Jean Goubault-Larrecq

Randomized complexity classes

Today: approximation problems, PCP

Today

- Approximation problems
- The class **PCP**
- **MAX3SAT** is not ε-approximable iff **NP**=**PCP**
- The Arora-Safra theorem: **NP**=**PCP** (no proof…)

Approximation problems

Approximation

- Attempt to attack **NP**-complete problems, by relaxing requirements. E.g., **3SAT** is **NP**-complete. Instead, given $\varepsilon \in]0,1[$, let **MAX3SAT** be:
- **INPUT**: a finite set S of 3-clauses
  OUTPUT: an environment ϱ that satisfies $(1-\varepsilon)\mathrm{opt}(S)$
  where $\mathrm{opt}(S) \stackrel{\text{def}}{=} \max_{\varrho \text{ env.}} (\#\text{clauses of } S \text{ s.t. } \varrho \models S)$
- For which values of ε is that in **P**?

Maximization problems

- For each input x, [e.g., a set of 3-clauses]
  a finite set $F(x)$ of so-called
  feasible solutions [e.g., all assignments ϱ on the vars of S}]
- For each $y \in F(x)$,
  a **value** $c(y)$ [e.g., #clauses satisfied by ϱ]
- Goal: estimate $\mathrm{opt}(x) \stackrel{\text{def}}{=} \max_{y \in F(x)} c(y)$
- **ε-approximable** iff can find $y \in F(x)$ / $c(y) \geq (1-\varepsilon)\mathrm{opt}(x)$ in polynomial time
- **Defn.** The **approximation threshold** $= \inf_{\varepsilon\text{-approximable}} \varepsilon$

Minimization problems

- For each input x,
  a finite set $F(x)$ of so-called
  feasible solutions
- For each $y \in F(x)$,
  a **cost** $c(y)$
- Goal: estimate $\mathrm{opt}(x) \stackrel{\text{def}}{=} \min_{y \in F(x)} c(y)$
- **ε-approximable** iff can find $y \in F(x)$ / $c(y) \leq 1/(1-\varepsilon).\mathrm{opt}(x)$ in polynomial time
- **Defn.** The **approximation threshold** = $\inf_{\varepsilon\text{-approximable}} \varepsilon$

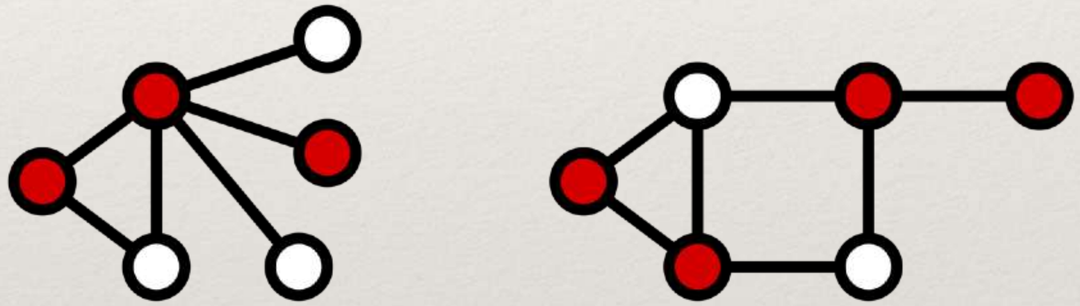
Optimization problems

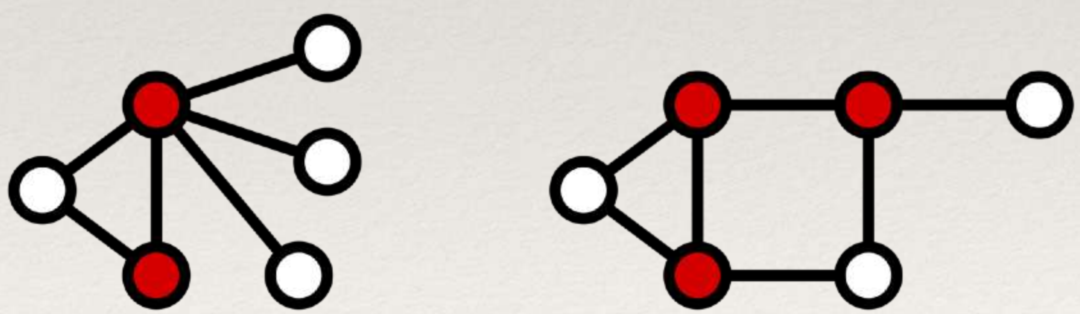
- Optimization = maximization or minimization
- **ε-approximable** iff can find $y \in F(x)$ /
  $$|c(y) - \mathrm{opt}(x)| \,/\, \max(c(y), \mathrm{opt}(x)) \le \varepsilon$$
  in polynomial time (ugly formula, but generalizes the previous formulae)
- **Defn.** The **approximation threshold** = $\inf_{\varepsilon\text{-approximable}} \varepsilon$
- Let us see, through a few examples, that this can be pretty much any number in [0,1].

NODE COVER

NODE COVER

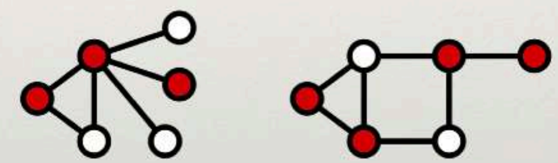
- **INPUT**: an undirected graph $G \stackrel{\text{def}}{=} (V, E)$
  FEASIBLE SOL.: **node covers**, i.e., subsets $C \subseteq V$
  such that every edge $u - v$ meets C
  (u or v or both are in C)
  COST: card(C)



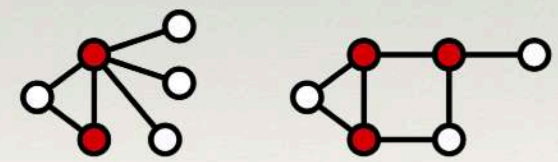


NODE COVER

INPUT: an undirected graph $G \stackrel{\text{def}}{=} (V, E)$
FEASIBLE SOL.: **node covers**, i.e., subsets $C \subseteq V$
such that every edge $u - v$ meets C
(u or v or both are in C)
COST: card(C)



By Miym - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=6017739

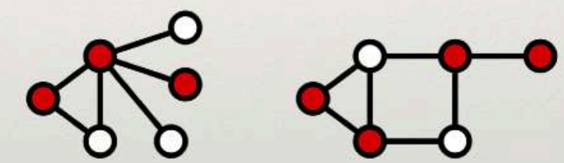


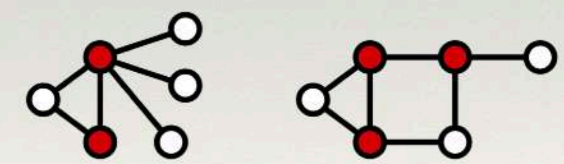
By Miym - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=6017749

- The associated **decision problem**:
  INPUT: G, a budget k
  QUESTION: does G have a node cover C with card(C) $\leq k$?
- is **NP**-complete.
- What is the approximation threshold of **NODE COVER**?
- Hint: the best known approximation algorithm is also one of the dumbest… and no, picking a vertex to be put in the cover, removing all incident edges, and going on is not dumb enough

NODE COVER is ½-approximable

INPUT: an undirected graph $G \stackrel{\text{def}}{=} (V, E)$
FEASIBLE SOL.: **node covers**, i.e., subsets $C \subseteq V$
such that every edge u — v meets C
(u or v or both are in C)
COST: card(C)





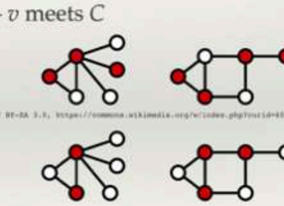
- **Algorithm**: (init: C:=∅);
  pick an **edge** u — v,
  add **both** u and v to C,
  then remove u and v
  and all incident edges, and proceed until no edge left.
- Let M be the set of edges picked by the algorithm.
  M is a **matching**: a vertex-disjoint collection of edges
- card(C)=2.card(M)

NODE COVER is ½-approximable

- Given a node cover C', every edge of M meets C' at a **distinct** vertex
- So $\mathrm{card}(M) \leq \mathrm{card}(C')$
- Since $\mathrm{card}(C) = 2.\mathrm{card}(M)$, $\mathrm{card}(C) \leq 2.\ \mathrm{card}(C')$
- Hence **NODE COVER** is ½-approximable.
  (½ is in fact the best we can do, unless **P=NP**)

- **Algorithm:** (init: $C := \varnothing$);
  pick an **edge** $u - v$,
  add **both** u and v to C,
  then remove u and v
  and all incident edges, and proceed until no edge left.
- Let M be the set of edges picked by the algorithm.
  M is a **matching**: a vertex-disjoint collection of edges
- $\mathrm{card}(C)=2.\mathrm{card}(M)$

INPUT: an undirected graph $G = (V, E)$
FEASIBLE SOL.: **node covers**, i.e., subsets $C \subseteq V$
such that every edge $u - v$ meets C
(u or v or both are in C)
COST: $\mathrm{card}(C)$

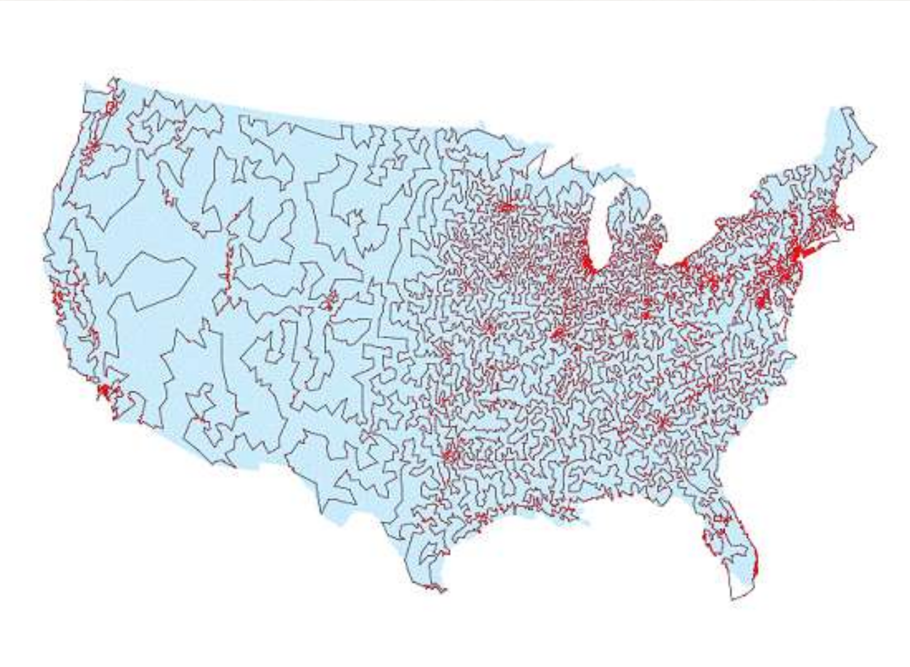


- **ε-approximable** iff can find $y \in F(x)$ / $c(y) \geq (1-\varepsilon)\mathrm{opt}(x)$ in polynomial time
- **Defn.** The **approximation threshold** $= \inf_{\varepsilon\text{-approximable}} \varepsilon$

The traveling salesman problem (TSP)

TSP

- **INPUT**: a matrix $D \stackrel{\text{def}}{=} (d_{ij})_{1 \le i,j \le n}$ of 'distances' between cities
  (only constraint: $d_{ii}=0$)
  FEASIBLE SOL.: **tours**, i.e., permutations π of $\{1,\ldots,n\}$
  COST: $d_{\pi(1)\pi(2)}+d_{\pi(2)\pi(3)}+\ldots+d_{\pi(n-1)\pi(n)}+d_{\pi(n)\pi(1)}$
- Decision problem (is cost ≤ some given budget?) is **NP**-complete
- ε-approximable for **no** $\varepsilon \in \,]0,1[$ unless **P**=**NP**. Hence approximation threshold is 1 (worst possible!)



13,509 U.S. cities with populations of more than 500 people connected optimally

http://www.crpc.rice.edu/CRPC/newsletters/sum98/news_tsp.html

TSP is not approximable

- We use the fact that **HAMILTONIAN CYCLE**:
  INPUT: an undirected graph $G \stackrel{\text{def}}{=} (V, E)$
  QUESTION: $\exists$cycle in G going once through each vertex?
  is **NP**-complete
- We build a poly time reduction from **HAMILTONIAN CYCLE** to (the decision form) of **TSP**,
- showing that if **TSP** is ε-approximable, then **HAMILTONIAN CYCLE** is in **P**, hence **P**=**NP**.

TSP is not approximable

HAMILTONIAN CYCLE:
INPUT: an undirected graph $G \stackrel{\text{def}}{=} (V, E)$
QUESTION: ∃cycle in G going once through each vertex?
is **NP**-complete

- Given G [$N \stackrel{\text{def}}{=} \text{card}(V)$] and
  $M > 1/(1-\varepsilon).N$,
  let $d_{ij} \stackrel{\text{def}}{=} 1$ if edge i — j,
  M if no edge. Defines an instance D of **TSP**.
- Tour π: cost $= N$ if Hamiltonian cycle, $\geq M$ otherwise
- Assume an ε-approximation (poly time) algorithm A for TSP
- If G has a Hamiltonian cycle, opt(D) $= N$
  A(D) will find a tour of cost $\leq 1/(1-\varepsilon).\text{opt}(D) < M$,
  hence a Hamiltonian cycle, in poly time
- Hence **HAMILTONIAN CYCLE** is in **P**, so **P**=**NP**.

KNAPSACK

KNAPSACK

- **INPUT**: prices v_i and weights w_i, $1 \le i \le n$, a max weight W
  (all are natural numbers)
  FEASIBLE SOL.: a subset $S \subseteq \{1,\ldots,n\}$ / $\Sigma_{i \in S}\, w_i \le W$
  COST: $\Sigma_{i \in S}\, v_i$
- Decision problem (is cost ≤ some given budget?) is **NP**-complete
- ε-approximable for **every** $\varepsilon \in]0,1[$.
  Approximation threshold is 0 (best possible!)

KNAPSACK

INPUT: prices v_i and weights w_i, $1 \leq i \leq n$, a max weight W
FEASIBLE SOL.: a subset $S \subseteq \{1,\ldots,n\}$ / $\Sigma_{i \in S} w_i \leq W$
COST: $\Sigma_{i \in S} v_i$

- A well-known **dynamic programming** algorithm for KNAPSACK:
- Let $V \stackrel{\text{def}}{=} \Sigma_{i=1}^{n} v_i$, and, for all $1 \leq j \leq n$ and $0 \leq v \leq V$:

$$W(j,v) = \min \{\Sigma_{i \in S} w_i \mid S \subseteq \{1,\ldots,j\}, \Sigma_{i \in S} v_i = v\}$$

- Then $W(j,v) = \min(W(j-1,v), W(j-1,v-v_j)+w_i)$ if $v \geq v_j$
  $W(j-1,v)$ otherwise
  can be computed in time $O(nV)$

exponential in size(V)=O(log V) if numbers in binary

- Finally, find largest v such that $W(n,v) \leq W$.

KNAPSACK

- A well-known **dynamic programming** algorithm for KNAPSACK:
- Let $V \overset{\text{def}}{=} \Sigma_{i=1}^{n} v_i$, and, for all $1 \leq j \leq n$ and $0 \leq v \leq V$:
  $W(j,v) = \min \{\Sigma_{i \in S}\, w_i \mid S \subseteq \{1,\ldots,j\},\ \Sigma_{i \in S}\, v_i = v\}$
- Then $W(j,v) = \min(W(j-1,v),\ W(j-1,v-v_j)+w_i)$ if $v \geq v_j$
  $W(j-1,v)$ otherwise
  can be computed in time $O(nV)$
- Finally, find largest v such that $W(n,v) \leq W$.

INPUT: prices v_i and weights w_i, $1 \leq i \leq n$, a max weight W
FEASIBLE SOL.: a subset $S \subseteq \{1,\ldots,n\}$ / $\Sigma_{i \in S}\, w_i \leq W$
COST: $\Sigma_{i \in S}\, v_i$

- Do all computations on values (i.e., v, v_i) by only keeping the
  ***k* most significant bits**
  of each number and
  rounding down
- I.e., represent v_i by the k-bit number $\lfloor v_i / 2^{k0-k} \rfloor$ [$k0 \overset{\text{def}}{=}$ #bits in nV]
- replace all values v by k-bit approximations v' ($v \approx 2^{k0-k}\, v'$)
- replace computation of $v - v_i$ by $v' - \lfloor v_i / 2^{k0-k} \rfloor$

KNAPSACK

INPUT: prices v_i and weights w_i, $1 \le i \le n$, a max weight W
FEASIBLE SOL.: a subset $S \subseteq \{1,\ldots,n\}$ / $\Sigma_{i \in S}\, w_i \le W$
COST: $\Sigma_{i \in S}\, v_i$

- A well-known **dynamic programming** algorithm for KNAPSACK:
- Let $V \stackrel{\text{def}}{=} \Sigma_{i=1}^{n} v_i$, and, for all $1 \le j \le n$ and $0 \le v \le V$:
  $W(j,v) = \min \{\Sigma_{i \in S}\, w_i \mid S \subseteq \{1,\ldots,j\},\ \Sigma_{i \in S}\, v_i = v\}$
- Then $W(j,v) = \min(W(j-1,v),\ W(j-1,v-v_j)+w_i)$ if $v \ge v_j$
  $W(j-1,v)$ otherwise
  can be computed in time $O(nV)$
- Finally, find largest v such that $W(n,v) \le W$.

- We choose
  $k \stackrel{\text{def}}{=} \lceil \mathrm{size}(nV) - \log_2(\varepsilon V / n) \rceil$
  $= \log_2 (n^2 / \varepsilon) + O(1)$
- There are now at most
  $2^k = O(n^2 / \varepsilon)$ different values
  instead of $O(V)$,
  hence times goes down to $O(n\, 2^k) = O(n^3 / \varepsilon)$
- And final value is between $(1-\varepsilon)$opt and opt
  (see lecture notes for details, Prop. 2.7).

MAX3SAT

MAXSAT

- **INPUT**: a finite list *S* of clauses
  FEASIBLE SOL.: an environment ϱ
  VALUE: #clauses satisfied by ϱ

- Decision problem (is value ≥ some given goal?)
  is **NP**-complete

- ε-approximable for **which** ε ∈]0,1[?
  Let me give you the best known (and silliest) algorithm…

Johnson's algorithm

- Rough idea: while there is a variable A left, decide to set A to 1 (true) or 0 (false) depending on which of

  E(#clauses of $S[A{:=}1]$ satisfied by ϱ) and

  E(#clauses of $S[A{:=}0]$ satisfied by ϱ) is larger,

  where ϱ is drawn at random.

- $S[A{:=}1]$: remove clauses where A occurs positively,

  remove $\neg A$ from remaining clauses

- $S[A{:=}0]$: remove clauses where $\neg A$ occurs

  (i.e., A occurs negatively),

  remove A from remaining clauses

Johnson's algorithm

- In reality: we compare

  E(#clauses of $S[A{:=}1]$ **not** satisfied by ϱ) and

  E(#clauses of $S[A{:=}0]$ **not** satisfied by ϱ)

- If the first one is smaller, set A to 1, $S := S[A{:=}1]$

- Otherwise, set A to 0, $S := S[A{:=}0]$

Computing E(#clauses of S not satisfied by ϱ)

- Let $S \stackrel{\text{def}}{=} [C_1, \ldots, C_m]$, each C_j being a clause or $\top$
  $E(S) \stackrel{\text{def}}{=} E(\#j \ / \ C_j$ not satisfied by ϱ, ϱ uniformly random)
- $E(S) = \Sigma_{j=1}{}^{m}\, E([C_j]) = \Sigma_{j=1}{}^{m}\, Pr_\varrho(\text{not } \varrho \vDash C_j)$
  [linearity of expectation]
- If C_j is a tautology $A \vee \neg A \vee \ldots$ (or $\top$), $Pr_\varrho(\text{not } \varrho \vDash C) = 0$
  else $Pr_\varrho(\text{not } \varrho \vDash C) = 1/2^{|C|}$,
  e.g., $Pr_\varrho(\text{not } \varrho \vDash A \vee \neg B \vee \neg C) = 1/8$

The key observation

- **Claim.** $E(S) = ½(E(S[A{:=}1]) + E(S[A{:=}0]))$
- *Proof.* By linearity of expectation, enough to check it for a single clause C_j
- If C_j tautology, $0 = ½(0+0)$, otherwise…
- If $C_j = A \vee rest$, $E(C_j) = 1/2^{|C_j|} = ½\, 1/2^{|rest|}$,

$$E(C_j[A{:=}1]) = E(\top) = 0$$
$$E(C_j[A{:=}0]) = E(rest) = 1/2^{|rest|}$$

- Similarly if $C_j = \neg A \vee rest$
- If neither A nor $\neg A$ occurs in C_j, $C_j[A{:=}1] = C_j[A{:=}0] = C_j$. □

Decreasing expectations

- **Claim.** $\mathrm{E}(S) = ½(\mathrm{E}(S[A{:=}1]) + \mathrm{E}(S[A{:=}0]))$
- List all the variables as $A_0, \ldots, A_n$. Set $S_0 \stackrel{\text{def}}{=} S$ $(= [C_1, \ldots, C_m])$

$\mathrm{E}(S_0[A_1{:=}1]) \leq \mathrm{E}(S_0[A_1{:=}0])$? — yes: set $A_1{:=}1$, $S_1{:=}S[A_1{:=}1]$; no: set $A_1{:=}0$, $S_1{:=}S[A_1{:=}0]$

$\mathrm{E}(S_1[A_2{:=}1]) \leq \mathrm{E}(S_1[A_2{:=}0])$? — yes: set $A_2{:=}1$, $S_2{:=}S_1[A_2{:=}1]$; no: set $A_2{:=}0$, $S_2{:=}S_1[A_2{:=}0]$

$\mathrm{E}(S_2[A_3{:=}1]) \leq \mathrm{E}(S_2[A_3{:=}0])$? — yes: set $A_3{:=}1$, $S_3{:=}S_2[A_3{:=}1]$; no: set $A_3{:=}0$, $S_3{:=}S_2[A_3{:=}0]$

etc.

- By the claim, $\mathrm{E}(S_{i+1}) \leq \mathrm{E}(S_i)$. So $\mathrm{E}(S_n) \leq \mathrm{E}(S)$.
- Let ϱ be the final environment
  The only clauses in S_n are $\top$ (if $\varrho \vDash C_j$), or the empty clause $\bot$
- Note: $\mathrm{E}(S_n) = \#$empty clauses in S_n.
  So ϱ satisfies $m - \mathrm{E}(S_n) \geq m - \mathrm{E}(S)$ clauses in S.

MAXSAT is approximable

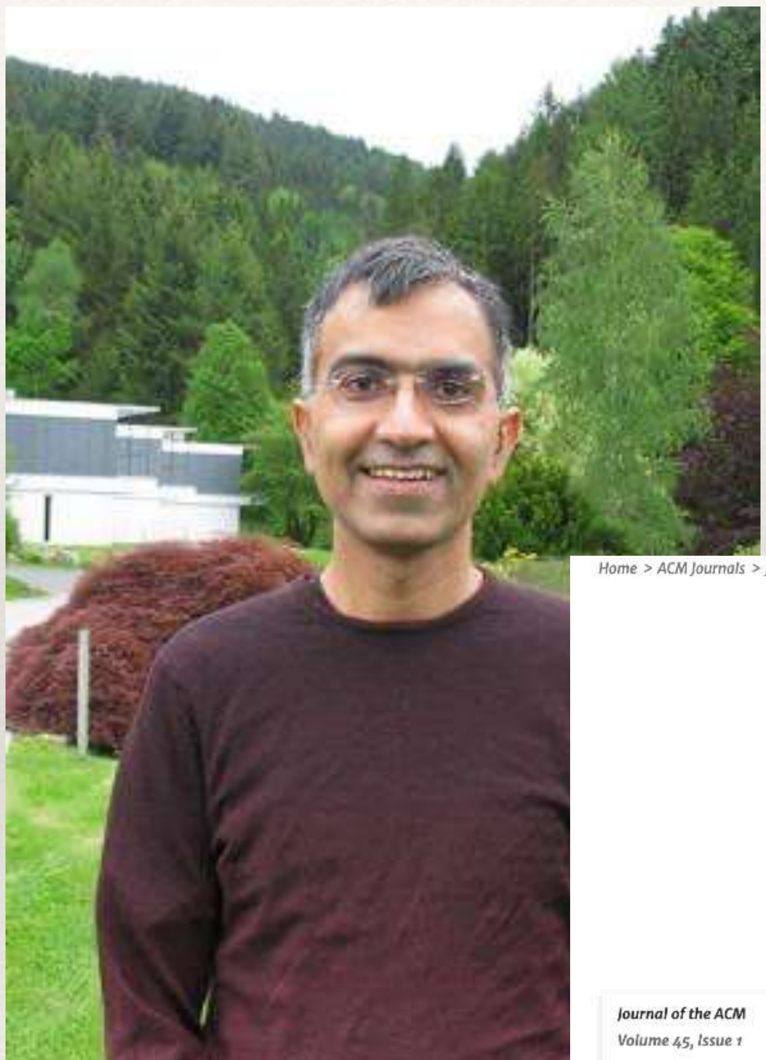
- ϱ satisfies m–E(S_n) ≥ m–E(S) clauses in $S = [C_1, \ldots, C_m]$
- If each non-tautological C_j has ≥k literals,
  $\Pr_\varrho(\text{not } \varrho \models C_j) \leq 1/2^k$ (=0 if tautological), so
  $E(S) = \Sigma_{j=1}{}^m \Pr_\varrho(\text{not } \varrho \models C_j) \leq m/2^k$
- Therefore ϱ satisfies $\geq m\,(1 - 1/2^k) \geq \text{opt}(S)\,(1 - 1/2^k)$ clauses in S:
- **Thm. MAXSAT** restricted to S /
  each non-tautological C_j has ≥k literals, is $1/2^k$-approximable.

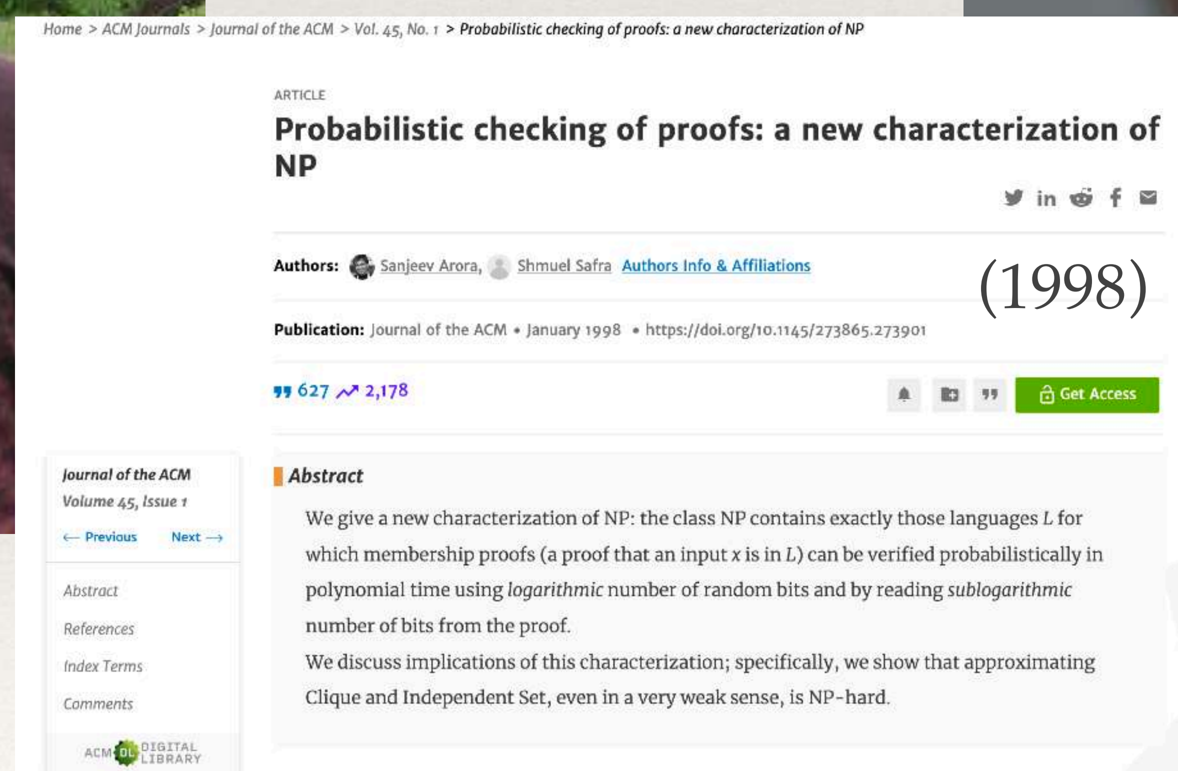
MAXSAT is approximable

- **Thm. MAXSAT** restricted to S /
  each non-tautological C_j has $\geq k$ literals, is $1/2^k$-approximable.
- One can always prepare S by eliminating unit clauses, so $k \geq 2$: **MAXSAT** is 1/4-approximable.
- If every clause in S has **at least** 3 literals, then 1/8-approximable.
- Hence **MAX=3SAT** (all clauses have exactly 3 literals) is **1/8-approximable**. It turns out that this is optimal.

PCP

Sanjeev Arora, Shmuel Safra





Home > ACM Journals > Journal of the ACM > Vol. 45, No. 1 > Probabilistic checking of proofs: a new characterization of NP

ARTICLE

Probabilistic checking of proofs: a new characterization of NP

Authors: Sanjeev Arora, Shmuel Safra Authors Info & Affiliations

Publication: Journal of the ACM • January 1998 • https://doi.org/10.1145/273865.273901

627 2,178

Get Access

Journal of the ACM
Volume 45, Issue 1
← Previous Next →

Abstract
References
Index Terms
Comments

ACM DIGITAL LIBRARY

Abstract

We give a new characterization of NP: the class NP contains exactly those languages L for which membership proofs (a proof that an input x is in L) can be verified probabilistically in polynomial time using *logarithmic* number of random bits and by reading *sublogarithmic* number of bits from the proof.

We discuss implications of this characterization; specifically, we show that approximating Clique and Independent Set, even in a very weak sense, is NP-hard.

(1998)

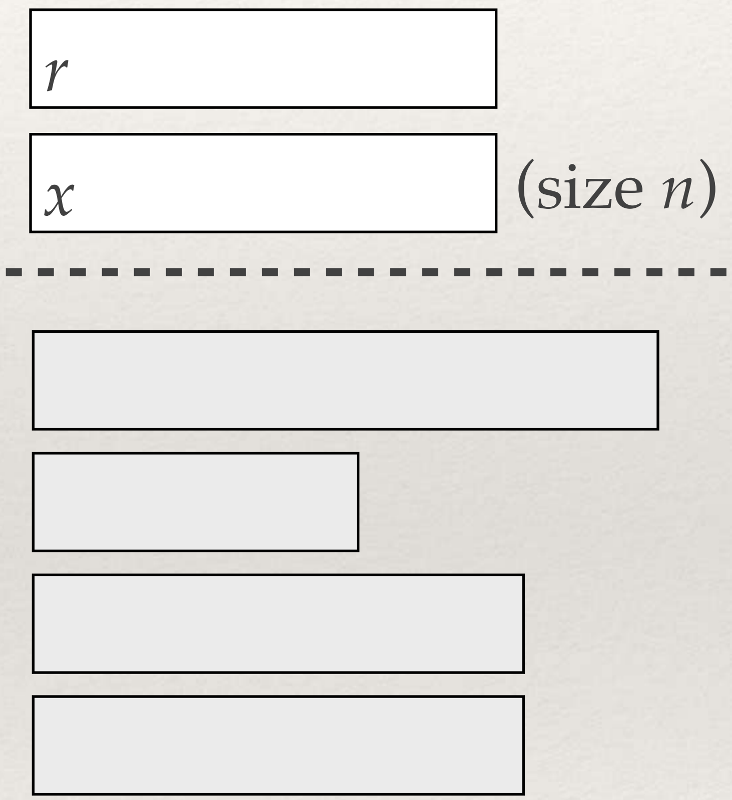
https://commons.wikimedia.org/wiki/File:Sanjeev_Arora.jpg#/media/Fichier:Sanjeev_Arora.jpg

https://simons.berkeley.edu/sites/default/files/styles/profile_main/public/dscn0007.jpg?itok=irtfi766

Reminder: randomized TMs

- Two **read-only** tapes

- As many **work tapes** as you need (but only a constant number!)



With the usual proviso:
head can only move right on random tape r

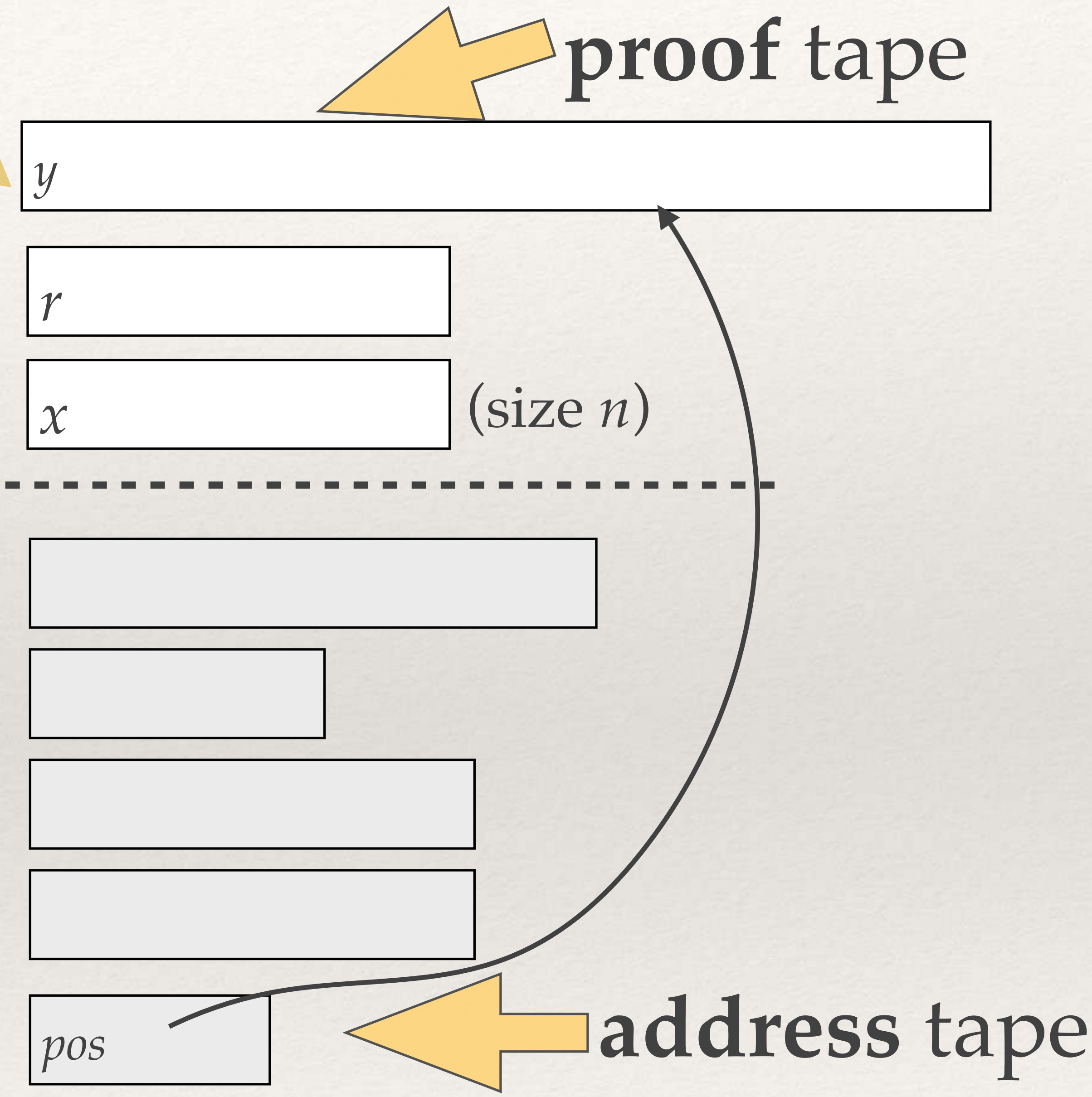
PCP machines

Proof tape is accessed in **random access** mode

Three

- ~~Two~~ **read-only** tapes

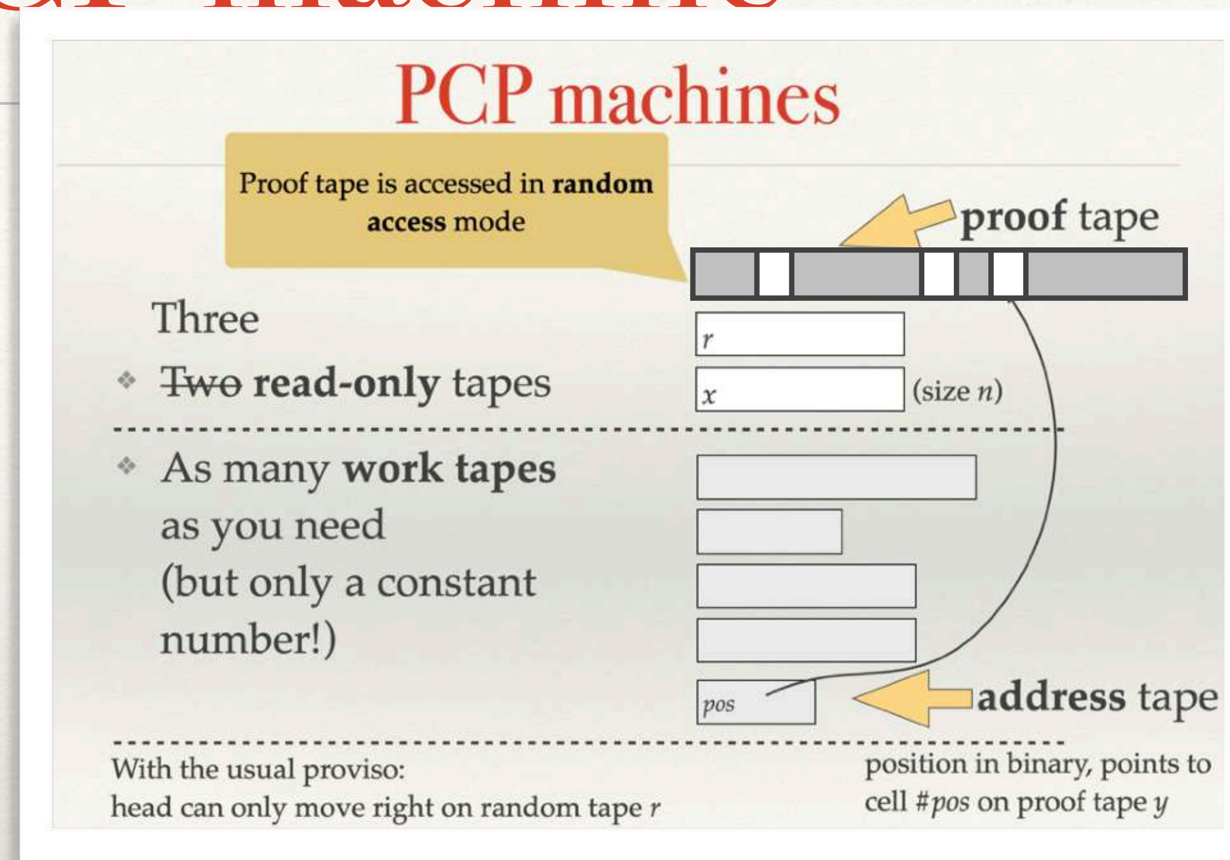
- As many **work tapes** as you need (but only a constant number!)



With the usual proviso:
head can only move right on random tape r

position in binary, points to cell #pos on proof tape y

Running a PCP machine



1. On input x, Merlin fills in proof tape y — but keeps it **masked** (= cryptographic **commitment**)

2. Arthur, only knowing $|y|$ (and x), computes k=$\mathbf{Q(n)}$ **positions** $p_1, \ldots, p_k$ on the proof tape in binary, in polynomial time, using $\mathbf{R(n)}$ **random bits**

3. Merlin **reveals** $y[p_1], \ldots, y[p_k]$

4. Arthur computes $f(y[p_1], \ldots, y[p_k]) \in \{$**accept, reject**$\}$ in time $T(n)$

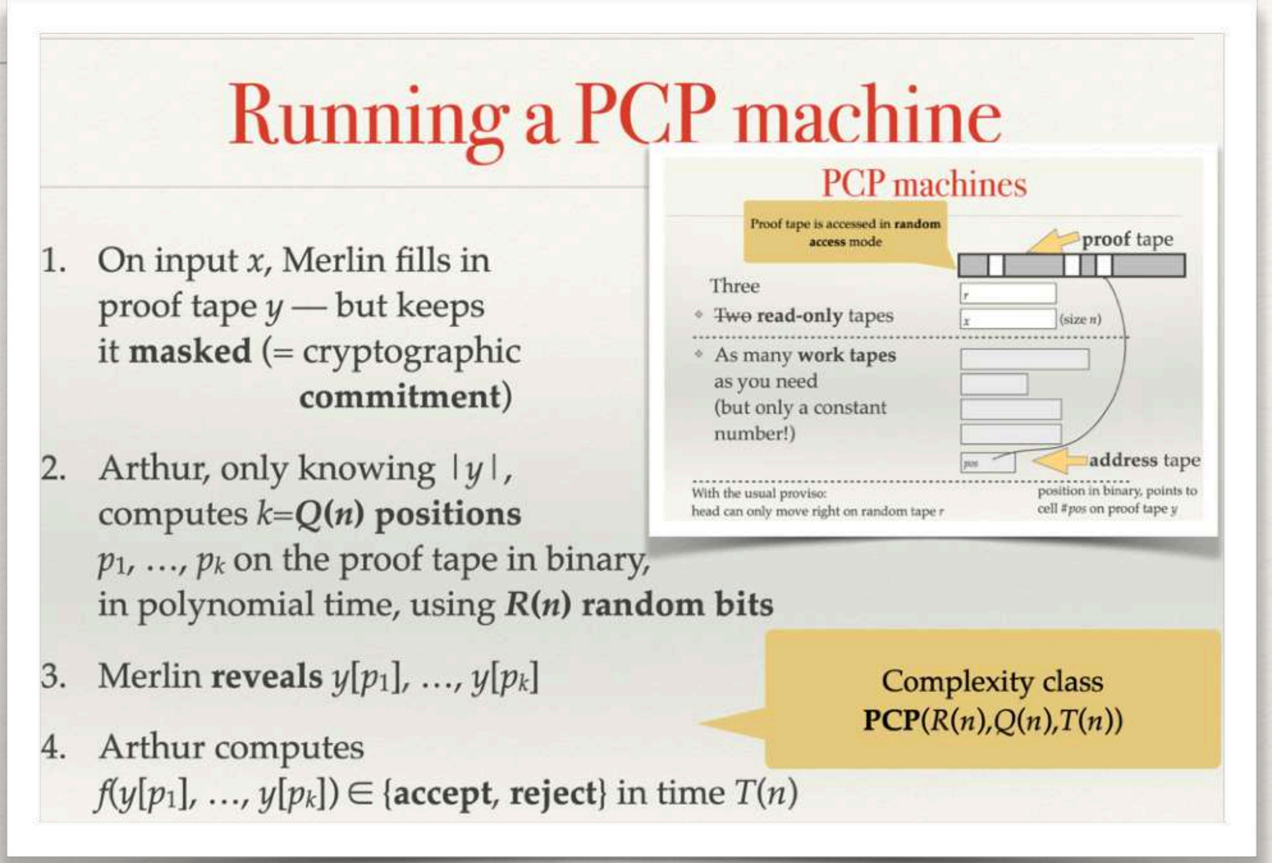
where f may also depend on x, and on the random bits of step 2

Acceptance conditions

- If $x \in L$, then Merlin can provide a proof tape y such that Arthur will **always** accept

- If $x \notin L$, then whichever proof tape Merlin provides, Arthur will reject **with probability ≥ ½**

- The languages L that can be decided this way form the complexity class **PCP**$(R(n),Q(n),T(n))$



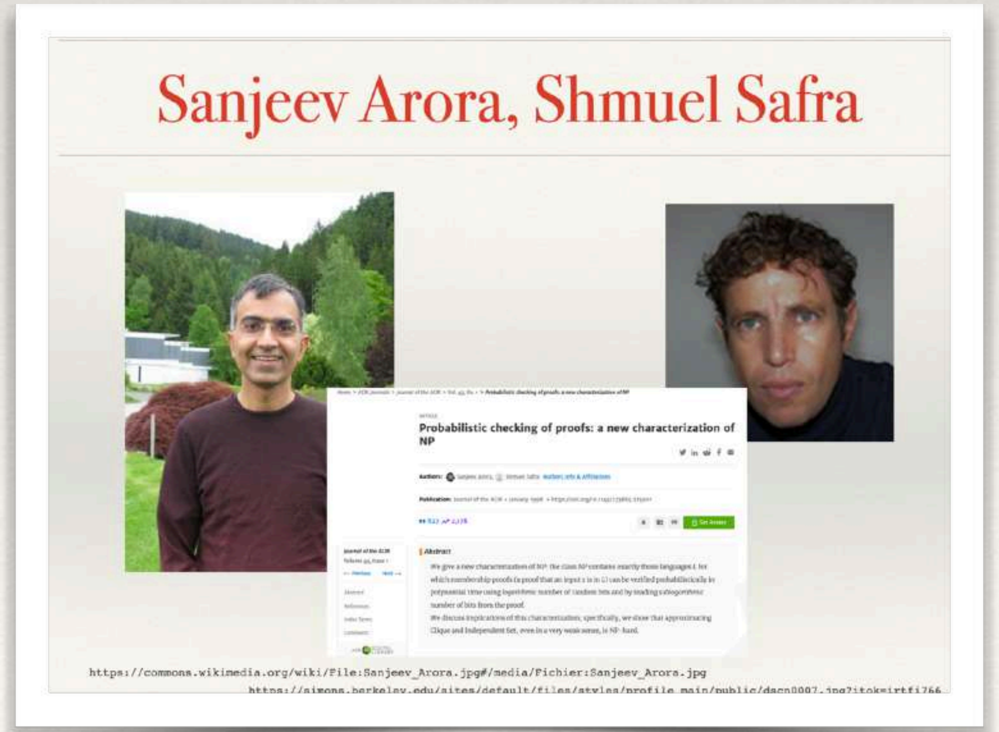
The Arora-Safra theorem

- **Theorem.** **NP** = **PCP**(O(log n), O(1), O(1))

NP=PCP,
for short

- I.e., one can decide every language in **NP** by running a PCP machine that:
  — asks $Q(n)$=**O(1) questions** (positions)
  — computed using only $R(n)$=**O(log n) random bits**, in poly time
  — and finally decides in $T(n)$=**O(1) time**.

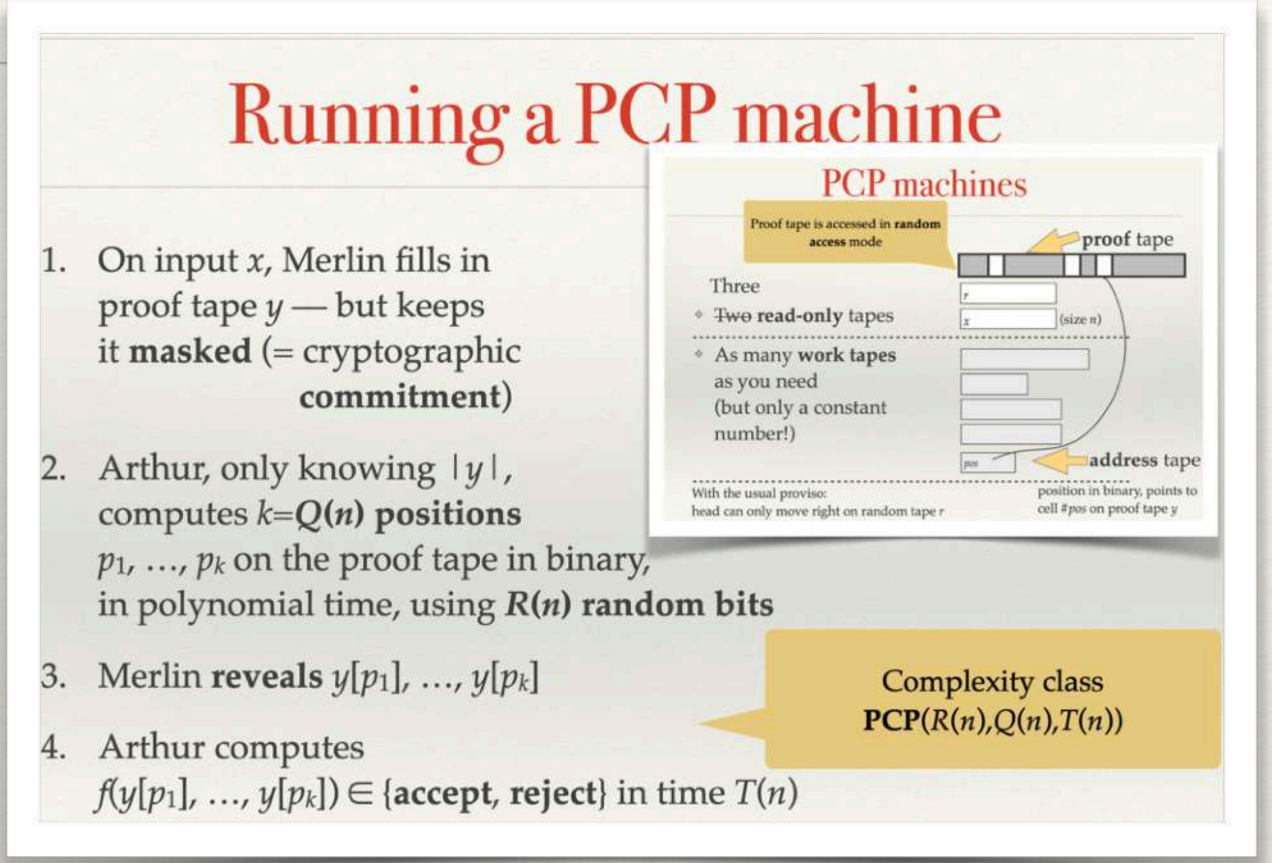
- Proof would require a whole term!



NP=PCP and the hardness of approximation

- What I will explain is that **NP=PCP** is equivalent to the ε-inapproximability of **MAX3SAT** for some ε>0
- Arora-Safra prove **NP=PCP**
- … and there is a simplified (still extremely complex) proof by Irit Dinur

The easy direction: **PCP** $\subseteq$ **NP**



- **Derandomize** naively: for every string of $R(n)$ random bits, simulate Arthur's computation
- If more than ½ of the simulations accept, then accept, else reject
- Works in time $2^{R(n)} \log R(n)\ \mathrm{poly}(n) + T(n)$
- So **PCP**$(R(n) \stackrel{\text{def}}{=} O(\log n),\ Q(n) \stackrel{\text{def}}{=} \text{whatever},\ T(n) \stackrel{\text{def}}{=} \mathrm{poly}(n)) \subseteq$ **NP**

PCP and the hardness of approximating **SAT**

MAX3SAT(ε)

- … is the following **promise problem**:
  INPUT: a finite set S of m propositional 3-clauses
  PROMISE: S satisfiable / $\mathrm{opt}(S) < (1-\varepsilon)m$
  QUESTION: which is true? [$\mathrm{opt}(S)$ = max #sat. clauses]
- We will see that:
  — if **3SAT** is ε-approximable then **MAX3SAT**(ε) is polytime decidable
  — (∃ ε>0, **MAX3SAT**(ε) is **NP**-hard) iff **NP**=**PCP**
- That was known before Arora-Safra.
  With Arora-Safra: ∃ ε>0, **3SAT** is not ε-approximable, unless **P**=**NP**

Note: **NP**-*complete* would not make sense for promise problems

If **3SAT** ε-approximable then **MAX3SAT**(ε) polytime

MAX3SAT(ε)

INPUT: a finite set S of m propositional 3-cla
PROMISE: S satisfiable / opt(S) < (1–ε)m
QUESTION: which is true?

- Given a (polytime) ε-approximation algorithm A for **3SAT**:
- For every instance S of **MAX3SAT**(ε),
  $\varrho \stackrel{\text{def}}{=} A(S)$ satisfies $\geq (1-\varepsilon).\text{opt}(S)$ clauses of S
- If S satisfiable, opt(S)=m, so ϱ satisfies $\geq (1-\varepsilon)m$ clauses
- Otherwise, ϱ satisfies $< (1-\varepsilon)m$ clauses by the promise
- Hence comparing #clauses satisfied by $\varrho \stackrel{\text{def}}{=} A(S)$ with $(1-\varepsilon)m$ yields a polytime algorithm deciding **MAX3SAT**(ε). □

(∃ ε>0, MAX3SAT(ε) **NP**-hard)

iff **NP=PCP**:

the left to right direction

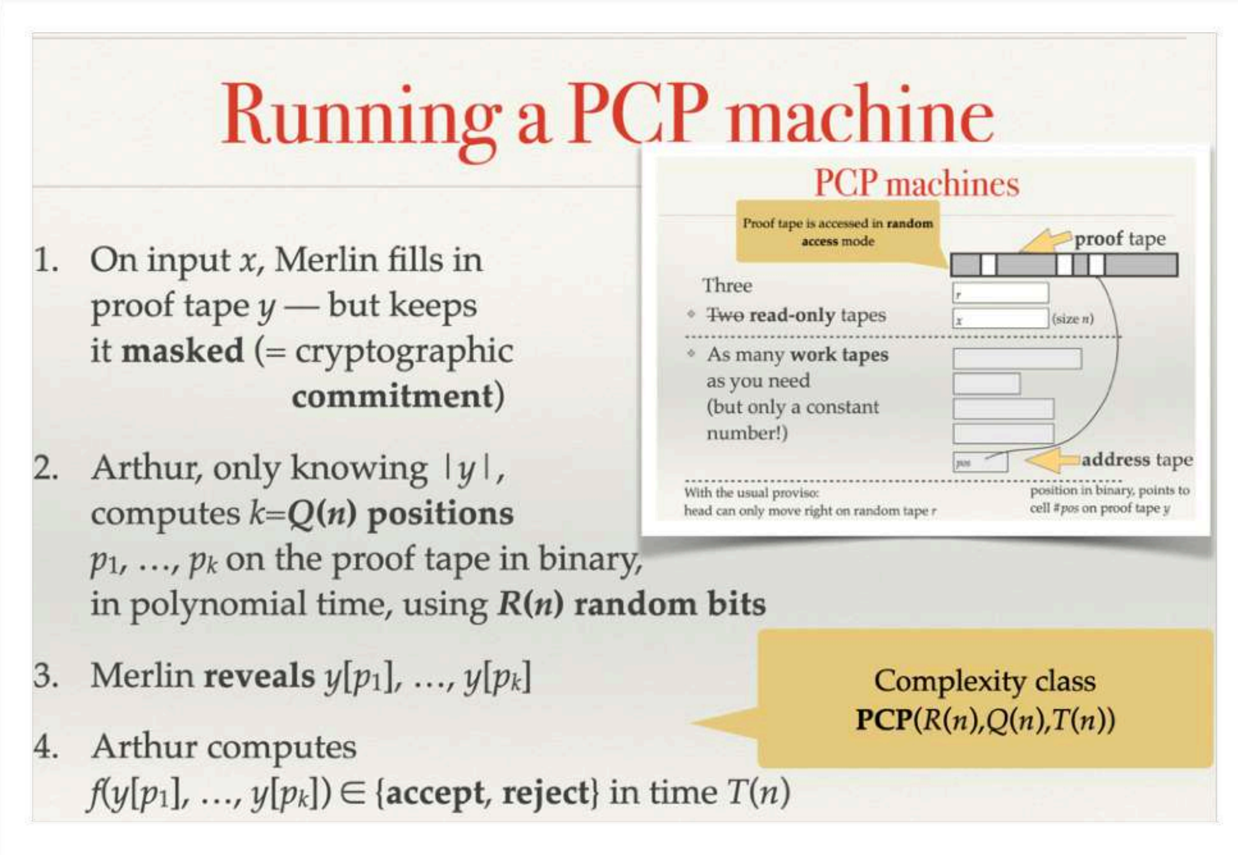
If ∃ ε>0, **MAX3SAT**(ε) **NP**-hard then **NP**=**PCP**

- We already know **PCP** ⊆ **NP**. Conversely, let L be any language in **NP**.

MAX3SAT(ε)

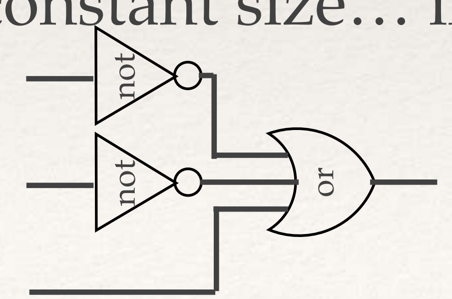
INPUT: a finite set S of m propositional 3-cla
PROMISE: S satisfiable / opt(S) < (1–ε)m
QUESTION: which is true?

- ∃polytime reduction from L to **MAX3SAT**(ε), since **MAX3SAT**(ε) **NP**-hard by assumption
- **PCP** is closed under polytime reductions (important!)
- So it suffices to exhibit a **PCP** machine deciding **MAX3SAT**(ε)



If ∃ ε>0, MAX3SAT(ε) NP-hard then NP=PCP

- A **PCP** machine deciding **MAX3SAT**(ε). Let $S \stackrel{\text{def}}{=} [C_1,\ldots,C_m]$

1. Merlin fills in y with ϱ
2. Arthur chooses C_j at random, (say $+A_{32} \vee -A_{71} \vee -A_{239}$) and gives the corresponding 3 positions (here: 32, 71, 239)
3. Merlin reveals the corresponding truth values
4. Arthur evaluates C_j using a precompiled circuit, of constant size… in time O(1) (here, ) accepts if true, rejects if false

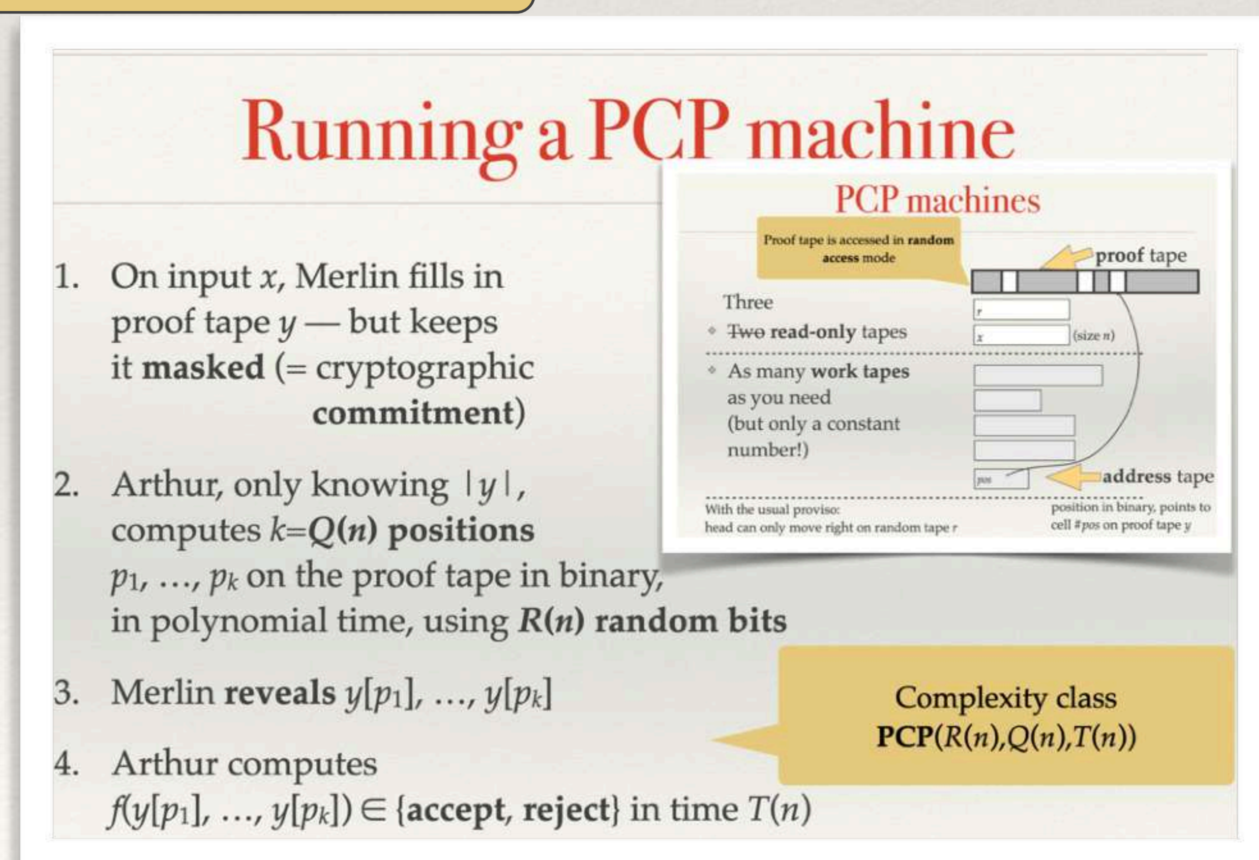
MAX3SAT(ε)

INPUT: a finite set S of m propositional 3-cla

PROMISE: S satisfiable / opt(S) < (1–ε)m

QUESTION: which is true?

Oops… and precompiles a circuit that evaluates C_j, to be used in step 4



If ∃ ε>0, MAX3SAT(ε) NP-hard then NP=PCP

- Uses $R(n)=O(\log n)$ random bits: just one number j $(1 \le j \le m)$ at random
- $Q(n)=O(1)$ (indeed, $=3$)
  $T(n)=O(1)$
- If S satisfiable, then Merlin can produce a satisfying assignment, so Arthur will accept
- If $\mathrm{opt}(S) < (1-\varepsilon)m$, then whatever ϱ is given, $\Pr_j(\varrho \vDash C_j) < (1-\varepsilon)$

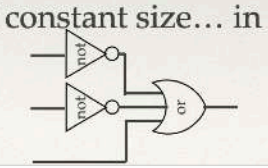
Shoot! We needed ½ here…

Note: j is random here, not ϱ as in Johnson's algorithm

MAX3SAT(ε)

INPUT: a finite set S of m propositional 3-cla
PROMISE: S satisfiable / $\mathrm{opt}(S) < (1-\varepsilon)m$
QUESTION: which is true?

- A **PCP** machine deciding **MAX3SAT**(ε). Let $S \triangleq [C_1,\ldots,C_m]$

1. Merlin fills in y with ϱ
2. Arthur chooses C_j at random, (say $+A_{32} \vee -A_{71} \vee -A_{239}$) and gives the corresponding 3 positions (here: 32, 71, 239)
3. Merlin reveals the corresponding truth values
4. Arthur decides using a precompiled circuit, of constant size… in time O(1) (here, )

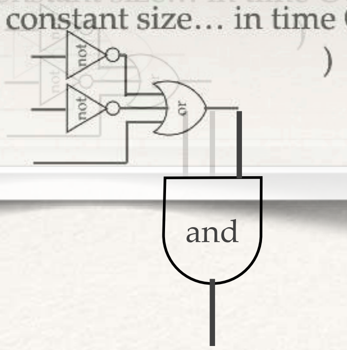
If ∃ $\varepsilon>0$, MAX3SAT(ε) NP-hard then NP=PCP

- We solve the problem using **parallel repetition** (k times)
  $k \stackrel{\text{def}}{=} \lceil -\log 2 / \log(1-\varepsilon) \rceil$
- Uses $R(n)=O(\log n)$ random bits: just k numbers j ($1 \le j \le m$) at random
- $Q(n)=O(1)$ (indeed, $3k$) $T(n)=O(1)$
- If S satisfiable, then Merlin can produce a satisfying assignment, so Arthur will accept
- If $\mathrm{opt}(S) < (1-\varepsilon)m$, then whatever ϱ is given, $\Pr_{j_1,\ldots,j_k}(\varrho \vDash C_{j_1} \text{ and } \ldots \text{ and } \varrho \vDash C_{j_k}) \le (1-\varepsilon)^k \le ½$

MAX3SAT(ε)

INPUT: a finite set S of m propositional 3-cla
PROMISE: S satisfiable / $\mathrm{opt}(S) < (1-\varepsilon)m$
QUESTION: which is true?

- A PCP machine deciding MAX3SAT(ε). Let $S \stackrel{\text{def}}{=} [C_1,\ldots,C_m]$
  1. Merlin fills in y with ϱ
  2. Arthur chooses C_j at random, (say $+A_{32} \vee -A_{71} \vee -A_{239}$) and gives the corresponding 3 positions (here: 32, 71, 239)
  3. Merlin reveals the corresponding truth values
  4. Arthur decides using a precompiled circuit, of constant size… in time O(1) (here,)



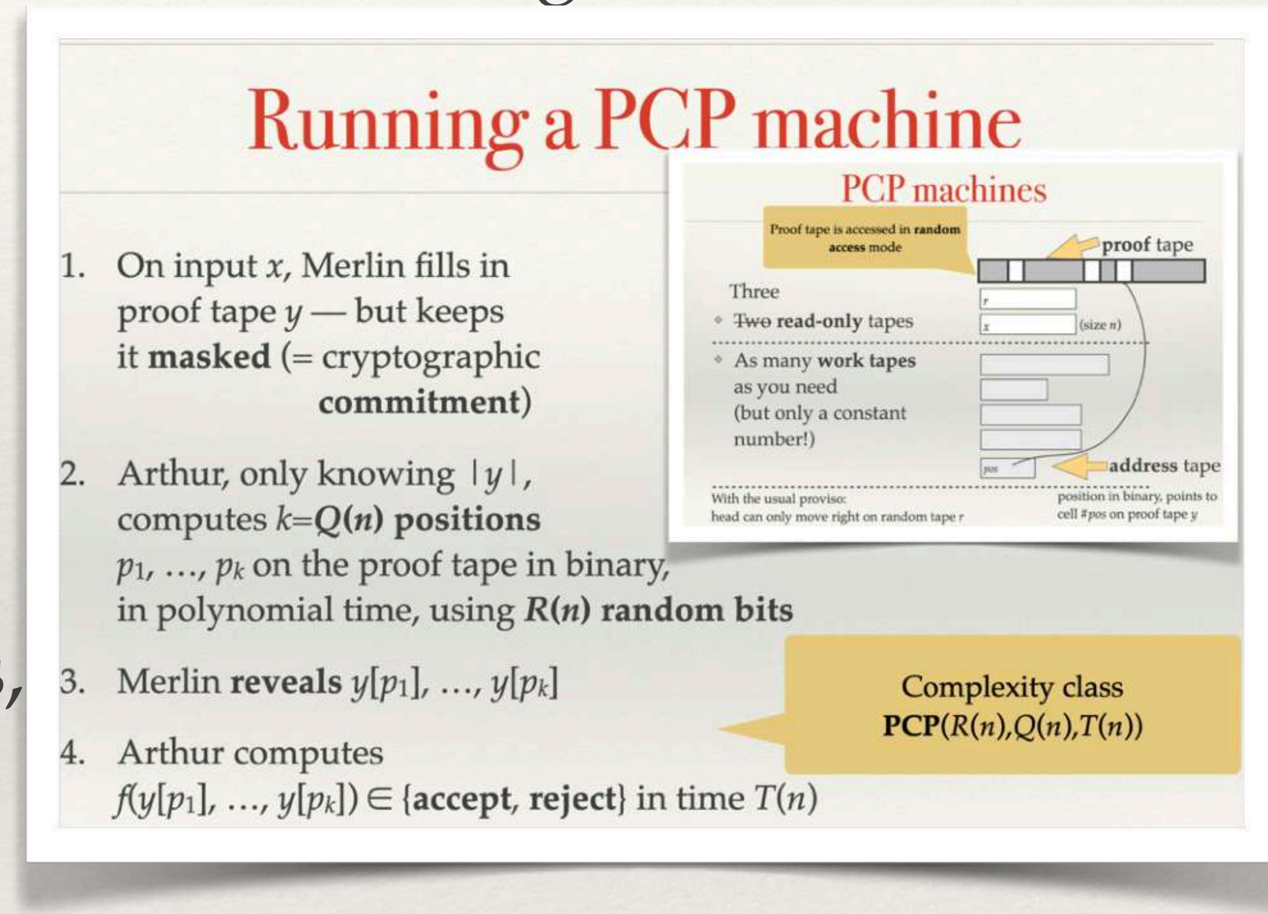
$(\exists\ \varepsilon > 0$, **MAX3SAT**$(\varepsilon)$ **NP**-hard)

iff **NP=PCP**:

the right to left direction

If NP=PCP then ∃ ε>0, MAX3SAT(ε) NP-hard

- Assume **NP**=**PCP**. Then there is a **PCP**($R(n) \stackrel{\text{def}}{=} k \log n$,O(1),O(1)) machine $\mathcal{M}$ deciding **SAT**



- We look for a polytime reduction from **SAT** to **MAX3SAT**(ε), for some ε>0

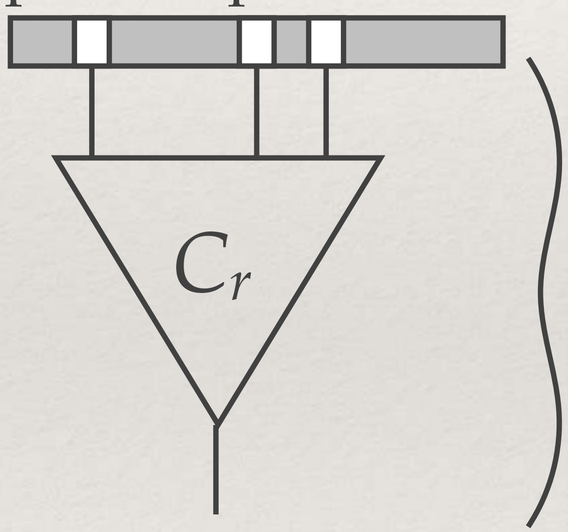
- Let us look at $\mathcal{M}(x)$'s possible runs, for each $R(n)$-bit word r drawn at random in step 2

- (Note that the assumption that $T(n)$=O(1) in step 4 is superfluous: f only has a constant #inputs, and can always be encoded by a constant-size circuit, evaluated in time O(1)…)

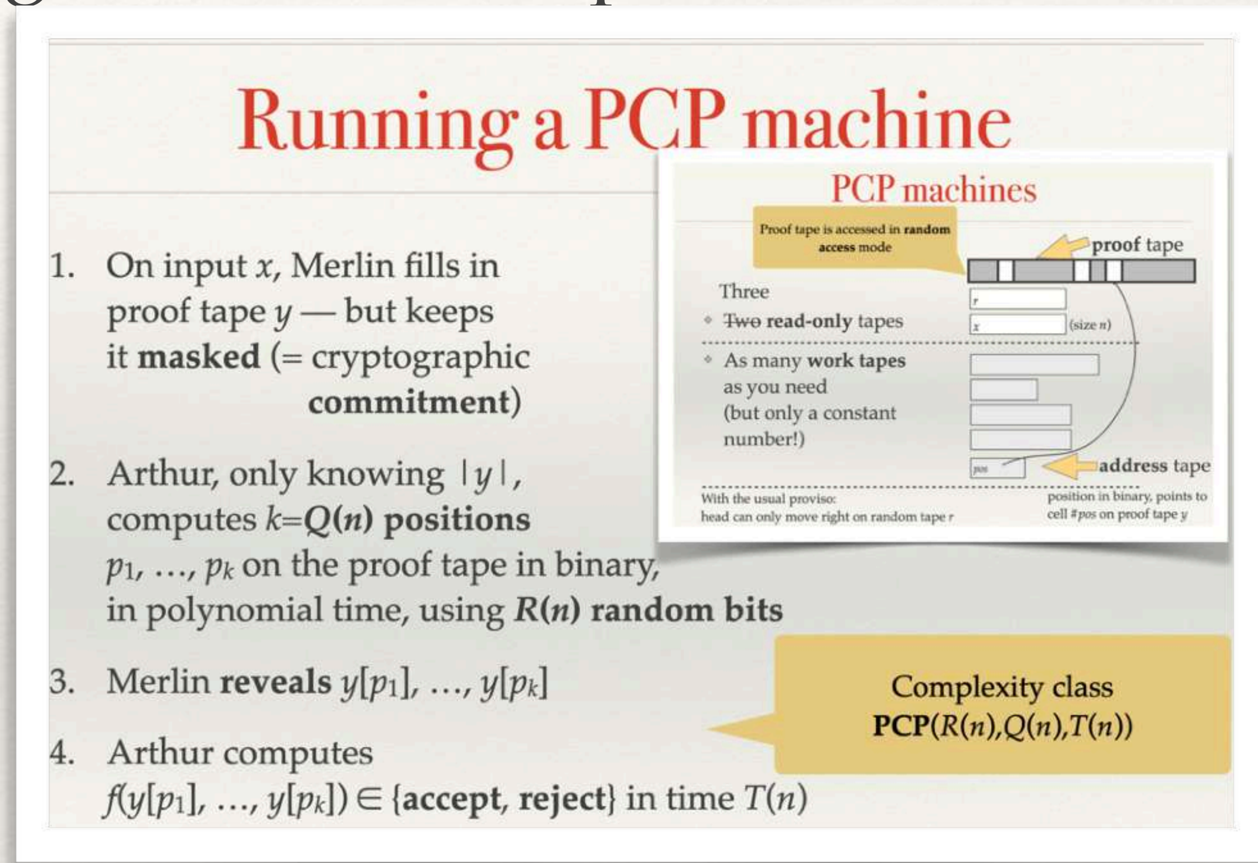
If NP=PCP then ∃ ε>0, MAX3SAT(ε) NP-hard

- For each $k \log_2 n$-bit random string r, Arthur computes O(1) positions, and a constant-size circuit C_r (say, $\leq G$ fan-in 2 gates)

proof tape



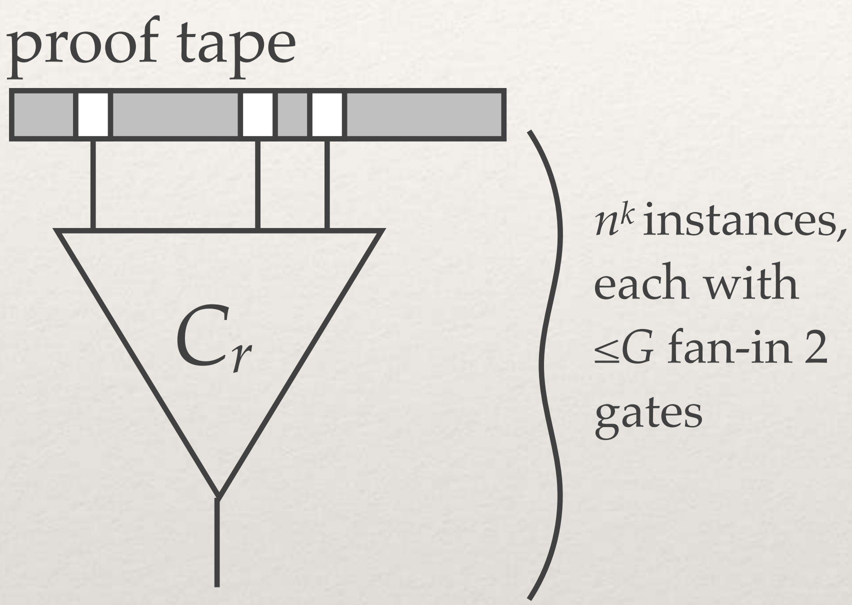
$2^{k \log_2 n} = n^k$ instances

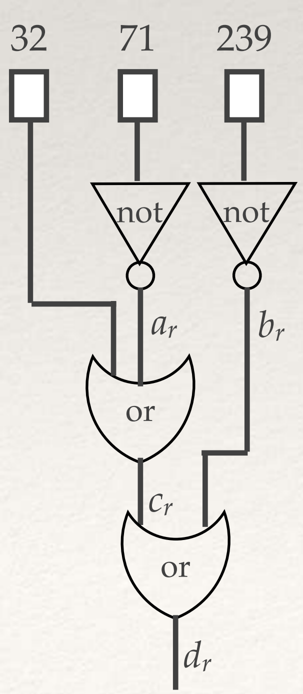


- If input S satisfiable, then Merlin can provide y / output wire of C_r is true, for every r
- Otherwise, whatever y, $\geq \frac{1}{2} n^k$ output wires false

If NP=PCP then ∃ ε>0, MAX3SAT(ε) NP-hard

- We now encode those circuits as a **3SAT** formula, e.g.:



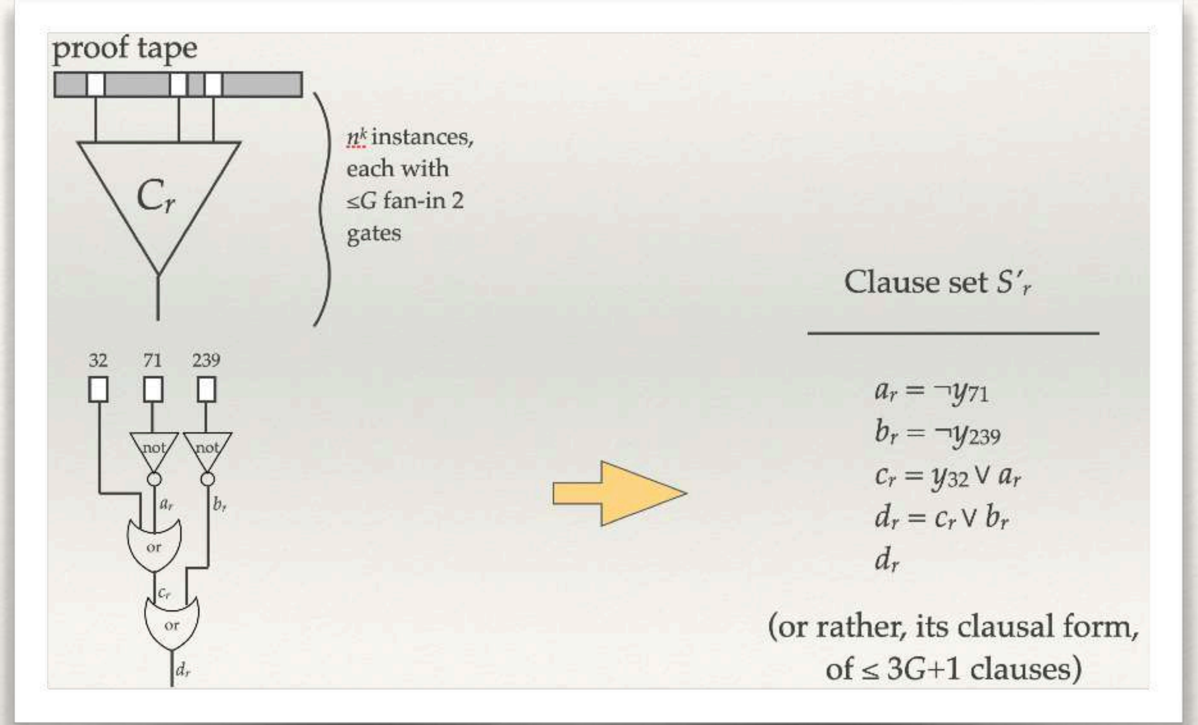


Clause set S'_r

$$a_r = \neg y_{71}$$
$$b_r = \neg y_{239}$$
$$c_r = y_{32} \vee a_r$$
$$d_r = c_r \vee b_r$$
$$d_r$$

(or rather, its clausal form,
of $\leq 3G+1$ clauses)

If NP=PCP then ∃ ε>0, MAX3SAT(ε) NP-hard



- If S satisfiable, then Merlin can provide y / $\forall r$, output(C_r) true, so $S' \stackrel{\text{def}}{=} \bigwedge_r S_r$ is satisfiable
- Otherwise, whatever y, there is a set I of $\geq \frac{1}{2}n^k$ values of r / $\forall r \in I$, output(C_r) false
- hence $\forall \varrho$ (giving truth values to each y_i and to every auxiliary var.), at least one clause in each S'_r, $r \in I$, must be unsatisfied by ϱ
- so opt(S') ≤ #clauses in $S' - \frac{1}{2}n^k$, and #clauses in $S' \leq (3G+1)n^k$, so opt(S')/#clauses in $S' \leq 1-(\frac{1}{2}n^k)/((3G+1)n^k) = 1-1/(6G+2)$
- Therefore S' is an instance of **MAX3SAT**(ε), with $\varepsilon \stackrel{\text{def}}{=} 1/(6G+2)$

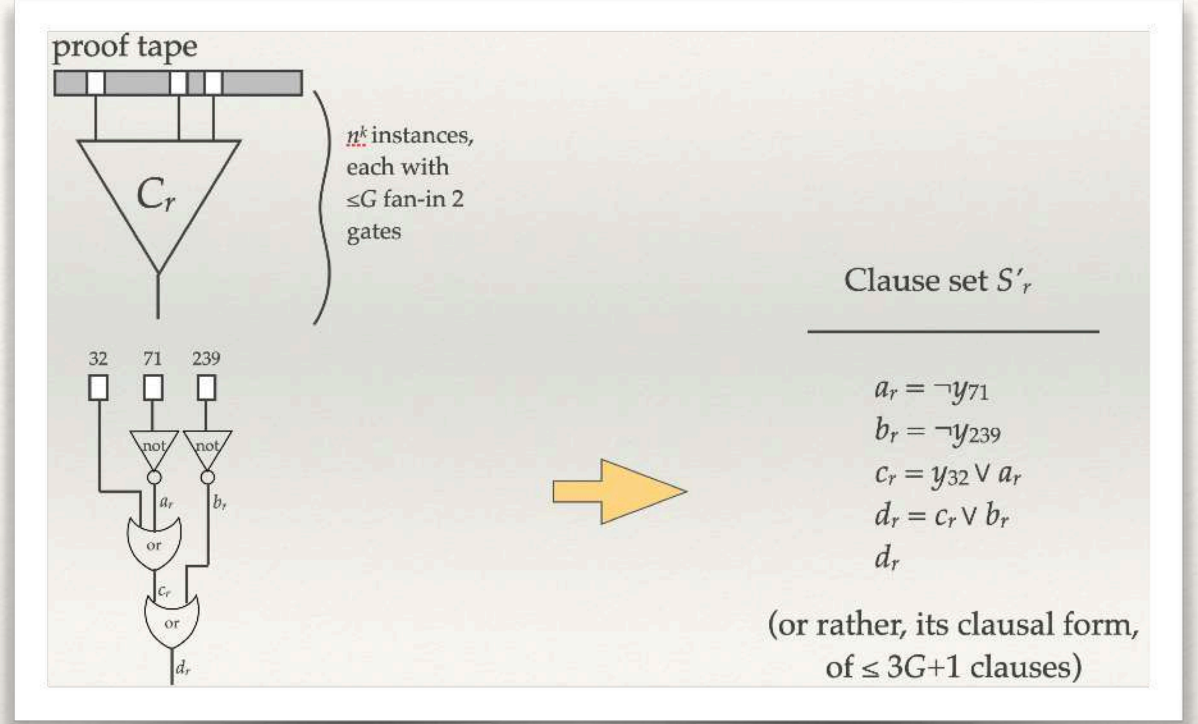
If NP=PCP then ∃ ε>0, MAX3SAT(ε) NP-hard

- **Summary**: If S satisfiable,

  then $S' \stackrel{\text{def}}{=} \bigwedge_r S_r$ is satisfiable

  Else, $\text{opt}(S') \leq (1-\varepsilon).\#$clauses in S'

  where $\varepsilon \stackrel{\text{def}}{=} 1/(6G+2)$



- Additionally, each C_r can be computed in polynomial time (simulating Arthur's computation),
  and computing S'_r from C_r also takes polynomial time

- Hence we have found a polytime reduction from **SAT** to **MAX3SAT**(ε) (assuming **NP**=**PCP**). □

Irit Dinur

Par נובאמחת — Travail personnel,

https://commons.wikimedia.org/w/index.php?curid=72395593

Home > ACM Journals > Journal of the ACM > Vol. 54, No. 3 > The PCP theorem by gap amplification

(2007)

ARTICLE

The PCP theorem by gap amplification

Author: Irit Dinur Authors Info & Affiliations

Publication: Journal of the ACM • June 2007 • https://doi.org/10.1145/1236457.1236459

The PCP Theorem by Gap Amplification

Irit Dinur*

September 26, 2005

Abstract

We describe a new proof of the PCP theorem that is based on a combinatorial amplification lemma. The *unsat value* of a set of constraints $\mathcal{C} = \{c_1, \ldots, c_n\}$, denoted $\text{UNSAT}(\mathcal{C})$, is the smallest fraction of unsatisfied constraints, ranging over all possible assignments for the underlying variables.

We prove a new combinatorial amplification lemma that doubles the unsat-value of a constraint-system, with only a linear blowup in the size of the system. Iterative application of this lemma yields a proof for the PCP theorem.

The amplification lemma relies on a new notion of "graph powering" that can be applied to systems of constraints. This powering amplifies the unsat-value of a constraint system provided that the underlying graph structure is an expander.

We also apply the amplification lemma to construct PCPs and locally-testable codes whose length is linear up to a *polylog* factor, and whose correctness can be probabilistically verified by making a *constant* number of queries. Namely, we prove $SAT \in PCP_{\frac{1}{2},1}[\log_2(n \cdot \text{poly}\log n), O(1)]$. This answers an open question of Ben-Sasson et al. (STOC '04).

- Simplified proof…
  I will only give a rough sketch
- Uses expander graphs, « powering » on random walks, Hadamard codes, etc.

Constraint graph satisfiability

- Instead of **MAX3SAT**(ε), Dinur uses:
- **Defn.** A **constraint graph** is an undirected graph (V, E) plus a set of constraints $c(e) \subseteq \Sigma\times\Sigma$, one for each edge e … where Σ is a finite set of values, or **colors**, that each vertex may assume under a **color assignment**
- Question: is there a color assignment satisfying all the edge constraints?
- **NP**-complete, generalizes **3-COLORABILITY**

The gap

- The **gap** of an unsatisfiable contraint graph is
min (#unsatisfied edge constraints) / m [$m \stackrel{\text{def}}{=}$ #edges]
- We start with an unsatisfiable constraint graph G
- … of gap $\geq 1/m$
- and we modify it so as to increase its gap
until we reach a **constant** non-zero number
- Applied to a **satisfiable** constraint graph,
the modifications will preserve satisfiability.

Graph expanders

- A **graph expander** is a family of undirected graphs with « good connectivity »

- **Defn.** The **edge expansion** $h(G)$ of a graph G is
  min (#edges between S and its complement / #S)
  over subsets S of $<n/2$ vertex of G [$n \stackrel{\text{def}}{=}$ #vertices]

- A **graph expander** is a family of graphs G_n, $n \in \mathbf{N}$,
  — each regular of constant degree d_0
  — with n vertices each
  — such that $\boxed{h(G_n) \geq h_0\text{, a positive constant}}$

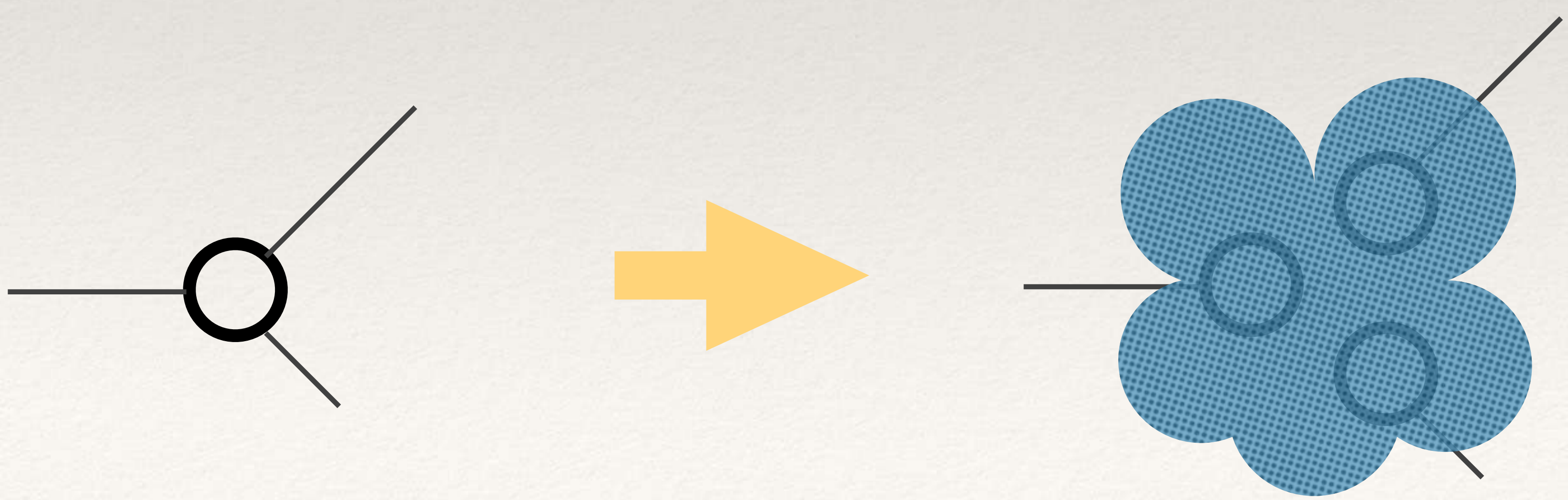
This exists, and G_n can even be produced in polynomial time (in n)

Graph expanders

- A **graph expander** is a family of undirected graphs with « good connectivity »
- A **random walk** on a graph expander is rapidly mixing, namely: just doing a few steps gets you exponentially close to the stationary distribution

1. Sparsification

- First step: make G **sparse enough**
  (so as to allow step 2 to apply; the important step is step 3)
  precisely: make it **regular** and of small enough degree d
- Gap decreases by a **constant** factor only
- Replace every vertex (degree, say, k) by
  a **graph expander** of degree d–1 with k vertices

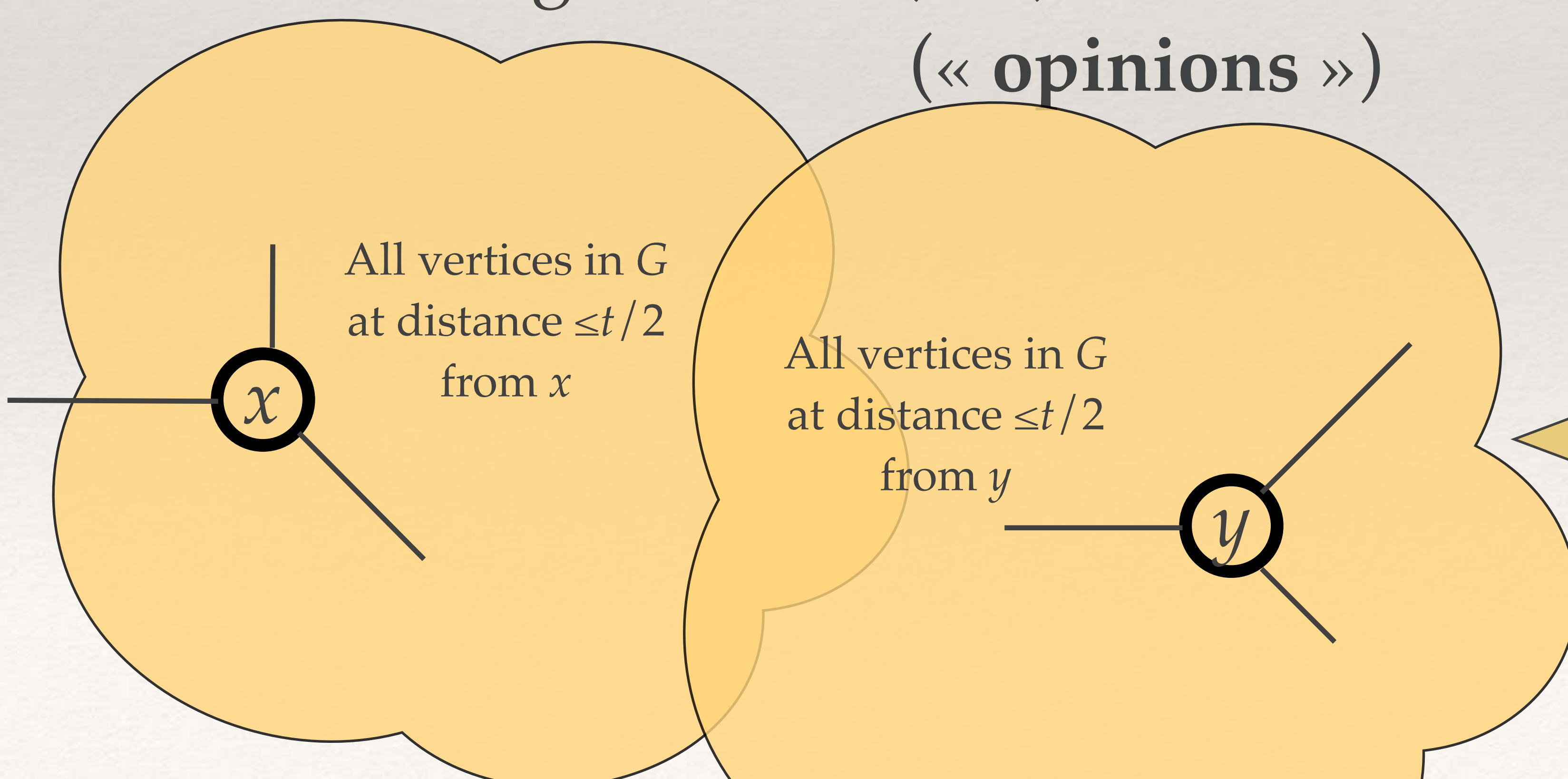


2. Expanderize

- First step: make G **an expander**
  (so as to allow step 3 to apply)
- By taking the union with a good expander
- Gap (also) decreases by a **constant** factor (only)

3. Amplify the gap

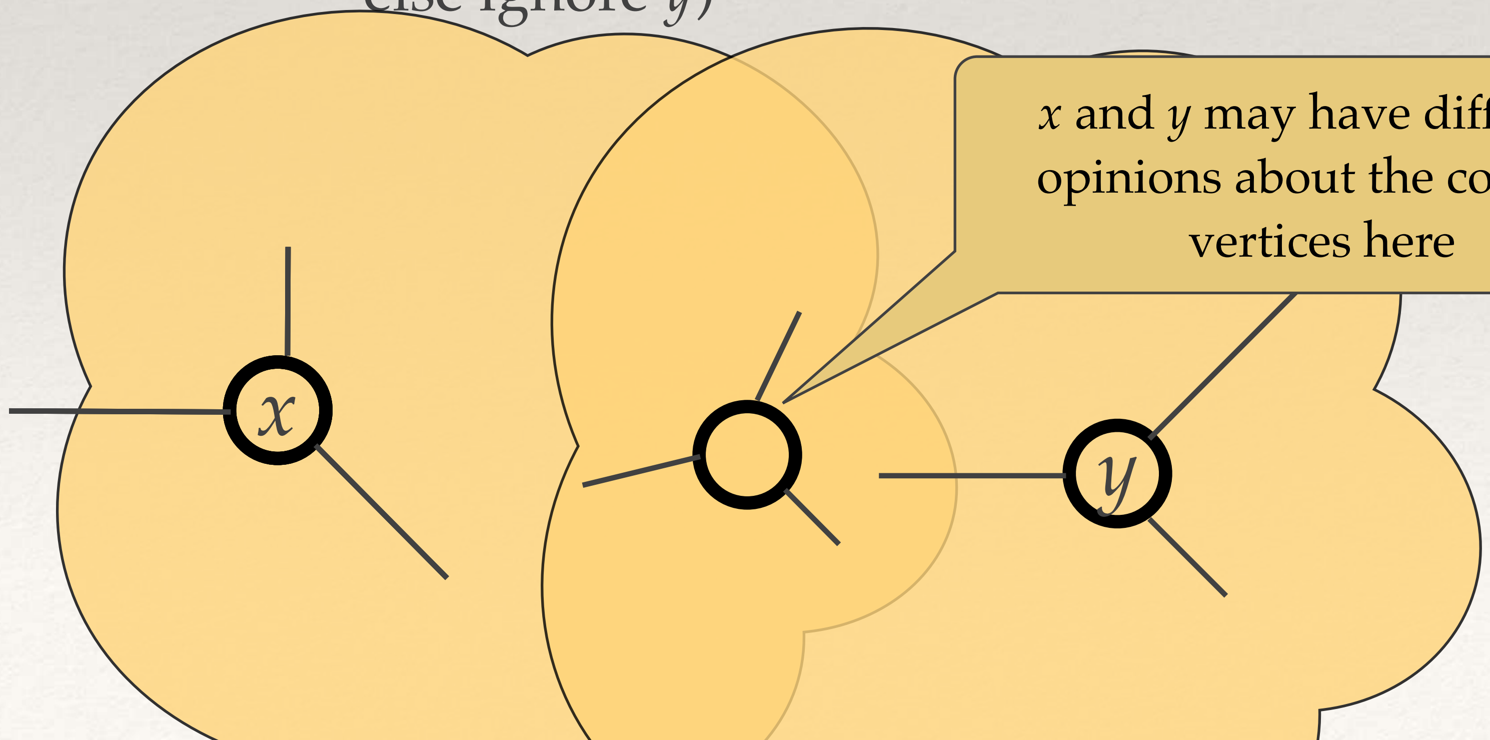
- This is the difficult step.
- Fix a constant $t>0$, and build a new constraint graph G^t whose single edges simulate **paths** of t edges in G
  (there are as many edges between x and y in G^t as paths in G)
- Encode distance $\leq t/2$ neighborhoods around each vertex
  New colors = assignment of (old) colors to vertices in those nbds (« **opinions** »)



Need $O(|\Sigma|$^$(t^{d/2}))$ new colors to express lists of [original] colors on all possible simulated paths

3. Amplify the gap

- Problem: close vertices in G may be assigned incompatible opinions (**consistency** problem)
- Correctness proof: given a color assignment on G^t,
  build back a color assignment on G:
  color of x (in G) $\stackrel{\text{def}}{=}$ **most likely** result as given by:
  (do **random walk** in G starting from x;
  stops at y with probability 1;
  if y is in neighborhood of x
  then return opinion of y on what the color of x should be
  else ignore y)



3. Amplify the gap

- The analysis is a bit complex, but:
- **Gap** is (finally) amplified, by roughly $\sqrt{t}$ while gap$\leq 1/t$
- … although we need O(| Σ | ^($t^{d/2}$)) new colors to solve consistency (express lists of [original] colors on all possible simulated paths)

4. Alphabet reduce

- Reduce back the alphabet of colors **constant size** (2^6, i.e. 64)
- By encoding constraints through assignment testers
  assignments are encoded by **Hadamard error-correcting** codes
  [correct many errors, but exponentially large —
  which is not a problem here because this will be the exponential of a constant…]
- Decreases back gap by some constant factor
- … and repeat steps 1—4 until gap becomes larger than a constant
  (requires O(log m) iterations)

Dinur's algorithm summarized

Step	Main Ideas	Effects	Proof Techniques
Degree Reduce	Split every vertex in to many vertices,and introduce an Expander cloud with equality constraints among the split vertices.	Size ↑ a $O(d)$ factor, Gap decreases by a constant factor, Alphabet remains same	Basic expansion property of expanders
Expanderize	Superimpose a constant degree expander with trivial constraints, on to the constraint graph G	Size ↑ a factor of 2 to 3, Gap decreases by a constant factor, Alphabet remains same	Existence of constant degree expanders and Property that Expander + Graph gives an expander.
Gap-Amplification	Each vertex's value is its opinion,on the values of vertices at a distance $< t$,Add edges corresponding to consistency on random walks	Size ↑ by a large constant factor ,Gap increases by $O(t)$, Alphabet size becomes $\|\Sigma\|^{O(d^t)}$	Properties of random walks on the graph
Alphabet-Reduce	Encode the assignment with error correcting codes, Build a circuit that checks if assignment satisfies and is a valid codeword, Use an assignment tester for the circuit	Size ↑ a constant factor, Gap decreases by a constant factor,Alphabet size reduced to 2^6	Hadamard codes, Linearity Testing, Fourier Analysis

Table 1: Proof of PCP

https://courses.cs.washington.edu/courses/cse533/05au/

and…

That's it, folks!

- I hope you enjoyed the material of the course!