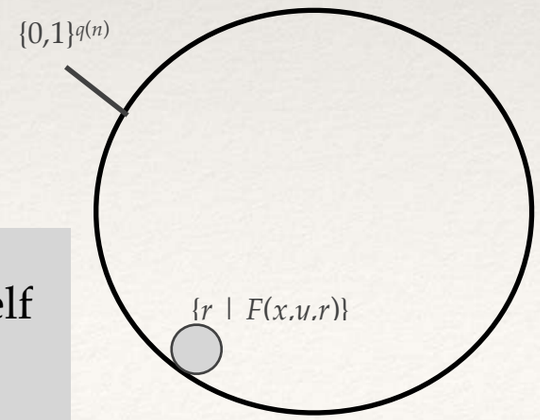
  (**independence**)
- $\le \sum_{y,r'} (1/2^n)^k$
  $= 2^{p(n)+q(n)-nk}$
  $\le 1/2^{g(n)}$. □



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MA ⊆ AM

- **Thm 3.12 (Babai). MA ⊆ AM.**

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- *Proof.* Let $L \in$ **MA**.
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Ah yes, case (2) only applies for n large enough.
For small values of n, tabulate.

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- **Lemma 3.11. MAM ⊆ AM.**

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Intermission: promise problems

Promise problems: example

- Look back at, say, **SAT**:
  INPUT: a clause set *S*
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- Some input words may **fail to parse** as clause sets.
- Hence, really, what we are interested in is…

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- INPUT: a word w
  PROMISE: w parses as a clause set S
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- E.g., for **SAT**, both are **NP**-complete.

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Promise problems are sometimes useful

- Let **BPP′** be the promise version of **BPP**, i.e.:
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- (There is no known **BPP**-complete problem.)

Promise versions of Arthur-Merlin games

- All the classes in the Arthur-Merlin hierarchy have analogues as **promise problems**:

  ε′, A′, M′, MA′, AM′, etc.

- $(L^+, L^-) \in$ **AMAM…′** iff

  for every polynomial $g(n)$,

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- **Thm 3.12′. MA′ ⊆ AM′.**
  Lemma 3.11′. MAM′ ⊆ AM′.

(same proof as before!)

The Arthur-Merlin hierarchy collapses

The A-M' hierarchy collapses

- We will show by induction on the length of w that $\mathbf{w'} \subseteq \mathbf{AM'}$.
- Obvious if this length is 0.
- We will then look at the first letter of w, either `A` or `M`.

$\mathbf{w'} \subseteq \mathbf{AM'}$: (1) w starts with $\mathtt{A}$

- Let $w \stackrel{\text{def}}{=} \mathtt{A}\ w_2$, and let $(L^+,L^-) \in \mathbf{w'}$… first, a useful lemma:

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- Let $w \stackrel{\text{def}}{=} \mathtt{A}\, w_2$, and let $(L^+,L^-) \in \mathbf{w'}$… first, a useful lemma:

- **Square root lemma.** Let $0<\varepsilon<1$, and X be a non-negative real-valued random variable with finite expectation.
(i) If $E(X)\leq\varepsilon$ then $\Pr(X\leq\sqrt{\varepsilon})\geq 1-\sqrt{\varepsilon}$
(ii) If $E(X)\geq 1-\varepsilon$ then $\Pr(X\geq 1-\sqrt{\varepsilon})\geq 1-\sqrt{\varepsilon}$.

I.e., if the expectation of X is **very large**, then X is **large**, with **high probability**.

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Theorem (Markov's inequality).
Let X be a **non-negative real-valued** ran
with **finite** expectation $E(X)$. For eve
$\Pr(X \geq a.E(X)) \leq 1/a$.

- *Proof.* (i) Let $a \stackrel{\text{def}}{=} 1/\sqrt{\varepsilon}$, so $a.E(X)\leq\sqrt{\varepsilon}$.

$$\Pr(X>\sqrt{\varepsilon}) \leq \Pr(X \geq \sqrt{\varepsilon}) \leq \Pr(X\geq a.E(X)) \leq \sqrt{\varepsilon}.$$

(ii) Use (i) with X replaced by $1-X$. □

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$\mathbf{w'} \subseteq \mathbf{AM'}$: (1) w starts with $\mathtt{A}$ (1/5)

- Let $w \stackrel{\text{def}}{=} \mathtt{A}\, w_2$, and let $(L^+,L^-) \in \mathbf{w'}$.
  — if $x \in L^+$, then $(\mathrm{E}r,\, F(x,r)) \geq 1 - 1/2^{2g(n)+2}$
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i.e., $(L^+,L^-) \in \mathbf{w'}$ is decided by a formula of the form $\mathrm{E}r, \exists y_1, \mathrm{E}r_2, \exists y_2, \ldots$ $F(x,r)$

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We reduce the error preventively. This will be needed.

- Beware: F is not a predicate, so expectation ≠ probability

w' ⊆ AM': (1) w starts with A (1/5)

i.e., $(L^+,L^-) \in \mathbf{w'}$ is decided by a formula of the form $\mathrm{E}r, \exists y_1, \mathrm{E}r_2, \exists y_2, \ldots$ $F(x,r)$

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Square root lemma. Let $0<\varepsilon<1$, and X be a non-n real-valued random variable with finite expectat
(i) If $\mathrm{E}(X) \leq \varepsilon$ then $\Pr(X \leq \sqrt{\varepsilon}) \geq 1–\sqrt{\varepsilon}$
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w' ⊆ AM': (1) w starts with A (2/5)

- Let $w \stackrel{\text{def}}{=} \mathtt{A}\, w_2$, and let $(L^+,L^-) \in \mathbf{w'}$.
  — if $x \in L^+$, then $(\mathrm{E}r,\, F(x,r)) \geq 1\text{–}1/2^{2g(n)+2}$
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- If $x \in L^+$, then by (ii) $\Pr_r(F(x,r) \geq 1\text{–}1/2^{g(n)+1}) \geq 1\text{–}1/2^{g(n)+1}$
- If $x \in L^-$, then by (i) $\Pr_r(F(x,r) \leq 1/2^{g(n)+1}) \geq 1\text{–}1/2^{g(n)+1}$

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w' ⊆ AM': (1) w starts with A (3/5)

- Let $w \stackrel{\text{def}}{=} \mathtt{A}\, w_2$, and let $(L^+,L^-) \in \mathbf{w'}$.
  — if $x \in L^+$, then $(\mathrm{E}r,\, F(x,r)) \geq 1\text{–}1/2^{2g(n)+2}$
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- With high probability on r ($\geq 1\text{–}1/2^{g(n)+1}$),
  — if $x \in L^+$, then $F(x,r) \geq 1\text{–}1/2^{g(n)+1}$
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$\mathbf{w}' \subseteq \mathbf{AM}'$: (1) w starts with $\mathtt{A}$ (3/5)

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- With high probability on r ($\geq 1\text{–}1/2^{g(n)+1}$),
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w' ⊆ AM': (1) w starts with A (4/5)

- Since $(D^+,D^-) \in \mathbf{AM'}$, for some $D \in \mathbf{P}$:
  — if $x\#r \in D^+$, then
  $\Pr_{r'}(\exists y',\ x\#r\#r'\#y' \in D) \geq 1-1/2^{g(n)+1}$
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- Let $w \triangleq \mathtt{A}\, w_2$, and let $(L^+,L^-) \in \boldsymbol{w'}$.
  — if $x \in L^+$, then $(\mathrm{E}r,\ F(x,r)) \geq 1-1/2^{2g(n)+2}$
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- With high probability on r ($\geq 1-1/2^{g(n)+1}$),
  — if $x \in L^+$, then $F(x,r) \geq 1-1/2^{g(n)+1}$
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- Let $D^+ \triangleq \{x\#r \mid F(x,r) \geq 1-1/2^{g(n)+1}\}$
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  (D^+,D^-) is a promise language in $\boldsymbol{w_2'}$.
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- If $x \in L^-$ then $(\exists y',\ x\#r\#r'\#y' \in D)$ holds:
  — with prob. $\leq 1/2^{g(n)+1}$ (on r') if $x\#r \in D^-$,
  — and $x\#r \notin D^-$ happens (i.e., $F(x,r) > 1/2^{g(n)+1}$)
  with prob. $\leq 1/2^{g(n)+1}$ (on r)

  hence with prob. $\leq 1/2^{g(n)}$ total (on r, r')

- Let $w \triangleq \mathtt{A}\, w_2$, and let $(L^+,L^-) \in \boldsymbol{w'}$.
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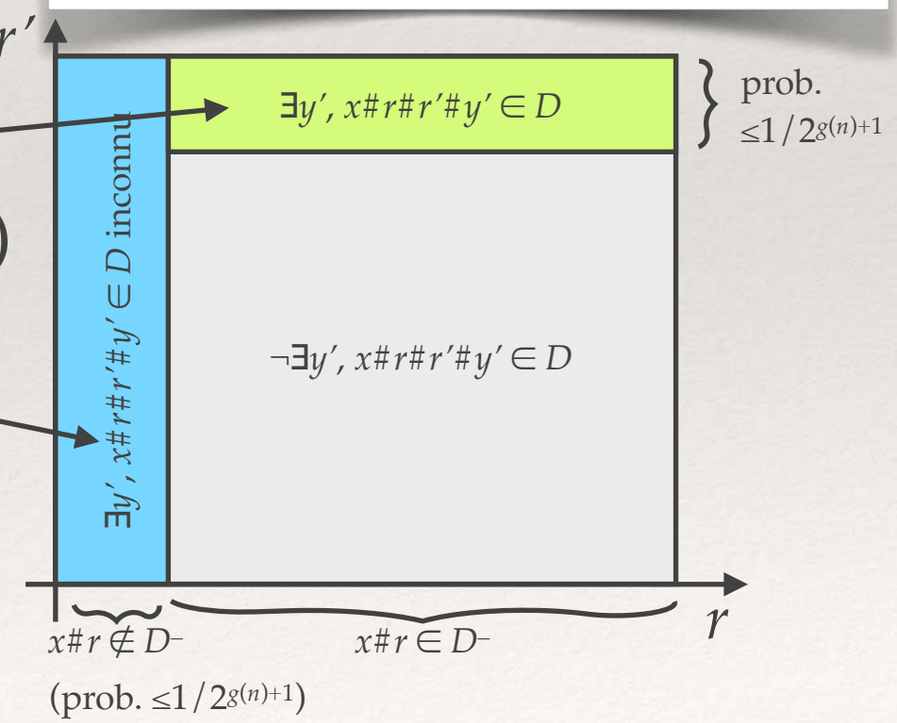

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- Hence $\Pr_{r,r'}(\exists y',\ x\#r\#r'\#y' \in D) \leq 1/2^{g(n)}$

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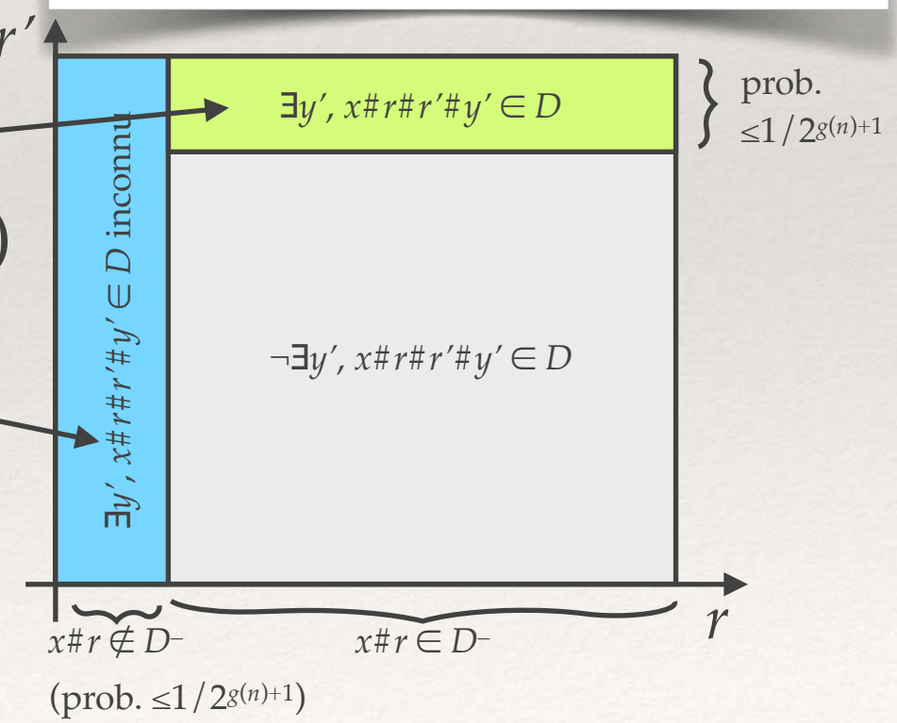

w' ⊆ AM': (1) w starts with A (5/5)

- In summary:
  - If $x \in L^-$ then $\Pr_{r,r'}(\exists y',\ x\#r\#r'\#y' \in D) \leq 1/2^{g(n)}$
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- Therefore (L^+,L^-) is in **AM'**.

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- In summary:
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  - If $x \in L^+$ then $\Pr_{r,r'}(\exists y',\ x\#r\#r'\#y' \in D) \geq 1-1/2^{g(n)}$
- Therefore (L^+,L^-) is in **AM'**.
- Since (L^+,L^-) was arbitrary in **w'**, **w'** $\subseteq$ **AM'**.

w' $\subseteq$ AM': (1) w starts with M (1/2)

This is simpler!

w’ $\subseteq$ **AM’**: (1) w starts with $\mathtt{M}$ (1/2)

- Let $w \stackrel{\text{def}}{=} \mathtt{M}\, w_2$, and let $(L^+,L^-) \in \mathbf{w'}$.
  — if $x \in L^+$, then for some y, $F(x,y) \geq 1–1/2^{g(n)}$
  — if $x \in L^-$ then for every y, $F(x,y) \leq 1/2^{g(n)}$

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- (D^+,D^-) is a promise language in $\mathbf{w_2}'$.

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No way we could use a single language
D^+=complement of D^-

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- (D^+,D^-) is a promise language in $\mathbf{w_2'}$.
- By induction hypothesis, (D^+,D^-) is in $\mathbf{AM'}$.

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No way we could use a single language D^+=complement of D^-

$\mathbf{w' \subseteq AM'}$: (1) w starts with $\mathtt{M}$ (2/2)

- Since $(D^+,D^-) \in \mathbf{AM'}$, for some $D \in \mathbf{P}$:
  — if $x\#y \in D^+$, then
  $(\mathrm{E}r', \exists y', x\#y\#r'\#y' \in D) \geq 1-1/2^{g(n)}$
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- (D^+,D^-) is a promise language in $\mathbf{w_2'}$.
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- If $x \in L^-$ then for every y, $x\#y \in D^-$, so
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- Let $w \stackrel{\text{def}}{=} \mathtt{M}\, w_2$, and let $(L^+,L^-) \in$ **w'**.
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- If $x \in L^+$ then for some y, $x\#y \in D^+$, so
  $(\exists y, \mathrm{E}r', \exists y', x\#y\#r'\#y' \in D) \geq 1-1/2^{g(n)}$
- If $x \in L^-$ then for every y, $x\#y \in D^-$, so
  $(\exists y, \mathrm{E}r', \exists y', x\#y\#r'\#y' \in D) \leq 1/2^{g(n)}$
- Hence (L^+,L^-) is in **MAM'**… hence in **AM'**! □

- Let $w \stackrel{\text{def}}{=} \mathtt{M}\, w_2$, and let $(L^+,L^-) \in$ **w'**.
  — if $x \in L^+$, then for some y, $F(x,y) \geq 1-1/2^{g(n)}$
  — if $x \in L^-$ then for every y, $F(x,y) \leq 1/2^{g(n)}$
- Let $D^+ \stackrel{\text{def}}{=} \{x\#y \mid F(x,y) \geq 1-1/2^{g(n)}\}$
  $D^- \stackrel{\text{def}}{=} \{x\#y \mid F(x,y) \leq 1/2^{g(n)}\}$
- (D^+,D^-) is a promise language in **w₂'**.
- By induction hypothesis, (D^+,D^-) is in **AM'**.

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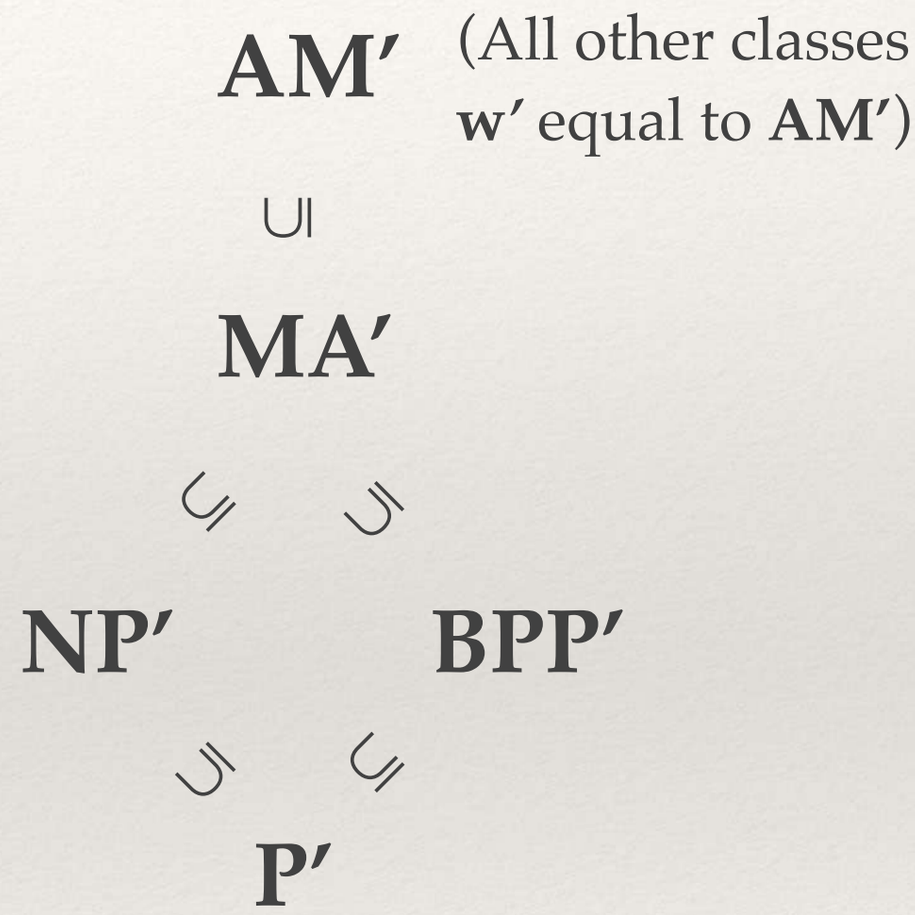
- So, for every word w of the form $w_1\mathtt{A}w_2\mathtt{M}w_3$, $\mathbf{w'=AM'}$.
- The remaining words are:
  — $w \in \mathtt{M}^+\mathtt{A}^+$: then $\mathbf{w' = MA'}$
  — $w \in \mathtt{M}^+$: then $\mathbf{w' = M'}$ ($=\mathbf{NP'}$)
  — $w \in \mathtt{A}^+$: then $\mathbf{w' = A'}$ ($=\mathbf{BPP'}$)
  — $w = \varepsilon$: then $\mathbf{w' = P'}$.

The Arthur-Merlin hierarchy collapses

- In summary:



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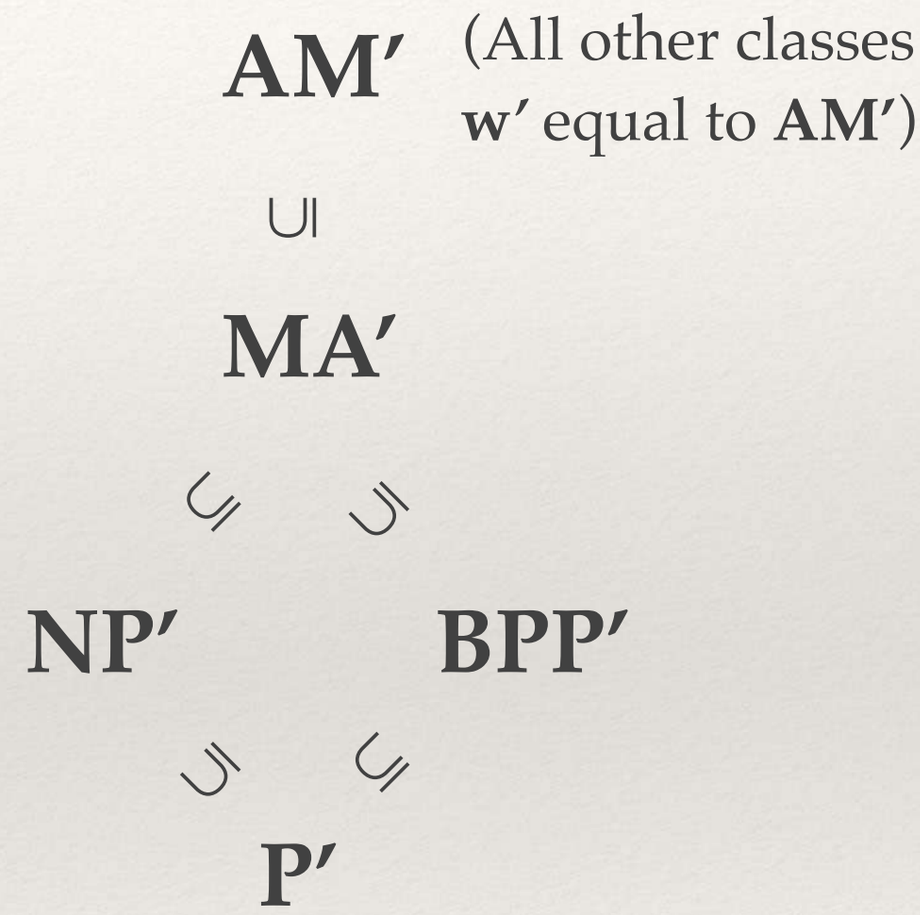

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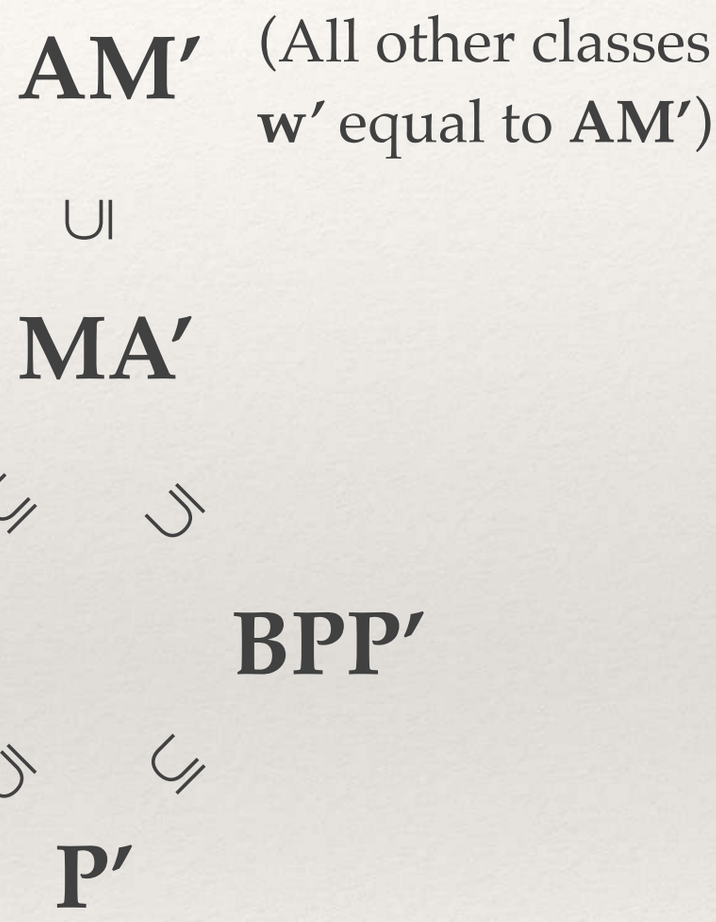

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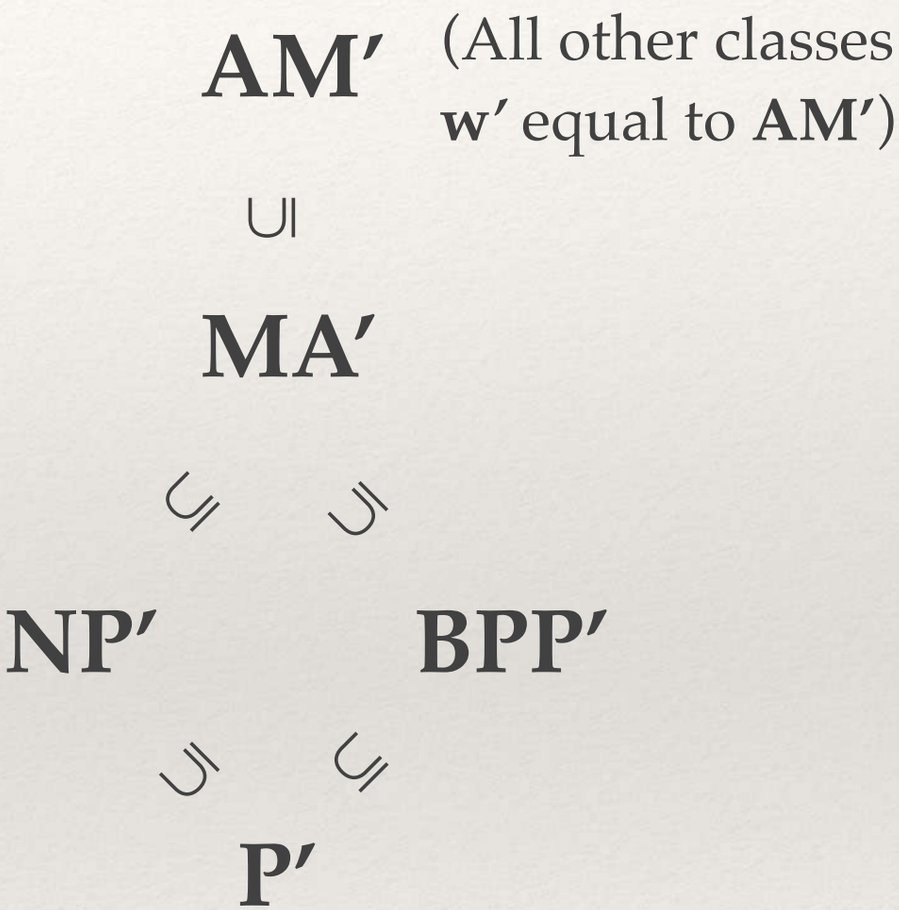
- Restricting to languages, we obtain…



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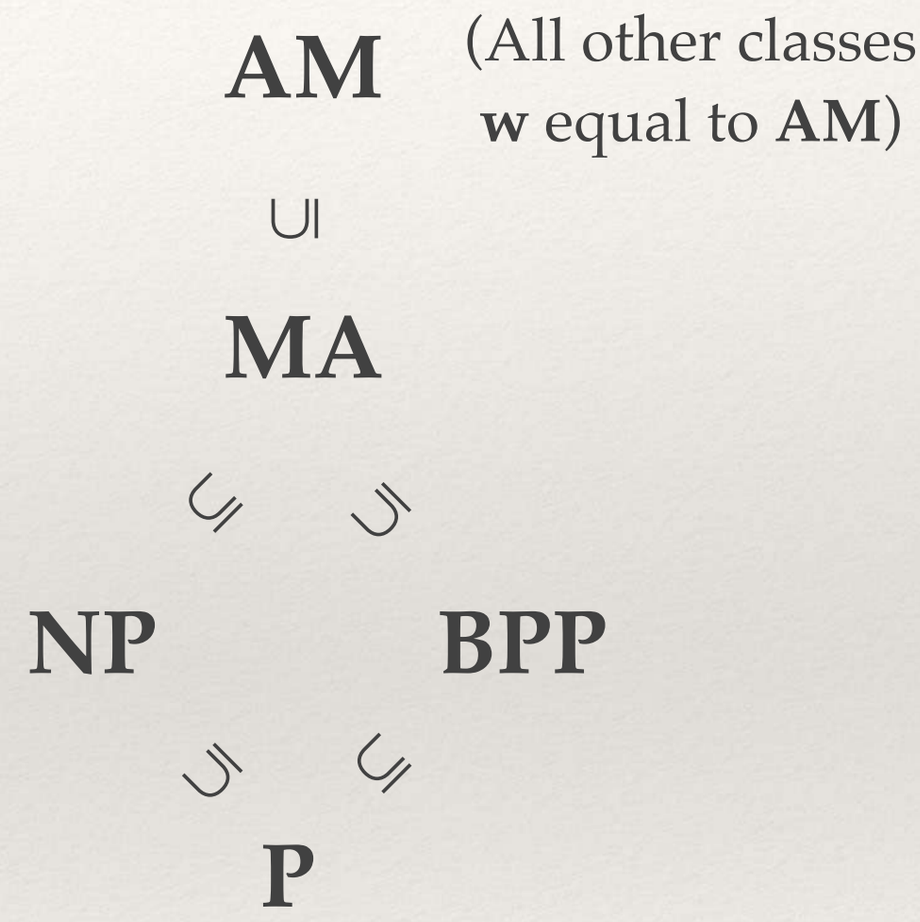

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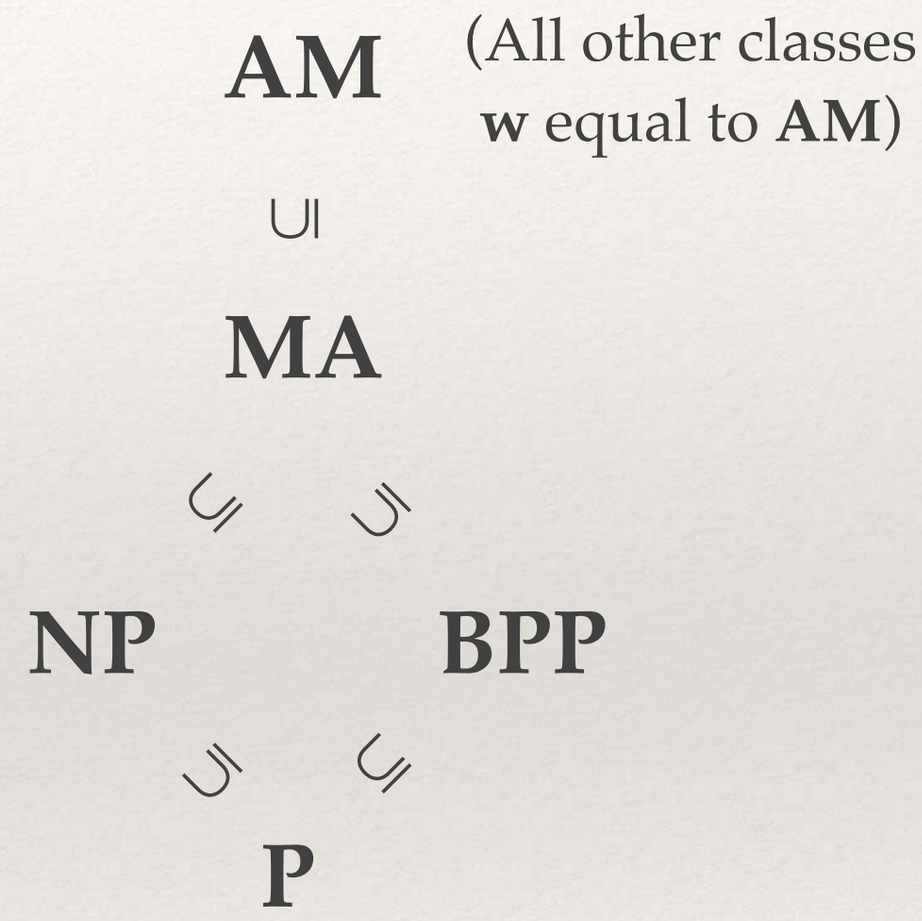

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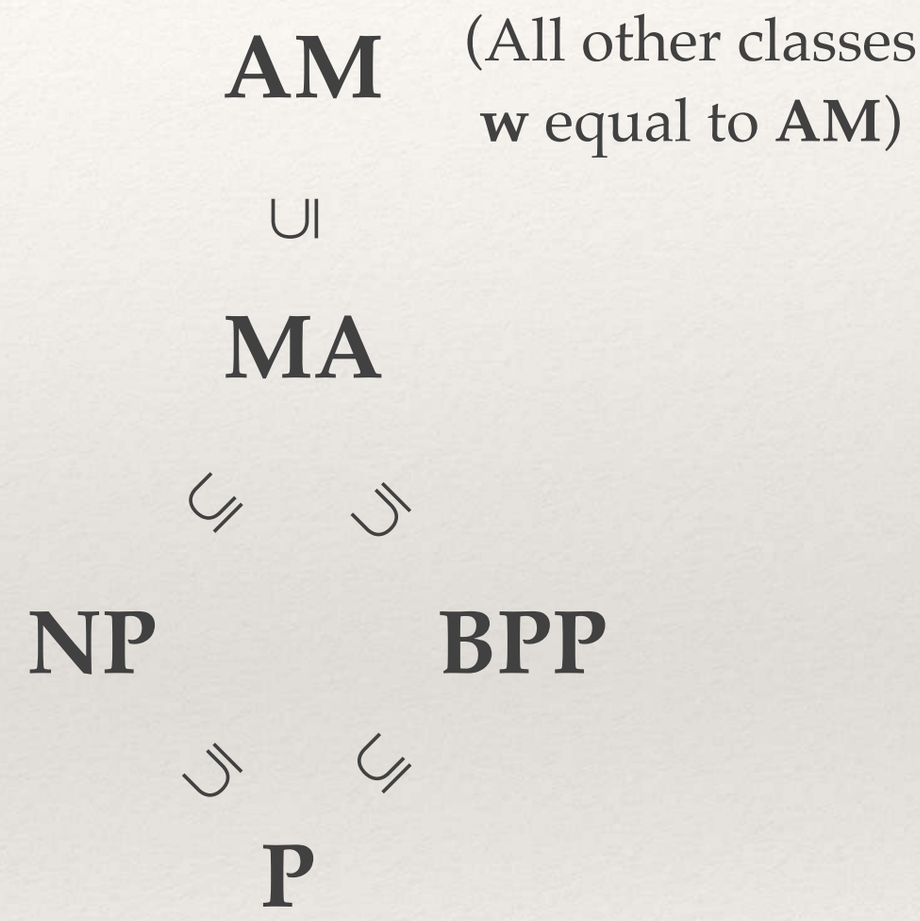

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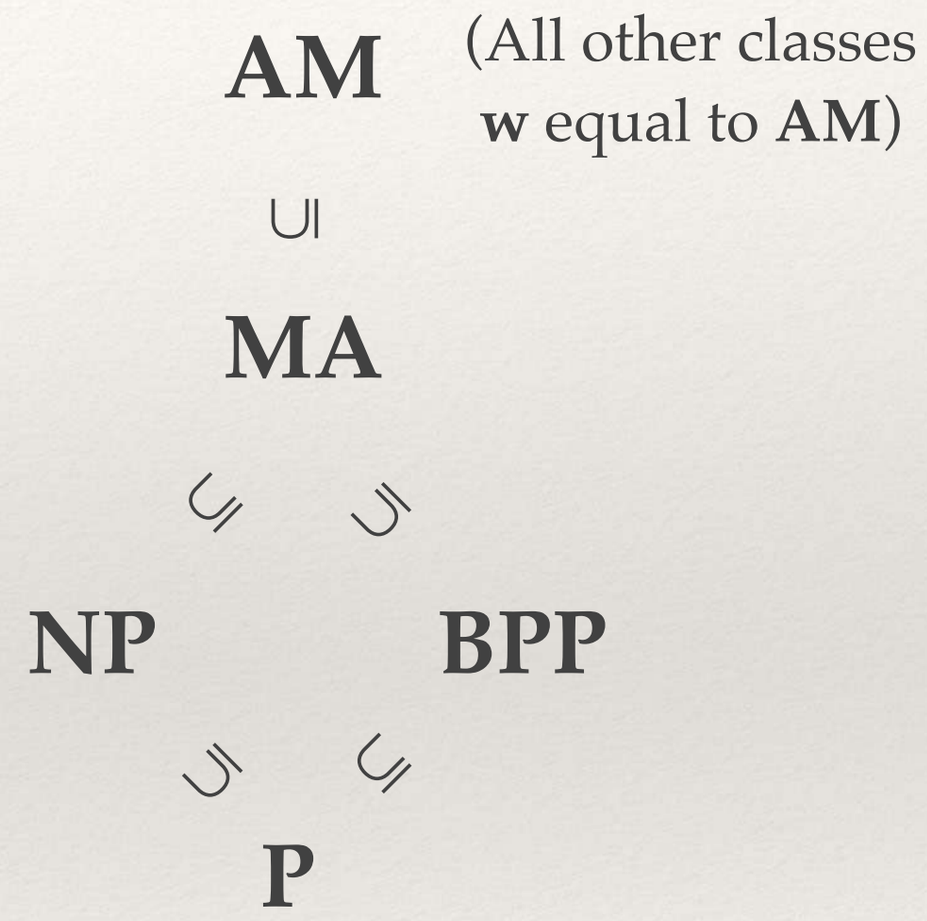

Variable number of turns $f(n)$… until now we only had a constant number of turns!

Next time…

Some more wonders!

- Sipser's coding lemmas
- **AM** is in the polynomial hierarchy
- The Goldwasser-Sipser theorem:
  public coins≡private coins
- The Boppana-Håstad-Zachos theorem:
  Graph Isomorphism is most certainly not **NP**-complete.