Initiation à la vérification
Basics of Verification


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Outline

1 Introduction

Models

Temporal Specifications

Satisfiability and Model Checking

More on Temporal Specifications
Need for formal verification methods

Critical systems

- Transport
- Energy
- Medicine
- Communication
- Finance
- Embedded systems
- ...
Disastrous software bugs


Mariner 1 probe, 1962

See http://en.wikipedia.org/wiki/Mariner_1

- Destroyed 293 seconds after launch
- Missing hyphen in the data or program? **No**!
- **Overbar** missing in the mathematical specification:
  \[
  \bar{R}_n : n\text{th smoothed} \text{ value of the time derivative of a radius.}
  \]
  Without the smoothing function indicated by the bar, the program treated normal minor variations of velocity as if they were serious, causing spurious corrections that sent the rocket off course.
## Disastrous software bugs

### Ariane 5 flight 501, 1996


- Destroyed 37 seconds after launch (cost: 370 millions dollars).
- Data conversion from a 64-bit floating point to 16-bit signed integer value caused a hardware exception (arithmetic overflow).
- Efficiency considerations had led to the disabling of the software handler (in Ada code) for this error trap.
- The fault occurred in the inertial reference system of Ariane 5. The software from Ariane 4 was re-used for Ariane 5 without re-testing.
- On the basis of those calculations the main computer commanded the booster nozzles, and somewhat later the main engine nozzle also, to make a large correction for an attitude deviation that had not occurred.
- The error occurred in a realignment function which was not useful for Ariane 5.
Disastrous software bugs

Spirit Rover (Mars Exploration), 2004


- Ceased communicating on January 21.
- Flash memory management anomaly: too many files on the file system
- Resumed to working condition on February 6.
Disastrous software bugs

Other well-known bugs


# Formal verifications methods

## Based on
- A formal model of the system
- A formal semantics of the modelling language
- A formal specification

## Complementary approaches
- Theorem prover
- Model checking
- Static analysis
- Test
Model Checking

- Purpose 1: *automatically* finding software or hardware bugs.
- Purpose 2: *prove correctness* of abstract models.
- Should be applied during design.
- Real systems can be analysed with abstractions.

E.M. Clarke  
E.A. Emerson  
J. Sifakis

Prix Turing 2007.
Model Checking

3 steps

- Constructing the model $M$ (transition systems)
- Formalizing the specification $\varphi$ (temporal logics)
- Checking whether $M \models \varphi$ (algorithmics)

Main difficulties

- Size of models (combinatorial explosion)
- Expressivity of models or logics
- Decidability and complexity of the model-checking problem
- Efficiency of tools

Challenges

- Extend models and algorithms to cope with more systems.
  Infinite systems, parameterized systems, probabilistic systems, concurrent systems, timed systems, hybrid systems, . . . See Modules 2.8 & 2.9
- Scale current tools to cope with real-size systems.
  Needs for modularity, abstractions, symmetries, . . .
References


Outline

Introduction

2 Models

- Transition Systems
- ... with Variables
- Concurrent Systems
- Synchronization and Communication

Temporal Specifications

Satisfiability and Model Checking

More on Temporal Specifications
Example: Golden face

Each coin has a golden face and a silver face. At each step, we may flip simultaneously the 3 coins of a line, column or diagonal. Is it possible to have all coins showing its golden face? If yes, what is the smallest number of steps.
Model and Specification

Example: Men, Wolf, Goat, Cabbage

Model = Transition system
- State = who is on which side of the river
- Transition = crossing the river
- Specification
  - Safety: Never leave WG or GC alone
  - Liveness: Take everyone to the other side of the river.
**Transition system or Kripke structure**

**Definition: TS**

\[ M = (S, \Sigma, T, I, AP, \ell) \]

- **S**: set of states (finite or infinite)
- **\( \Sigma \)**: set of actions
- **\( T \subseteq S \times \Sigma \times S \)**: set of transitions
- **\( I \subseteq S \)**: set of initial states
- **AP**: set of atomic propositions
- **\( \ell : S \rightarrow 2^{AP} \)**: labelling function.

Every discrete system may be described with a TS.

**Example: Digicode ABA**
Pb: How can we easily describe big systems?

Description Languages (high level)
- Programming languages
- Boolean circuits
- Modular description, e.g., parallel compositions
  problems: concurrency, synchronization, communication, atomicity, fairness, ...
- Petri nets (intermediate level)
- Transition systems (intermediate level)
  with variables, stacks, channels, ...
  synchronized products
- Logical formulae (low level)

Operational semantics
High level descriptions are translated (compiled) to low level (infinite) TS.
Transition systems with variables

Definition: TSV

\[ M = (S, \Sigma, V, (D_v)_{v \in V}, T, I, AP, \ell) \]

- Finite description with \( S, \Sigma, AP, \ell \) as before
- \( V \): set of (typed) variables, e.g., boolean, [0..4], \( \mathbb{N} \), ...
- Each variable \( v \in V \) has a domain \( D_v \) (finite or infinite). Let \( D = \prod_{v \in V} D_v \).
- Guard or Condition \( g \) with semantics \( \llbracket g \rrbracket \subseteq D \) (predicate)
  Symbolic descriptions: \( x < 5 \), \( x + y = 10 \), ...
- Instruction or Update \( f \) with semantics \( \llbracket f \rrbracket : D \to D \)
  Symbolic descriptions: \( x := 0 \), \( x := (y + 1)^2 \), ...
- \( T \subseteq S \times (\text{Guard} \times \Sigma \times \text{Update}) \times S \)
  Symbolic descriptions: \( s \xrightarrow{x < 50, ?\text{coin}, x := x + \text{coin}} s' \)
- \( I \subseteq S \times \text{Guard} \)
  Symbolic descriptions: \((s_0, x = 0)\)

Example: Vending machine

- coffee: 50 cents, orange juice: 1 euro, ...
- possible coins: 10, 20, 50 cents
- we may shuffle coin insertions and drink selection
Transition systems with variables

Semantics: low level TS

- \( S' = S \times D \)
- \( I' = \{(s, \nu) \mid \exists (s, g) \in I \text{ with } \nu \models g \} \)
- Transitions: \( T' \subseteq (S \times D) \times \Sigma \times (S \times D) \)

\[ \frac{s \xrightarrow{g,a,f} s' \land \nu \models g}{(s, \nu) \xrightarrow{a} (s', f(\nu))} \]

SOS: Structural Operational Semantics

- \( \text{AP'} \): we may use atomic propositions in \( \text{AP} \) or guards such as \( x > 0 \).

Programs = Kripke structures with variables

- Program counter = states
- Instructions = transitions
- Variables = variables

Example: GCD
Example: Digicode

1. \( \text{cpt} = 0 \)
2. \( \text{cpt} < n \) if \( B, C \) then \( \text{cpt} += 1 \), else \( \text{cpt} < n \) if \( A \) then \( \text{cpt} += 1 \)
3. \( \text{cpt} = n \) if \( C \) then \( \text{cpt} += 1 \), else \( \text{cpt} = n \) if \( A, C \) then \( \text{cpt} += 1 \)
4. \( \text{cpt} = n \) if \( B, C \) then \( \text{cpt} += 1 \)
5. \( \text{cpt} = n \) if \( B, C \) then \( \text{cpt} += 1 \)

States:
- State 1: \( \text{cpt} = 0 \)
- State 2: \( \text{cpt} < n \) on \( A \)
- State 3: \( \text{cpt} = n \) on \( A, C \)
- State 4: OPEN
- State 5: ERROR

Transitions:
- From State 1: To State 2 on \( A \), or to State 5 on \( B, C \)
- From State 2: To State 3 on \( B \)
- From State 3: To State 1 on \( A \), or to State 4 on \( B, C \)
- From State 5: Back to State 1 on \( B, C \)
Only variables

The state is nothing but a special variable: \( s \in \mathcal{V} \) with domain \( D_s = \mathcal{S} \).

**Definition: TSV**

\[
M = (\mathcal{V}, (D_v)_{v \in \mathcal{V}}, T, I, AP, \ell)
\]

- \( D = \prod_{v \in \mathcal{V}} D_v \),
- \( I \subseteq D, T \subseteq D \times D \)

**Symbolic representations with logic formulae**

- \( I \) given by a formula \( \psi(\nu) \)
- \( T \) given by a formula \( \varphi(\nu, \nu') \)
  - \( \nu \): values before the transition
  - \( \nu' \): values after the transition
- Often we use boolean variables only: \( D_v = \{0, 1\} \)
- Concise descriptions of boolean formulae with Binary Decision Diagrams.

**Example: Boolean circuit: modulo 8 counter**

\[
\begin{align*}
    b'_0 & = \neg b_0 \\
    b'_1 & = b_0 \oplus b_1 \\
    b'_2 & = (b_0 \land b_1) \oplus b_2
\end{align*}
\]
Modular description of concurrent systems

\[ M = M_1 \parallel M_2 \parallel \cdots \parallel M_n \]

Semantics

- Various semantics for the parallel composition \( \parallel \)
- Various communication mechanisms between components: Shared variables, FIFO channels, Rendez-vous, ...
- Various restrictions

Atomic propositions are inherited from the local systems.

Example: Elevator with 1 cabin, 3 doors, 3 calling devices

- Cabin:
- Door for level \( i \):
- Call for level \( i \):
Synchronized products (semantics)

**Definition: General product**

- **Components:** \( M_i = (S_i, \Sigma_i, T_i, I_i, AP_i, \ell_i) \)
- **Product:** \( M = (S, \Sigma, T, I, AP, \ell) \) with \( S = \prod_i S_i \), \( \Sigma = \prod_i (\Sigma_i \cup \{\varepsilon\}) \), and \( I = \prod_i I_i \)

\[
T \text{ defined by } \frac{\forall i, \; p_i \xrightarrow{a_i} q_i \in T_i \lor (a_i = \varepsilon \land p_i = q_i)}{(p_1, \ldots, p_n) \xrightarrow{(a_1, \ldots, a_n)} (q_1, \ldots, q_n)}
\]

\[AP = \bigcup_i AP_i \text{ and } \ell(p_1, \ldots, p_n) = \bigcup_i \ell(p_i)\]

**Synchronized products: restrictions of the general product.**

**Parallel compositions:** 2 special cases

- **Synchronous:** \( \Sigma_{sync} = \prod_i \Sigma_i \)
- **Asynchronous:** \( \Sigma_{async} = \bigcup_i \Sigma'_i \) with \( \Sigma'_i = \{\varepsilon\}^{i-1} \times \Sigma_i \times \{\varepsilon\}^{n-i} \)

**Restrictions**

- **on states:** \( S_{restrict} \subseteq S \)
- **on labels:** \( \Sigma_{restrict} \subseteq \Sigma \)
- **on transitions:** \( T_{restrict} \subseteq T \)
**Shared variables**

**Definition:** Asynchronous product + shared variables

\[ \bar{s} = (s_1, \ldots, s_n) \] denotes a tuple of states

\[ \nu \in D = \prod_{v \in \mathcal{V}} D_v \] is a valuation of variables.

**Semantics (SOS)**

\[
\nu \models g \land s_i \xrightarrow{g,a,f} s_i' \land s_j' = s_j \quad \text{for } j \neq i
\]

\[
(\bar{s}, \nu) \xrightarrow{a} (\bar{s}', f(\nu))
\]

**Example:** Mutual exclusion for 2 processes satisfying

- **Safety:** never simultaneously in critical section (CS).
- **Liveness:** if a process wants to enter its CS, it eventually does.
- **Fairness:** if process 1 wants to enter its CS, then process 2 will enter its CS at most once before process 1 does.

using shared variables but without further restrictions: the atomicity is

- testing or reading or writing a single variable at a time
- no test-and-set: \[ \{x = 0; x := 1\} \]
Peterson’s algorithm (1981)

Process \( i \):  // \( i \) is not a variable
    loop forever
        req[\( i \)] := true; turn := 1-\( i \)
        wait until (turn = \( i \) or req[\( 1-i \)] = false)
    Critical section
    req[\( i \)] := false

Exercise:
- Draw the concrete TS assuming the first two assignments are atomic.
- Is the algorithm still correct if we swape the first two assignments?
Atomicity

Example:

Initially $x = 1 \land y = 2$

Program $P_1$: $x := x + y \parallel y := x + y$

Program $P_2$: \[
\begin{pmatrix}
\text{Load}R_1, x \\
\text{Add}R_1, y \\
\text{Store}R_1, x
\end{pmatrix}
\parallel
\begin{pmatrix}
\text{Load}R_1, x \\
\text{Add}R_2, y \\
\text{Store}R_2, y
\end{pmatrix}
\]

Assuming each instruction is atomic, what are the possible results of $P_1$ and $P_2$?
Atomicity

Definition: Atomic statements: \texttt{atomic(ES)}

Elementary statements (no loops, no communications, no synchronizations)

\[
ES ::= \text{skip} \mid \text{await } c \mid x := e \mid ES ; ES \mid ES \square ES \\
| \text{when } c \text{ do } ES \mid \text{if } c \text{ then } ES \text{ else } ES
\]

Atomic statements: if the ES can be fully executed then it is executed in one step.

\[
(\bar{s}, \nu) \xrightarrow{ES} (\bar{s}', \nu') \quad \xrightarrow{\text{atomic}(ES)} (\bar{s}', \nu')
\]

Example: Atomic statements

- \texttt{atomic}(x = 0; x := 1) \quad \text{(Test and set)}
- \texttt{atomic}(y := y - 1; \text{await}(y = 0); y := 1) \text{ is equivalent to } \text{await}(y = 1)
Communication by Rendez-vous

Restriction on transitions is universal but too low-level.

Definition: Rendez-vous

- !m sending message m
- ?m receiving message m
- **SOS:** Structural Operational Semantics

Local actions

\[
\begin{align*}
    s_1 & \xrightarrow{a_1} s_1' \\
    (s_1, s_2) & \xrightarrow{a_1} (s_1', s_2)
\end{align*}
\]

\[
\begin{align*}
    s_2 & \xrightarrow{a_2} s_2' \\
    (s_1, s_2) & \xrightarrow{a_2} (s_1, s_2')
\end{align*}
\]

Rendez-vous

\[
\begin{align*}
    s_1 & \xrightarrow{!m} s_1' \wedge s_2 \xrightarrow{?m} s_2' \\
    (s_1, s_2) & \xrightarrow{m} (s_1', s_2')
\end{align*}
\]

\[
\begin{align*}
    s_1 & \xrightarrow{?m} s_1' \wedge s_2 \xrightarrow{!m} s_2' \\
    (s_1, s_2) & \xrightarrow{m} (s_1', s_2')
\end{align*}
\]

- It is a restriction on actions.
- Essential feature of process algebra.

Example: Elevator with 1 cabin, 3 doors, 3 calling devices

- ?up is uncontrollable for the cabin
- ?leave_i is uncontrollable for door i
- ?call_0 is uncontrollable for the system
Example: Leader election

We have $n$ processes on a directed ring, each having a unique id $\in \{1, \ldots, n\}$.

```plaintext
send(id)
loop forever
    receive(x)
    if (x = id) then STOP fi
    if (x > id) then send(x)
```
Definition: Channels

- Declaration:
  - $c : \text{channel } [k] \text{ of bool}$ size $k$
  - $c : \text{channel } [\infty] \text{ of int}$ unbounded
  - $c : \text{channel } [0] \text{ of colors}$ Rendez-vous

- Primitives:
  - empty($c$)
  - $c!e$ add the value of expression $e$ to channel $c$
  - $c?x$ read a value from $c$ and assign it to variable $x$

- Domain: Let $D_m$ be the domain for a single message.
  - $D_c = D_m^k$ size $k$
  - $D_c = D_m^*$ unbounded
  - $D_c = \{\varepsilon\}$ Rendez-vous

- Politics: FIFO, LIFO, BAG, ...
Channels

Semantics: (lossy) FIFO

Send

\[
{s_i} \xrightarrow{c!e} s'_i \land v'(c) = v(e) \cdot v(c)
\]

\[
(\bar{s}, v) \xrightarrow{c!e} (\bar{s}', v')
\]

Receive

\[
{s_i} \xrightarrow{c?x} s'_i \land v(c) = v'(c) \cdot v'(x)
\]

\[
(\bar{s}, v) \xrightarrow{c?e} (\bar{s}', v')
\]

Lossy send

\[
{s_i} \xrightarrow{c!e} s'_i
\]

\[
(\bar{s}, v) \xrightarrow{c!e} (\bar{s}', v)
\]

Implicit assumption: all variables that do not occur in the premise are not modified.

Exercises:

1. Implement a FIFO channel using rendez-vous with an intermediary process.
2. Give the semantics of a LIFO channel.
3. Model the alternating bit protocol (ABP) using a lossy FIFO channel.
   Fairness assumption: For each channel, if infinitely many messages are sent, then infinitely many messages are delivered.
High-level descriptions

Summary

- Sequential program = transition system with variables
- Concurrent program with shared variables
- Concurrent program with Rendez-vous
- Concurrent program with FIFO communication
- Petri net
- ...


Models: expressivity versus decidability

Remark: (Un)decidability

- Automata with 2 integer variables = Turing powerful
  Restriction to variables taking values in finite sets
- Asynchronous communication: unbounded fifo channels = Turing powerful
  Restriction to bounded channels or lossy channels

Remark: Some infinite state models are decidable

- Petri nets. Several unbounded integer variables but no zero-test.
- Pushdown automata. Model for recursive procedure calls.
- Timed automata.
- ...
Outline

Introduction

Models

3 Temporal Specifications
- General Definitions
- (Linear) Temporal Specifications
- Branching Temporal Specifications
- CTL*
- CTL

Satisfiability and Model Checking

More on Temporal Specifications
Static and dynamic properties

Example: Static properties

Mutual exclusion
Safety properties are often static.
They can be reduced to reachability.

Example: Dynamic properties

Every elevator request should be eventually granted.

The elevator should not cross a level for which a call is pending without stopping.
Temporal Structures

Definition: Flows of time

A flow of time is a strict order \((\mathbb{T}, <)\) where \(\mathbb{T}\) is the nonempty set of time points and \(<\) is an irreflexive transitive relation on \(\mathbb{T}\).

Example: Flows of time

- \((\{0, \ldots, n\}, <)\): Finite runs of sequential systems.
- \((\mathbb{N}, <)\): Infinite runs of sequential systems.
- \((\mathbb{R}, <)\): Runs of real-time sequential systems.
- Trees: Finite or infinite run-trees of sequential systems.
- Mazurkiewicz traces: Runs of distributed systems (partial orders).
- And also \((\mathbb{Z}, <)\) or \((\mathbb{Q}, <)\) or \((\omega^2, <)\), \(\ldots\)

Definition: Temporal Structures

Let \(\text{AP}\) be a set of atoms (atomic propositions) and let \(\mathcal{C}\) be a class of time flows. A temporal structure over \((\mathcal{C}, \text{AP})\) is a triple \((\mathbb{T}, <, \lambda)\) where \((\mathbb{T}, <)\) is a time flow in \(\mathcal{C}\) and \(\lambda: \mathbb{T} \to 2^{\text{AP}}\) labels time points with atomic propositions.

The temporal structure \((\mathbb{T}, <, \lambda)\) is also denoted \((\mathbb{T}, <, h)\) where \(h: \text{AP} \to 2^{\mathbb{T}}\) assigns time points to atomic propositions: \(h(p) = \{ t \in \mathbb{T} \mid p \in \lambda(t) \}\) for \(p \in \text{AP}\).
Let $M = (S, T, I, \text{AP}, \ell)$ be a Kripke structure.

**Definition: Runs as temporal structures**

An infinite run $\sigma = s_0 s_1 s_2 \cdots$ of $M$ with $(s_i, s_{i+1}) \in T$ for all $i \geq 0$ defines a *linear* temporal structure $\ell(\sigma) = (\mathbb{N}, <, \lambda)$ where $\lambda(i) = \ell(s_i)$ for $i \in \mathbb{N}$.

Such a temporal structure can be seen as an infinite word over $\Sigma = 2^{\text{AP}}$: $\ell(\sigma) = \ell(s_0)\ell(s_1)\ell(s_2)\cdots \in \Sigma^\omega$

**Linear specifications** only depend on runs.

Example: The printer manager is fair.

On each run, whenever some process requests the printer, it eventually gets it.

**Remark:**

Two Kripke structures having the same linear temporal structures satisfy the same linear specifications.
The system has an infinite active run, but it may always reach an inactive state.

Definition: Computation-tree or run-tree: unfolding of the TS

Let $M = (S, T, I, \text{AP}, \ell)$ be a Kripke structure. Wlog. $I = \{s_0\}$ is a singleton. Let $D$ be a finite set with $|D|$ the outdegree of the transition relation $T$.

The computation-tree of $M$ is an unordered tree $t : D^* \rightarrow S$ (partial map) s.t.

- $t(\varepsilon) = s_0$,
- For every node $u \in \text{dom}(t)$ labelled $s = t(u)$, if $T(s) = \{s_1, \ldots, s_k\}$ then $u$ has exactly $k$ children which are labelled $s_1, \ldots, s_k$.

Associated temporal structure $\ell(t) = (\text{dom}(t), <, \lambda)$ where

- $<$ is the strict prefix relation over $D^*$,
- and $\lambda(u) = \ell(t(u))$ for $u \in \text{dom}(t)$.

(Linear) runs of $M$ are branches of the computation-tree $t$. 

First-order Specifications

Definition: Syntax of FO(AP, <)

Let \( \text{Var} = \{x, y, \ldots\} \) be first-order variables.

\[
\varphi ::= \bot \mid p(x) \mid x = y \mid x < y \mid \neg \varphi \mid \varphi \lor \varphi \mid \exists x \varphi
\]

where \( p \in \text{AP} \).

Definition: Semantics of FO(AP, <)

Let \( w = (\mathbb{T}, <, \lambda) \) be a temporal structure over \( \text{AP} \).
Let \( \nu : \text{Var} \rightarrow \mathbb{T} \) be an assignment of first-order variables to time points.

\[
w, \nu \models p(x) \quad \text{if} \quad p \in \lambda(\nu(x))
\]

\[
w, \nu \models x = y \quad \text{if} \quad \nu(x) = \nu(y)
\]

\[
w, \nu \models x < y \quad \text{if} \quad \nu(x) < \nu(y)
\]

\[
w, \nu \models \exists x \varphi \quad \text{if} \quad w, \nu[x \mapsto t] \models \varphi \text{ for some } t \in \mathbb{T}
\]

where \( \nu[x \mapsto t] \) maps \( x \) to \( t \) and \( y \neq x \) to \( \nu(y) \).

Previous specifications can be written in FO(<) (except the branching one).
First-order vs Temporal

First-order logic

- FO(<) has a good expressive power
  ... but FO(<)-formulae are not easy to write and to understand.
- FO(<) is decidable
  ... but satisfiability and model checking are non elementary.

Temporal logics

- no variables: time is implicit.
- quantifications and variables are replaced by modalities.
- Usual specifications are easy to write and read.
- Good complexity for satisfiability and model checking problems.
- Good expressive power.

Linear Temporal Logic (LTL) over (\(\mathbb{N}, <\)) introduced by Pnueli (1977) as a convenient specification language for verification of systems.
Temporal Specifications

**Definition: Syntax of TL(AP, SU, SS)**

\[ \varphi ::= \bot \mid p \ (p \in \text{AP}) \mid \neg \varphi \mid \varphi \lor \varphi \mid \varphi \text{ SU } \varphi \mid \varphi \text{ SS } \varphi \]

**Definition: Semantics:** \( w = (\mathbb{T}, <, \lambda) \) temporal structure and \( i \in \mathbb{T} \)

\[
\begin{align*}
  w, i \models p & \quad \text{if} \quad p \in \lambda(i) \\
  w, i \models \neg \varphi & \quad \text{if} \quad w, i \not\models \varphi \\
  w, i \models \varphi \lor \psi & \quad \text{if} \quad w, i \models \varphi \text{ or } w, i \models \psi \\
  w, i \models \varphi \text{ SU } \psi & \quad \text{if} \quad \exists k \ i < k \text{ and } w, k \models \psi \text{ and } \forall j \ (i < j < k \rightarrow w, j \models \varphi) \\
  w, i \models \varphi \text{ SS } \psi & \quad \text{if} \quad \exists k \ i > k \text{ and } w, k \models \psi \text{ and } \forall j \ (i > j > k \rightarrow w, j \models \varphi)
\end{align*}
\]

Previous specifications can be written in TL(AP, SU, SS) (except the branching one).
## Temporal Specifications

### Definition: non-strict versions of until and since

\[
\varphi \mathcal{U} \psi \overset{\text{def}}{=} \psi \lor (\varphi \land \varphi \mathcal{S} \mathcal{U} \psi) \quad \varphi \mathcal{S} \psi \overset{\text{def}}{=} \psi \lor (\varphi \land \varphi \mathcal{S} \mathcal{S} \psi)
\]

\[w, i \models \varphi \mathcal{U} \psi \quad \text{if} \quad \exists k \ i \leq k \text{ and } w, k \models \psi \text{ and } \forall j \ (i \leq j < k \rightarrow w, j \models \varphi)\]

\[w, i \models \varphi \mathcal{S} \psi \quad \text{if} \quad \exists k \ i \geq k \text{ and } w, k \models \psi \text{ and } \forall j \ (i \geq j > k \rightarrow w, j \models \varphi)\]

### Definition: Derived modalities

\[\mathcal{X} \varphi \overset{\text{def}}{=} \perp \mathcal{S} \mathcal{U} \varphi \quad \text{Next} \quad \mathcal{Y} \varphi \overset{\text{def}}{=} \perp \mathcal{S} \mathcal{S} \varphi \quad \text{Yesterday}\]

\[w, i \models \mathcal{X} \varphi \quad \text{if} \quad \exists k \ i < k \text{ and } w, k \models \varphi \text{ and } \neg \exists j \ (i < j < k)\]

\[w, i \models \mathcal{Y} \varphi \quad \text{if} \quad \exists k \ i > k \text{ and } w, k \models \varphi \text{ and } \neg \exists j \ (i > j > k)\]

\[\mathcal{S} \mathcal{F} \varphi \overset{\text{def}}{=} \top \mathcal{S} \mathcal{U} \varphi \quad \mathcal{S} \mathcal{P} \varphi \overset{\text{def}}{=} \top \mathcal{S} \mathcal{S} \varphi\]

\[\mathcal{F} \varphi \overset{\text{def}}{=} \top \mathcal{U} \varphi \quad \mathcal{P} \varphi \overset{\text{def}}{=} \top \mathcal{S} \varphi\]

\[\mathcal{G} \varphi \overset{\text{def}}{=} \neg \mathcal{F} \neg \varphi \quad \mathcal{H} \varphi \overset{\text{def}}{=} \neg \mathcal{P} \neg \varphi\]

\[\varphi \mathcal{W} \psi \overset{\text{def}}{=} (\mathcal{G} \varphi) \lor (\varphi \mathcal{U} \psi) \quad \text{Weak Until}\]

\[\varphi \mathcal{R} \psi \overset{\text{def}}{=} (\mathcal{G} \psi) \lor (\psi \mathcal{U} (\varphi \land \psi)) \quad \text{Release}\]
Temporal Specifications

<table>
<thead>
<tr>
<th>Example: Specifications on the time flow ((\mathbb{N}, &lt;))</th>
</tr>
</thead>
<tbody>
<tr>
<td>- <strong>Safety:</strong> (G) good</td>
</tr>
<tr>
<td>- <strong>MutEx:</strong> (\neg F(crit_1 \land crit_2))</td>
</tr>
<tr>
<td>- <strong>Liveness:</strong> (G F) active</td>
</tr>
<tr>
<td>- <strong>Response:</strong> (G(request \rightarrow F\ grant))</td>
</tr>
<tr>
<td>- <strong>Response':</strong> (G(request \rightarrow (\neg request SU\ grant)))</td>
</tr>
<tr>
<td>- <strong>Release:</strong> (reset \ R) alarm</td>
</tr>
<tr>
<td>- <strong>Strong fairness:</strong> (GF\ request) \rightarrow (GF grant)</td>
</tr>
<tr>
<td>- <strong>Weak fairness:</strong> (FG\ request) \rightarrow (GF grant)</td>
</tr>
</tbody>
</table>
Discrete linear time flows

Definition: discrete linear time flows \((\mathbb{T}, <)\)

A linear time flow is discrete if \(SF \, \mathbb{T} \rightarrow X \, \mathbb{T}\) and \(SP \, \mathbb{T} \rightarrow Y \, \mathbb{T}\) are valid formulae.

\((\mathbb{N}, <)\) and \((\mathbb{Z}, <)\) are discrete.

\((\mathbb{Q}, <)\) and \((\mathbb{R}, <)\) are not discrete.

Exercise: For discrete linear time flows \((\mathbb{T}, <)\)

\[\varphi \, SU \, \psi \equiv X(\varphi \, U \, \psi)\]
\[\varphi \, SS \, \psi \equiv Y(\varphi \, S \, \psi)\]
\[\neg X \, \varphi \equiv \neg X \, \mathbb{T} \lor X \, \neg \varphi\]
\[\neg Y \, \varphi \equiv \neg Y \, \mathbb{T} \lor Y \, \neg \varphi\]
\[\neg (\varphi \, U \, \psi) \equiv (G \, \neg \psi) \lor (\neg \psi \, U \, (\neg \varphi \land \neg \psi))\]
\[\equiv \neg \psi \, W \, (\neg \varphi \land \neg \psi)\]
\[\equiv \neg \varphi \, R \, \neg \psi\]
Model checking for linear behaviors

Definition: Model checking problem

Input: A Kripke structure $M = (S, T, I, AP, \ell)$
A formula $\varphi \in \text{LTL}(AP, SU, SS)$

Question: Does $M \models \varphi$?

- Universal MC: $M \models \forall \varphi$ if $\ell(\sigma), 0 \models \varphi$ for all initial infinite runs $\sigma$ of $M$.
- Existential MC: $M \models \exists \varphi$ if $\ell(\sigma), 0 \models \varphi$ for some initial infinite run $\sigma$ of $M$.

\[
M \models \forall \varphi \quad \text{iff} \quad M \not\models \exists \neg \varphi
\]

Theorem [11, Sistla, Clarke 85], [10, Lichtenstein & Pnueli 85]
The Model checking problem for LTL is PSPACE-complete. Proof later
Weaknesses of linear behaviors

Example:

\( \varphi \): Whenever \( p \) holds, it is possible to reach a state where \( q \) holds.

\( \varphi \) cannot be checked on linear behaviors.

We need to consider the computation-trees.
Weaknesses of FO specifications

Example:

\( \psi: \) The system has an infinite active run, but it may always reach an inactive state.

\( \psi \) cannot be expressed in FO.

We need quantifications on runs:

\[ \psi = \text{EG}(\text{Active} \land \text{EF} \, \neg \text{Active}) \]

- E: for some infinite run
- A: for all infinite runs
**MSO Specifications**

**Definition: Syntax of MSO(AP, <)**

\[ \varphi ::= \bot | p(x) | x = y | x < y | x \in X | \neg \varphi | \varphi \lor \varphi | \exists x \varphi | \exists X \varphi \]

where \( p \in \text{AP} \), \( x, y \) are first-order variables and \( X \) is a second-order variable.

**Definition: Semantics of MSO(AP, <)**

Let \( w = (\mathbb{T}, <, \lambda) \) be a temporal structure over \( \text{AP} \).

An assignment \( \nu \) maps first-order variables to time points in \( \mathbb{T} \) and second-order variables to sets of time points.

The semantics of first-order constructs is unchanged.

\[ w, \nu \models x \in X \quad \text{if} \quad \nu(x) \in \nu(X) \]
\[ w, \nu \models \exists X \varphi \quad \text{if} \quad w, \nu[X \mapsto T] \models \varphi \text{ for some } T \subseteq \mathbb{T} \]

where \( \nu[X \mapsto T] \) maps \( X \) to \( T \) and keeps unchanged the other assignments.
MSO vs Temporal

**MSO logic**

- MSO(≤) has a good expressive power
  
  ... but MSO(≤)-formulae are not easy to write and to understand.

- MSO(≤) is decidable on computation trees
  
  ... but satisfiability and model checking are non elementary.

**We need a temporal logic**

- with no explicit variables,

- allowing quantifications over runs,

- usual specifications should be easy to write and read,

- with good complexity for satisfiability and model checking problems,

- with good expressive power.

**Computation Tree Logic CTL^∗ introduced by Emerson & Halpern (1986).**
**Definition: Syntax of the Computation Tree Logic CTL\(^*\)(AP, SU)**

\[ \varphi ::= \bot \mid p \ (p \in AP) \mid \neg \varphi \mid \varphi \lor \varphi \mid \varphi \text{ SU } \varphi \mid \text{E } \varphi \mid \text{A } \varphi \]

We may also add the past modality SS

**Definition: Semantics of CTL\(^*\)(AP, SU)**

Let \( M = (S, T, I, AP, \ell) \) be a Kripke structure. Let \( \sigma = s_0 s_1 s_2 \cdots \) be an infinite run of \( M \).

\[
\begin{align*}
M, \sigma, i \models p \quad &\text{if } \quad p \in \ell(s_i) \\
M, \sigma, i \models \varphi \text{ SU } \psi \quad &\text{if } \quad \exists k > i, \ M, \sigma, k \models \psi \text{ and } \forall i < j < k, \ M, \sigma, j \models \varphi \\
M, \sigma, i \models \text{E}\varphi \quad &\text{if } \quad M, \sigma', i \models \varphi \text{ for some infinite run } \sigma' \text{ such that } \sigma'[i] = \sigma[i] \\
M, \sigma, i \models \text{A}\varphi \quad &\text{if } \quad M, \sigma', i \models \varphi \text{ for all infinite runs } \sigma' \text{ such that } \sigma'[i] = \sigma[i]
\end{align*}
\]

where \( \sigma[i] = s_0 \cdots s_i \).

**Remark:**

\( \sigma'[i] = \sigma[i] \) means that future is branching but past is not.
Example: Some specifications

- $\text{EF } \varphi$: $\varphi$ is possible
- $\text{AG } \varphi$: $\varphi$ is an invariant
- $\text{AF } \varphi$: $\varphi$ is unavoidable
- $\text{EG } \varphi$: $\varphi$ holds globally along some path

Remark: Some equivalences

- $A \varphi \equiv \neg E \neg \varphi$
- $E (\varphi \lor \psi) \equiv E \varphi \lor E \psi$
- $A (\varphi \land \psi) \equiv A \varphi \land A \psi$
Definition: Existential and universal model checking

Let \( M = (S, T, I, AP, \ell) \) be a Kripke structure and \( \varphi \in \text{CTL}^* \) a formula.

\[
M \models \exists \varphi \quad \text{if} \quad M, \sigma, 0 \models \varphi \quad \text{for some initial infinite run} \ \sigma \ \text{of} \ \ M.
\]

\[
M \models \forall \varphi \quad \text{if} \quad M, \sigma, 0 \models \varphi \quad \text{for all initial infinite runs} \ \sigma \ \text{of} \ \ M.
\]

Remark: \( M \models \forall \varphi \iff M \not\models \exists \neg \varphi \)

Definition: Model checking problems \( MC_{\text{CTL}^*}^{\forall} \) and \( MC_{\text{CTL}^*}^{\exists} \)

Input: A Kripke structure \( M = (S, T, I, AP, \ell) \) and a formula \( \varphi \in \text{CTL}^* \)

Question: Does \( M \models \forall \varphi \)? \quad or \quad Does \( M \models \exists \varphi \)?

Theorem:

The model checking problem for \( \text{CTL}^* \) is PSPACE-complete. \quad Proof later
**State formulae and path formulae**

**Definition: State formulae**

\( \varphi \in \text{CTL}^* \) is a **state formula** if \( \forall M, \sigma, \sigma', i, j \) such that \( \sigma(i) = \sigma'(j) \) we have

\[
M, \sigma, i \models \varphi \iff M, \sigma', j \models \varphi
\]

If \( \varphi \) is a state formula and \( M = (S, T, I, \text{AP}, \ell) \), define

\( M, s \models \varphi \) if \( M, \sigma, 0 \models \varphi \) for some infinite run \( \sigma \) of \( M \) with \( \sigma(0) = s \)

and

\[
\llbracket \varphi \rrbracket^M = \{ s \in S \mid M, s \models \varphi \}
\]

**Example: State formulae**

Atomic propositions are state formulae:

\[
\llbracket p \rrbracket = \{ s \in S \mid p \in \ell(s) \}
\]

State formulae are closed under boolean connectives.

\[
\llbracket \neg \varphi \rrbracket = S \setminus \llbracket \varphi \rrbracket \\
\llbracket \varphi_1 \lor \varphi_2 \rrbracket = \llbracket \varphi_1 \rrbracket \cup \llbracket \varphi_2 \rrbracket
\]

Formulae of the form \( E \varphi \) or \( A \varphi \) are state formulae, provided \( \varphi \) is future.

**Remark:**

\( M \models \exists \varphi \) iff \( I \cap \llbracket E \varphi \rrbracket \neq \emptyset \)

\( M \models \forall \varphi \) iff \( I \subseteq \llbracket A \varphi \rrbracket \)

**Definition: Alternative syntax**

State formulae:

\[
\varphi ::= \bot \mid p \ (p \in \text{AP}) \mid \neg \varphi \mid \varphi \lor \varphi \mid E \psi \mid A \psi
\]

Path formulae:

\[
\psi ::= \varphi \mid \neg \psi \mid \psi \lor \psi \mid \psi \text{SU} \psi
\]
Definition: Computation Tree Logic \( \text{CTL}(\text{AP}, X, U) \)

Syntax:

\[
\phi ::= \bot \mid p \ (p \in \text{AP}) \mid \neg \phi \mid \phi \lor \phi \mid \text{EX} \phi \mid \text{AX} \phi \mid \text{E} \phi \cup \phi \mid \text{A} \phi \cup \phi
\]

The semantics is inherited from \( \text{CTL}^* \).

Remark: All CTL formulae are state formulae

\[
[\phi]^M = \{ s \in S \mid M, s \models \phi \}
\]

Examples: Macros

- \( \text{EF} \phi = \text{E} \top \cup \phi \) and \( \text{AG} \phi = \neg \text{EF} \neg \phi \)
- \( \text{AF} \phi = \text{A} \top \cup \phi \) and \( \text{EG} \phi = \neg \text{AF} \neg \phi \)
- \( \text{AG}(\text{req} \rightarrow \text{EF grant}) \)
- \( \text{AG}(\text{req} \rightarrow \text{AF grant}) \)
**Definition: Semantics**

All CTL-formulae are state formulae. Hence, we have a simpler semantics. Let $M = (S, T, I, AP, \ell)$ be a Kripke structure **without deadlocks** and let $s \in S$.

<table>
<thead>
<tr>
<th>$M, s \models p$</th>
<th>if $p \in \ell(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M, s \models \text{EX } \varphi$</td>
<td>if $\exists s \to s'$ with $M, s' \models \varphi$</td>
</tr>
<tr>
<td>$M, s \models \text{AX } \varphi$</td>
<td>if $\forall s \to s'$ we have $M, s' \models \varphi$</td>
</tr>
<tr>
<td>$M, s \models \text{E } \varphi \text{ U } \psi$</td>
<td>if $\exists s = s_0 \to s_1 \to s_2 \to \cdots s_k$ finite path, with $M, s_k \models \psi$ and $M, s_j \models \varphi$ for all $0 \leq j &lt; k$</td>
</tr>
<tr>
<td>$M, s \models \text{A } \varphi \text{ U } \psi$</td>
<td>if $\forall s = s_0 \to s_1 \to s_2 \to \cdots$ infinite paths, $\exists k \geq 0$ with $M, s_k \models \psi$ and $M, s_j \models \varphi$ for all $0 \leq j &lt; k$</td>
</tr>
</tbody>
</table>
Example:

\[
\begin{align*}
[\text{EX } p] &= \\
[\text{AX } p] &= \\
[\text{EF } p] &= \\
[\text{AF } p] &= \\
[\text{E } q \cup r] &= \\
[\text{A } q \cup r] &= 
\end{align*}
\]
Remark: Equivalent formulae

- $AX \varphi \equiv \neg EX \neg \varphi,$
- $\neg(\varphi U \psi) \equiv G \neg \psi \lor (\neg \psi U (\neg \varphi \land \neg \psi))$
- $A \varphi U \psi \equiv \neg EG \neg \psi \land \neg E(\neg \psi U (\neg \varphi \land \neg \psi))$
- $AG(\text{req} \rightarrow F\text{ grant}) \equiv AG(\text{req} \rightarrow AF\text{ grant})$
- $A GF \varphi \equiv AG AF \varphi$
- $EF G \varphi \equiv EF EG \varphi$
- $EG EF \varphi \not\equiv E GF \varphi \not\equiv EG AF \varphi$
- $AF AG \varphi \not\equiv AF G \varphi \not\equiv AF EG \varphi$
- $EG EX \varphi \not\equiv EG X \varphi \not\equiv EG AX \varphi$
Model checking of $\text{CTL}$

**Definition: Existential and universal model checking**

Let $M = (S, T, I, AP, \ell)$ be a Kripke structure and $\varphi \in \text{CTL}$ a formula.

$M \models \exists \varphi$ if $M, s \models \varphi$ for some $s \in I$.

$M \models \forall \varphi$ if $M, s \models \varphi$ for all $s \in I$.

**Remark:**

$M \models \exists \varphi$ iff $I \cap \llbracket \varphi \rrbracket \neq \emptyset$

$M \models \forall \varphi$ iff $I \subseteq \llbracket \varphi \rrbracket$

$M \models \forall \varphi$ iff $M \not\models \exists \neg \varphi$

**Definition: Model checking problems $\text{MC}_{\text{CTL}}^\forall$ and $\text{MC}_{\text{CTL}}^\exists$**

**Input:** A Kripke structure $M = (S, T, I, AP, \ell)$ and a formula $\varphi \in \text{CTL}$

**Question:** Does $M \models \forall \varphi$? or Does $M \models \exists \varphi$?

**Theorem:**

Let $M = (S, T, I, AP, \ell)$ be a Kripke structure and $\varphi \in \text{CTL}$ a formula.

The model checking problem $M \models \exists \varphi$ is decidable in time $O(|M| \cdot |\varphi|)$
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Outline

Introduction

Models

Temporal Specifications

4 Satisfiability and Model Checking

- CTL
- Fair CTL
- Büchi automata
- From LTL to BA
- LTL
- CTL*

More on Temporal Specifications
Model checking of CTL

Theorem

Let $M = (S, T, I, AP, \ell)$ be a Kripke structure and $\varphi \in CTL$ a formula. The model checking problem $M \models \varphi$ is decidable in time $O(|M| \cdot |\varphi|)$.

Proof:

Compute $[[\varphi]] = \{s \in S \mid M, s \models \varphi\}$ by induction on the formula.

The set $[[\varphi]]$ is represented by a boolean array: $L[s][\varphi] = \top$ if $s \in [[\varphi]]$.

The labelling $\ell$ is encoded in $L$: for $p \in AP$ we have $L[s][p] = \top$ if $p \in \ell(s)$.

For each $t \in S$, the set $T^{-1}(t)$ is represented as a list.

for all $t \in S$ do for all $s \in T^{-1}(t)$ do ... od takes time $O(|T|)$. 

Model checking of $\text{CTL}$

**Definition: procedure semantics($\varphi$)**

- **case** $\varphi = \neg \varphi_1$
  
  $\text{semantics}(\varphi_1)$
  
  $[\varphi] := S \setminus [\varphi_1]$ \hspace{1cm} $O(|S|)$

- **case** $\varphi = \varphi_1 \lor \varphi_2$
  
  $\text{semantics}(\varphi_1); \text{semantics}(\varphi_2)$
  
  $[\varphi] := [\varphi_1] \cup [\varphi_2]$ \hspace{1cm} $O(|S|)$

- **case** $\varphi = \text{EX} \varphi_1$
  
  $\text{semantics}(\varphi_1)$
  
  $[\varphi] := \emptyset$
  
  for all $t \in [\varphi_1]$ do for all $s \in T^{-1}(t)$ do $[\varphi] := [\varphi] \cup \{s\}$ \hspace{1cm} $O(|S|) \hspace{1cm} O(|T|)$

- **case** $\varphi = \text{AX} \varphi_1$
  
  $\text{semantics}(\varphi_1)$
  
  $[\varphi] := S$
  
  for all $t \notin [\varphi_1]$ do for all $s \in T^{-1}(t)$ do $[\varphi] := [\varphi] \setminus \{s\}$ \hspace{1cm} $O(|S|) \hspace{1cm} O(|T|)$
Definition: procedure semantics(φ)

\[ \text{case } \varphi = E \varphi_1 U \varphi_2 \]

\[ \text{semantics}(\varphi_1); \text{semantics}(\varphi_2) \]

\[ \text{Todo} := [\varphi_2] \quad // \text{the “todo” set Todo is implemented with a list} \]

\[ \text{Good} := [\varphi_2] \quad // \text{the “result” is computed in the array Good} \]

\[ \text{while } \text{Todo} \neq \emptyset \text{ do} \]

\[ \text{Invariant 1: } [\varphi_2] \cup \text{Todo} \subseteq \text{Good} \subseteq [E \varphi_1 U \varphi_2] \]

\[ \text{take } t \in \text{Todo}; \text{Todo} := \text{Todo} \setminus \{t\} \quad O(1) \]

\[ \text{for all } s \in T^{-1}(t) \text{ do} \quad |T| \text{ times} \]

\[ \text{if } s \in [\varphi_1] \setminus \text{Good} \text{ then} \]

\[ \text{Todo} := \text{Todo} \cup \{s\}; \text{Good} := \text{Good} \cup \{s\} \quad O(1) \]

\[ \text{od} \]

\[ [\varphi] := \text{Good} \quad O(|S|) \]

\[ \text{Good is only used to make the invariant clear. It can be replaced by } [\varphi]. \]
Model checking of **CTL**

**Definition: procedure semantics(φ)**

```plaintext
case φ = Aφ₁ U φ₂
  semantics(φ₁); semantics(φ₂)
  Todo := [φ₂]  // the “todo” set Todo is implemented with a list
  Good := [φ₂]  // the “result” is computed in the array Good
for all s ∈ S do c[s] := |T(s)|
while Todo ≠ ∅ do
  Invariant 1: [φ₂] U Todo ⊆ Good ⊆ [Aφ₁ U φ₂]
  Invariant 2: ∀s ∈ S, c[s] = |T(s) \ (Good \ Todo)|
    take t ∈ Todo; Todo := Todo \ {t}
    for all s ∈ T⁻¹(t) do
      c[s] := c[s] − 1
      if c[s] = 0 ∧ s ∈ [φ₁] \ Good then
        Todo := Todo ∪ {s}; Good := Good ∪ {s}
  od
  [[φ]] := Good
```

**O(|S| + |T|)**

**O(|S|)**

**O(|S|)**

**O(|S|)**

**O(1)**

**O(1)**

**O(1)**

**O(|S|)**

Good is only used to make the invariant clear. It can be replaced by [[φ]].
Complexity of $\text{CTL}$

**Definition: $\text{SAT}(\text{CTL})$**

- **Input:** A formula $\varphi \in \text{CTL}$
- **Question:** Existence of a model $M$ and a state $s$ such that $M, s \models \varphi$?

**Theorem: Complexity**

- The model checking problem for $\text{CTL}$ is PTIME-complete.
- The satisfiability problem for $\text{CTL}$ is EXPTIME-complete.
Example: Fairness

Only fair runs are of interest

- Each process is enabled infinitely often: \( \bigwedge_i GF\text{run}_i \)
- No process stays ultimately in the critical section: \( \bigwedge_i \neg FG\text{CS}_i = \bigwedge_i GF\neg\text{CS}_i \)

Definition: Fair Kripke structure

\[ M = (S, T, I, AP, \ell, F_1, \ldots, F_n) \text{ with } F_i \subseteq S. \]

An infinite run \( \sigma \) is fair if it visits infinitely often each \( F_i \).
**fair CTL**

**Definition: Syntax of fair-CTL**

\[ \phi ::= \bot \mid p \ (p \in AP) \mid \neg \phi \mid \phi \lor \phi \mid E_f \cdot X \phi \mid A_f \cdot X \phi \mid E_f \cdot U \phi \mid A_f \cdot U \phi \]

**Definition: Semantics as a fragment of CTL*\right)**

Let \( M = (S, T, I, AP, \ell, F_1, \ldots, F_n) \) be a fair Kripke structure.

Then,

\[
E_f \phi = E(fair \land \phi) \quad \text{and} \quad A_f \phi = A(fair \rightarrow \phi)
\]

where

\[ fair = \bigwedge_i GF F_i \]

**Lemma: CTL_f cannot be expressed in CTL**
Proof: $CTL_f$ cannot be expressed in $CTL$

Consider the Kripke structure $M_k$ defined by:

![Diagram](image)

- $M_k, 2k \models EGF p$ but $M_k, 2k - 2 \not\models EGF p$
- If $\varphi \in CTL$ and $|\varphi| \leq m \leq k$ then
  \[ M_k, 2k \models \varphi \iff M_k, 2m \models \varphi \]
  \[ M_k, 2k - 1 \models \varphi \iff M_k, 2m - 1 \models \varphi \]

If the fairness condition is $\ell^{-1}(p)$ then $E_f \top$ cannot be expressed in $CTL$. 
Theorem

The model checking problem for $\text{CTL}_f$ is decidable in time $O(|M| \cdot |\varphi|)$

Proof: Computation of $\text{Fair} = \{s \in S \mid M, s \models E_f \top\}$

Compute the SCC of $M$ with Tarjan's algorithm (in time $O(|M|)$).

Let $S'$ be the union of the (non trivial) SCCs which intersect each $F_i$.

Then, $\text{Fair}$ is the set of states that can reach $S'$.

Note that reachability can be computed in linear time.
Model checking of $\text{CTL}_f$

Proof: Reductions

$E_f X \varphi = E X (\text{Fair} \land \varphi)$ and $E_f \varphi \ U \psi = E \varphi \ U (\text{Fair} \land \psi)$

It remains to deal with $A_f \varphi \ U \psi$.

We have

$$A_f \varphi \ U \psi = \neg E_f \ G \neg \psi \land \neg E_f (\neg \psi \ U (\neg \varphi \land \neg \psi))$$

Hence, we only need to compute the semantics of $E_f \ G \varphi$.

Proof: Computation of $E_f \ G \varphi$

Let $M_\varphi$ be the restriction of $M$ to $[\varphi]_f$.

Compute the SCC of $M_\varphi$ with Tarjan's algorithm (in linear time).

Let $S'$ be the union of the (non trivial) SCCs of $M_\varphi$ which intersect each $F_i$.

Then, $M, s \models E_f \ G \varphi$ iff $M, s \models E \varphi \ U S'$ iff $M_\varphi, s \models EF S'$.

This is again a reachability problem which can be solved in linear time.
Büchi automata

**Definition:**

A Büchi automaton (BA) is a tuple \( A = (Q, \Sigma, I, T, F) \) where

- \( Q \): finite set of states
- \( \Sigma \): finite set of labels
- \( I \subseteq Q \): set of initial states
- \( T \subseteq Q \times \Sigma \times Q \): set of transitions (non-deterministic)
- \( F \subseteq Q \): set of final (repeated) states

**Run:** \( \rho = q_0, a_0, q_1, a_1, q_2, a_2, q_3, \ldots \) with \( (q_i, a_i, q_{i+1}) \in T \) for all \( i \geq 0 \).

\( \rho \) is initial if \( q_0 \in I \).

\( \rho \) is final (successful) if \( q_i \in F \) for infinitely many \( i \)'s.

\( \rho \) is accepting if it is both initial and final.

\[ \mathcal{L}(A) = \{ a_0 a_1 a_2 \ldots \in \Sigma^\omega \mid \exists \rho = q_0, a_0, q_1, a_1, q_2, a_2, q_3, \ldots \text{ accepting run} \} \]

A language \( L \subseteq \Sigma^\omega \) is \( \omega \)-regular if it can be accepted by some Büchi automaton.
Büchi automata

Examples:

Infinitely many $a$’s:

Finitely many $a$’s:

Whenever $a$ then later $b$: 
Büchi automata

Properties

Büchi automata are closed under union, intersection, complement.

- Union: trivial
- Intersection: easy (exercise)
- Complement: difficult

Let $L = \Sigma^* (a\Sigma^{n-1}b \cup b\Sigma^{n-1}a)\Sigma^\omega$

Any non deterministic Büchi automaton for $\Sigma^\omega \setminus L$ has at least $2^n$ states.
Theorem: Büchi

Let \( L \subseteq \Sigma^\omega \) be a language. The following are equivalent:

- \( L \) is \( \omega \)-regular
- \( L \) is \( \omega \)-rational, i.e., \( L \) is a finite union of languages of the form \( L_1 \cdot L_2^\omega \) where \( L_1, L_2 \subseteq \Sigma^+ \) are rational.
- \( L \) is MSO-definable, i.e., there is a sentence \( \varphi \in \text{MSO}_\Sigma(\prec) \) such that \( L = L(\varphi) = \{ w \in \Sigma^\omega \mid w \models \varphi \} \).

Exercises:

1. Construct a BA for \( L(\varphi) \) where \( \varphi \) is the \( \text{FO}_\Sigma(\prec) \) sentence

\[
(\forall x, (P_a(x) \rightarrow \exists y > x, P_a(y))) \rightarrow (\forall x, (P_b(x) \rightarrow \exists y > x, P_c(y)))
\]

2. Given BA for \( L_1 \subseteq \Sigma^\omega \) and \( L_2 \subseteq \Sigma^\omega \), construct BA for

\[
\text{next}(L_1) = \Sigma \cdot L_1
\]

\[
\text{until}(L_1, L_2) = \{ uv \in \Sigma^\omega \mid u \in \Sigma^+ \land v \in L_2 \land u'u'' \in L_1 \text{ for all } u', u'' \in \Sigma^+ \text{ with } u = u'u'' \}.
\]
Generalized Büchi automata

Definition: final condition on states or on transitions

\[ A = (Q, \Sigma, I, T, F_1, \ldots, F_n) \] with \( F_i \subseteq Q \).

An infinite run \( \sigma \) is final (successful) if it visits infinitely often each \( F_i \).

\[ A = (Q, \Sigma, I, T, T_1, \ldots, T_n) \] with \( T_i \subseteq T \).

An infinite run \( \sigma \) is final if it uses infinitely many transitions from each \( T_i \).

Example: Infinitely many \( a \)'s and infinitely many \( b \)'s

Theorem:

1. GBA and BA have the same expressive power.
2. Checking whether a BA or GBA has an accepting run is NLOGSPACE-complete.
Unambiguous or prophetic Büchi automata

Definition: Unambiguous Büchi automata
A BA or GBA $\mathcal{A}$ is unambiguous if every word has at most one accepting run in $\mathcal{A}$.

Definition: Prophetic Büchi automata
A BA or GBA $\mathcal{A}$ is prophetic if every word has exactly one final run in $\mathcal{A}$.

Examples: UBA and PBA
- Finitely many $a$'s.
- $G(a \rightarrow F b)$ with $\Sigma = \{a, b, c\}$.

Proposition: Closure properties
Prophetic BA or GBA are closed under boolean operations (union, intersection, complement).

Theorem: Prophetic Büchi automata (Carton-Michel 2003)
Every $\omega$-regular language can be accepted by a prophetic BA.
Definition: SBT: Synchronous (letter to letter) Büchi transducer

Let $A$ and $B$ be two alphabets.

A synchronous Büchi transducer from $A$ to $B$ is a tuple $\mathcal{A} = (Q, A, I, T, F, \mu)$ where $(Q, A, I, T, F)$ is a Büchi automaton (input) and $\mu : T \to B$ is the output function. It computes the relation

$$\mathcal{A} = \{(u, v) \in A^\omega \times B^\omega \mid \exists \rho = q_0, a_0, q_1, a_1, q_2, a_2, q_3, \ldots \text{ accepting run with } u = a_0 a_1 a_2 \cdots \text{ and } v = \mu(\rho),$$

i.e., $v = b_0 b_1 b_2 \cdots$ with $b_i = \mu(q_i, a_i, q_{i+1})$ for $i \geq 0\}.$

If $(Q, A, I, T, F)$ is unambiguous then $\mathcal{A} : A^\omega \to B^\omega$ is a (partial) function, in which case we also write $\mathcal{A}(u) = v$ for $(u, v) \in \mathcal{A}$.

We will also use SGBT: synchronous transducers with generalized Büchi acceptance.

Example: Left shift with $A = B = \{a, b\}$

\[
\begin{align*}
\begin{array}{c}
\text{a/a} \\
\text{a/}\text{b} \\
\text{b/a}
\end{array}
\quad 
\begin{array}{c}
\text{a/}\text{b} \\
\text{b/}\text{b}
\end{array}
\end{align*}
\]
Definition: Composition

Let $A$, $B$, $C$ be alphabets.
Let $A = (Q, A, I, T, (F_i)_i, \mu)$ be an SGBT from $A$ to $B$.
Let $A' = (Q', B, I', T', (F'_j)_j, \mu')$ be an SGBT from $B$ to $C$.
Then $A \cdot A' = (Q \times Q', A, I \times I', T'', (F_i \times Q')_i, (Q \times F'_j)_j, \mu'')$ defined by:

$$\tau'' = (p, p') \xrightarrow{a} (q, q') \in T'' \text{ and } \mu''(\tau'') = c$$

iff

$$\tau = p \xrightarrow{a} q \in T \text{ and } \tau' = p' \xrightarrow{\mu(\tau)} q' \in T' \text{ and } c = \mu'(\tau')$$

is an SGBT from $A$ to $C$.

When the transducers define functions, we also denote the composition by $A' \circ A$.

Proposition: Composition

1. We have $[A \cdot A'] = [A] \cdot [A']$.
2. If $(Q, A, I, T, (F_i)_i)$ and $(Q', B, I', T', (F'_j)_j)$ are unambiguous (resp. prophetic) then $(Q \times Q', A, I \times I', T'', (F_i \times Q')_i, (Q \times F'_j)_j)$ is also unambiguous (resp. prophetic), and $
\forall u \in A^\omega$ we have $[A' \circ A](u) = [A']([A](u))$. 
**Product of Büchi transducers**

**Definition: Product**

Let $A$, $B$, $C$ be alphabets.

Let $A = (Q, A, I, T, (F_i)_i, \mu)$ be an SGBT from $A$ to $B$.

Let $A' = (Q', A, I', T', (F'_j)_j, \mu')$ be an SGBT from $A$ to $C$.

Then $A \times A' = (Q \times Q', A, I \times I', T'', (F_i \times Q')_i, (Q \times F'_j)_j, \mu'')$ defined by:

$$\tau'' = (p, p') \xrightarrow{a} (q, q') \in T'' \text{ and } \mu''(\tau'') = (b, c)$$

iff

$$\tau = p \xrightarrow{a} q \in T \text{ and } b = \mu(\tau) \text{ and } \tau' = p' \xrightarrow{a} q' \in T' \text{ and } c = \mu'(\tau')$$

is an SGBT from $A$ to $B \times C$.

**Proposition: Product**

We identify $(B \times C)^{\omega}$ with $B^{\omega} \times C^{\omega}$.

1. We have $[A \times A'] = \{(u, v, v') \mid (u, v) \in [A] \text{ and } (u, v') \in [A']\}$.

2. If $(Q, A, I, T, (F_i)_i)$ and $(Q', A, I', T', (F'_j)_j)$ are unambiguous (resp. prophetic) then $(Q \times Q', A, I \times I', T'', (F_i \times Q')_i, (Q \times F'_j)_j)$ is also unambiguous (resp. prophetic), and

   $\forall u \in A^{\omega}$ we have $[A \times A'](u) = ([A](u), [A'](u))$.
Subalphabets of $\sum = 2^{\text{AP}}$

Definition:

For a propositional formula $\xi$ over AP, we let $\Sigma_\xi = \{a \in \Sigma \mid a \models \xi\}$.

For instance, for $p, q \in \text{AP}$,

- $\Sigma_p = \{a \in \Sigma \mid p \in a\}$ and $\Sigma_{\neg p} = \Sigma \setminus \Sigma_p$
- $\Sigma_{p \land q} = \Sigma_p \cap \Sigma_q$ and $\Sigma_{p \lor q} = \Sigma_p \cup \Sigma_q$
- $\Sigma_{p \land \neg q} = \Sigma_p \setminus \Sigma_q$ ... 

Notation:

In automata, $s \xrightarrow{\Sigma_\xi} s'$ stands for the set of transitions $\{s\} \times \Sigma_\xi \times \{s'\}$.

To simplify the pictures, we use $s \xrightarrow{\xi} s'$ instead of $s \xrightarrow{\Sigma_\xi} s'$.

Example: $G(p \rightarrow F q)$

![Diagram of automaton with transitions labeled $\neg p \lor q$, $p \land \neg q$, $\neg q$, $q$, and states 1 and 2.]
Definition: Semantics of $\varphi \in \text{LTL}(\text{AP}, \text{SU}, \text{SS})$

Let $\Sigma = 2^{\text{AP}}$ and $B = \{0, 1\}$.

Define $\sem{\varphi} : \Sigma^\omega \to B^\omega$ by $\sem{\varphi}(u) = b_0 b_1 b_2 \cdots$ with $b_i = \begin{cases} 1 & \text{if } u, i \models \varphi \\ 0 & \text{otherwise.} \end{cases}$

Example:

$\sem{p \text{ SU } q} (\emptyset \{q\} \{p\} \emptyset \{p\} \{q\} \emptyset \{p\} \{p, q\} \emptyset^\omega) = 1001110110^\omega$

$\sem{X p} (\emptyset \{q\} \{p\} \emptyset \{p\} \{q\} \emptyset \{p\} \{p, q\} \emptyset^\omega) = 0101100110^\omega$

$\sem{F p} (\emptyset \{q\} \{p\} \emptyset \{p\} \{q\} \emptyset \{p\} \{p, q\} \emptyset^\omega) = 1111111110^\omega$

The aim is to compute $\sem{\varphi}$ with synchronous Büchi transducers (actually, SGBT).
**Synchronous Büchi transducer for \( p \text{ SU } q \)**

**Example: An SBT for \([p \text{ SU } q]\)***

![Diagram of SBT](image)

**Lemma: The input BA is prophetic**

For all \( u = a_0a_1a_2 \cdots \in \Sigma^\omega \), there is a unique final run \( \rho = s_0, a_0, s_1, a_1, s_2, a_2, s_3, \ldots \) of \( A \) on \( u \).

The run \( \rho \) satisfies for all \( i \geq 0, s_i = \begin{cases} 1 & \text{if } u, i \models q \\ 2 & \text{if } u, i \models \neg q \land (p \text{ U } q) \\ 3 & \text{if } u, i \models \neg (p \text{ U } q) \end{cases} \)

Hence, the SBT computes \([p \text{ SU } q]\).
Example: An SBT for $[p \mathsf{U} q]$

The automaton is prophetic (same input BA as for $p \mathsf{SU} q$).
This SBT computes $[p \mathsf{U} q]$. 
**Special cases of Until: Future and Next**

**Example:** \( F q = \top \land U q \) and \( X q = \bot \land SU q \)

![Diagram](diagram.png)

**Exercise:** Give SBT’s for the following formulae:

- \( SF q, SG q, p SR q, p SS q, Y q, G q, p R q, p S q, G(p \rightarrow F q) \).
From LTL to Büchi automata

Definition: SBT for LTL modalities

- \( \mathcal{A}_\top \) from \( \Sigma \) to \( \mathbb{B} = \{0, 1\} \):
  - \( \top \) from \( \Sigma \) to \( \mathbb{B} = \{0, 1\} \):
    - \( \circ \) from \( \Sigma \) to \( \mathbb{B} = \{0, 1\} \): \( 0 \) \( \xrightarrow{1} \) \( 0 \)

- \( \mathcal{A}_p \) from \( \Sigma \) to \( \mathbb{B} = \{0, 1\} \):
  - \( p \) from \( \Sigma \) to \( \mathbb{B} = \{0, 1\} \):
    - \( \circ \) from \( \Sigma \) to \( \mathbb{B} = \{0, 1\} \): \( 0 \) \( \xrightarrow{1} \) \( 0 \)

- \( \mathcal{A}_\neg \) from \( \mathbb{B} \) to \( \mathbb{B} \):
  - \( \neg \) from \( \mathbb{B} \) to \( \mathbb{B} \):
    - \( \circ \) from \( \mathbb{B} \) to \( \mathbb{B} \): \( 0 \) \( \xrightarrow{1} \) \( 1 \)

- \( \mathcal{A}_\lor \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \):
  - \( \lor \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \):
    - \( \circ \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \): \( 0 \) \( \xrightarrow{1} \) \( 0 \), \( 1 \) \( \xrightarrow{1} \) \( 1 \)

- \( \mathcal{A}_\land \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \):
  - \( \land \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \):
    - \( \circ \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \): \( 0 \) \( \xrightarrow{1} \) \( 0 \), \( 1 \) \( \xrightarrow{1} \) \( 1 \)
From LTL to Büchi automata

Definition: SBT for LTL modalities (cont.)

- \( A_{SU} \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \):
  - Prophetic

- \( A_{SS} \) from \( \mathbb{B}^2 \) to \( \mathbb{B} \):
  - Deterministic
  - Not prophetic
From LTL to Büchi automata

Definition: Translation from LTL to SGBT

For each \( \xi \in \text{LTL}(\text{AP},\text{SU},\text{SS}) \) we define inductively an SGBT \( A_\xi \) as follows:

- \( A_T \) and \( A_p \) for \( p \in \text{AP} \) are already defined
- \( A_{\neg \phi} = A_\neg \circ A_\phi \)
- \( A_{\phi \lor \psi} = A_\lor \circ (A_\phi \times A_\psi) \)
- \( A_{\phi \text{SS} \psi} = A_{\text{SS}} \circ (A_\phi \times A_\psi) \)
- \( A_{\phi \text{SU} \psi} = A_{\text{SU}} \circ (A_\phi \times A_\psi) \)

Theorem: Correctness of the translation

For each \( \xi \in \text{LTL}(\text{AP},\text{SU},\text{SS}) \), we have \( \llbracket A_\xi \rrbracket = \llbracket \xi \rrbracket \) and \( A_\xi \) is unambiguous.

Moreover, the number of states of \( A_\xi \) is at most \( 2 |\xi|_{\text{SS}} \cdot 3 |\xi|_{\text{SU}} \)
the number of acceptance conditions is \( |\xi|_{\text{SU}} \)
where \( |\xi|_{\text{SS}} \) (resp. \( |\xi|_{\text{SU}} \)) is the number of SS (resp. SU) occurring in \( \xi \).

Remark:

- If a subformula \( \phi \) occurs serveral time in \( \xi \), we only need one copy of \( A_\phi \).
- We may also use automata for other modalities: \( A_X \) (2 states), \( A_U \), \ldots
Useful simplifications

Reducing the number of temporal subformulae

\[(X \varphi) \land (X \psi) \equiv X(\varphi \land \psi)\]
\[(G \varphi) \land (G \psi) \equiv G(\varphi \land \psi)\]
\[(\varphi_1 \text{SU} \psi) \land (\varphi_2 \text{SU} \psi) \equiv (\varphi_1 \land \varphi_2) \text{SU} \psi\]
\[(X \varphi) \text{SU} (X \psi) \equiv X(\varphi \text{SU} \psi)\]
\[G \varphi \lor G \psi \equiv G(\varphi \lor \psi)\]
\[(\varphi_1 \text{SU} \psi_1) \lor (\varphi_1 \text{SU} \psi_2) \equiv \varphi \text{SU} (\psi_1 \lor \psi_2)\]

Merging equivalent states

Let \(A = (Q, \Sigma, I, T, (F_i)_i, \mu)\) be an SGBT and \(s_1, s_2 \in Q\).

We can merge \(s_1\) and \(s_2\) if they satisfy the same final conditions:

\[s_1 \in F_i \iff s_2 \in F_i\]

for all \(i\)

and they have the same outgoing transitions: \(\forall a \in \Sigma, \forall s \in Q,\)

\[\tau_1 = (s_1, a, s) \in T \iff \tau_2 = (s_2, a, s) \in T\]

and \(\mu(\tau_1) = \mu(\tau_2)\)
Other constructions

- Tableau construction. See for instance [16, Wolper 85]
  + : Easy definition, easy proof of correctness
  + : Works both for future and past modalities
  – : Inefficient without strong optimizations

- Using Very Weak Alternating Automata [17, Gastin & Oddoux 01].
  + : Very efficient
  – : Only for future modalities
  Online tool: http://www.lsv.ens-cachan.fr/~gastin/ltl2ba/

- Using reduction rules [7, Demri & Gastin 10].
  + : Efficient and produces small automata
  + : Can be used by hand on real examples
  – : Only for future modalities

- The domain is still very active.
Some References

    Checking that finite state concurrent programs satisfy their linear specification.

    The tableau method for temporal logic: An overview,

    The complexity of propositional linear temporal logic.

    Fast *LTL* to Büchi automata translation.
    http://www.lsv.ens-cachan.fr/~gastin/mes-publis.php

    Specification and Verification using Temporal Logics.
    In Modern applications of automata theory, IISc Research Monographs 2.
    http://www.lsv.ens-cachan.fr/~gastin/mes-publis.php
Satisfiability for LTL over \((\mathbb{N}, <)\)

Let \(AP\) be the set of atomic propositions and \(\Sigma = 2^{AP}\).

**Definition: Satisfiability problem**

**Input:** A formula \(\varphi \in LTL(AP, SU, SS)\)

**Question:** Existence of \(w \in \Sigma^\omega\) and \(i \in \mathbb{N}\) such that \(w, i \models \varphi\).

**Definition: Initial Satisfiability problem**

**Input:** A formula \(\varphi \in LTL(AP, SU, SS)\)

**Question:** Existence of \(w \in \Sigma^\omega\) such that \(w, 0 \models \varphi\).

Remark: \(\varphi\) is satisfiable iff \(F \varphi\) is *initially* satisfiable.

**Definition: (Initial) validity**

\(\varphi\) is valid iff \(\neg \varphi\) is not satisfiable.

**Theorem [11, Sistla, Clarke 85], [10, Lichtenstein & Pnueli 85]**

The satisfiability problem for LTL is PSPACE-complete.
Model checking for LTL

**Definition: Model checking problem**

**Input:** A Kripke structure \( M = (S, T, I, AP, \ell) \)
A formula \( \varphi \in \text{LTL}(AP, SU, SS) \)

**Question:** Does \( M \models \varphi \) ?

- **Universal MC:** \( M \models \forall \varphi \) if \( \ell(\sigma), 0 \models \varphi \) for all initial infinite runs of \( M \).
- **Existential MC:** \( M \models \exists \varphi \) if \( \ell(\sigma), 0 \models \varphi \) for some initial infinite run of \( M \).

\[
M \models \forall \varphi \iff M \not\models \exists \neg \varphi
\]

**Theorem** [11, Sistla, Clarke 85], [10, Lichtenstein & Pnueli 85]

The Model checking problem for LTL is PSPACE-complete
MC³(SU) \leq_P SAT(SU)

[11, Sistla & Clarke 85]

Let $M = (S, T, I, AP, \ell)$ be a Kripke structure and $\varphi \in LTL(AP, SU)$

Introduce new atomic propositions: $AP_S = \{at_s \mid s \in S\}$

Define $AP' = AP \uplus AP_S$ \quad $\Sigma' = 2^{AP'}$ \quad $\pi : \Sigma'^\omega \to \Sigma^\omega$ by $\pi(a) = a \cap AP$.

Let $w \in \Sigma'^\omega$. We have $w \models \varphi$ iff $\pi(w) \models \varphi$

Define $\psi_M \in LTL(AP', X, F)$ of size $O(|M|^2)$ by

\[
\psi_M = \left( \bigvee_{s \in I} at_s \right) \land G \left( \bigvee_{s \in S} \left( at_s \land \bigwedge_{t \neq s} \neg at_t \land \bigwedge_{p \in \ell(s)} p \land \bigwedge_{p \notin \ell(s)} \neg p \land \bigvee_{t \in T(s)} X at_t \right) \right)
\]

Let $w = a_0a_1a_2 \cdots \in \Sigma'^\omega$. Then, $w \models \psi_M$ iff there exists an initial infinite run $\sigma = s_0, s_1, s_2, \ldots$ of $M$ such that $\pi(w) = \ell(\sigma)$ and $a_i \cap AP_S = \{at_{s_i}\}$ for all $i \geq 0$.

Therefore, $M \models \exists \varphi$ iff $\psi_M \land \varphi$ is initially satisfiable

$M \models \forall \varphi$ iff $\psi_M \land \neg \varphi$ is not initially satisfiable

Remark: we also have $MC^3(X, F) \leq_P SAT(X, F)$. 
**Definition: QBF**

Input: A formula $\gamma = Q_1 x_1 \cdots Q_n x_n \gamma'$ with $\gamma' = \bigwedge_{1 \leq i \leq m} \bigvee_{1 \leq j \leq k_i} a_{ij}$ (CNF)

$Q_i \in \{\forall, \exists\}$ and $a_{ij} \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$.

Question: Is $\gamma$ valid?

**Definition:**

An assignment of the variables $\{x_1, \ldots, x_n\}$ is a word $v = v_1 \cdots v_n \in \{0, 1\}^n$. We write $v[i]$ for the prefix of length $i$.

Let $V \subseteq \{0, 1\}^n$ be a set of assignments.

- $V$ is valid (for $\gamma'$) if $v \models \gamma'$ for all $v \in V$,
- $V$ is closed (for $\gamma$) if $\forall v \in V$, $\forall 1 \leq i \leq n$ s.t. $Q_i = \forall$, $\exists v' \in V$ s.t. $v[i-1] = v'[i-1]$ and $v_i' = 1 - v_i$.

**Proposition:**

$\gamma$ is valid iff $\exists V \subseteq \{0, 1\}^n$ s.t. $V$ is nonempty valid and closed
Let $\gamma = Q_1 x_1 \cdots Q_n x_n \bigwedge_{1 \leq i \leq m} \bigvee_{1 \leq j \leq k_i} a_{ij}$ with $Q_i \in \{\forall, \exists\}$ and $a_{ij}$ literals.

Consider the KS $M$:

Let $\psi_{ij} = \begin{cases} G(x^f_k \rightarrow s_k \overline{R} a_{ij}) & \text{if } a_{ij} = x_k \\ G(x^t_k \rightarrow s_k \overline{R} a_{ij}) & \text{if } a_{ij} = \overline{x_k} \end{cases}$ and $\psi = \bigwedge_{i,j} \psi_{ij}$.

Let $\varphi_i = G(e_{i-1} \rightarrow (\neg s_{i-1} \cup x^t_i) \land (\neg s_{i-1} \cup x^f_i))$ and $\varphi = \bigwedge_{i | Q_i = \forall} \varphi_i$.

Then, $\gamma$ is valid iff $M \models_\exists \psi \land \varphi$. 

\[ QBF \leq_P MC^\exists(U) [11, \text{Sistla & Clarke 85}] \]
Complexity of LTL

Theorem: Complexity of LTL

The following problems are PSPACE-complete:

- $\text{SAT}(\text{LTL}(\text{SU}, \text{SS}))$, $\text{MC}^\forall(\text{LTL}(\text{SU}, \text{SS}))$, $\text{MC}^\exists(\text{LTL}(\text{SU}, \text{SS}))$
- $\text{SAT}(\text{LTL}(\text{X}, \text{F}))$, $\text{MC}^\forall(\text{LTL}(\text{X}, \text{F}))$, $\text{MC}^\exists(\text{LTL}(\text{X}, \text{F}))$
- $\text{SAT}(\text{LTL}(\text{U}))$, $\text{MC}^\forall(\text{LTL}(\text{U}))$, $\text{MC}^\exists(\text{LTL}(\text{U}))$
- The restriction of the above problems to a unique propositional variable

The following problems are NP-complete:

- $\text{SAT}(\text{LTL}(\text{F}))$, $\text{MC}^\exists(\text{LTL}(\text{F}))$
Complexity of $\text{CTL}^*$

**Definition: Syntax of the Computation Tree Logic $\text{CTL}^*$**

$$\varphi ::= \bot \mid p \ (p \in \text{AP}) \mid \neg \varphi \mid \varphi \lor \varphi \mid X \varphi \mid \varphi \lor \varphi \mid E \varphi \mid A \varphi$$

**Theorem**

The model checking problem for $\text{CTL}^*$ is PSPACE-complete

**Proof:**

PSPACE-hardness: follows from $\text{LTL} \subseteq \text{CTL}^*$.

PSPACE-easiness: reduction to LTL-model checking by inductive eliminations of path quantifications.
Satisfiability for $\text{CTL}^*$

**Definition:** \( \text{SAT}(\text{CTL}^*) \)

- **Input:** A formula \( \varphi \in \text{CTL}^* \)
- **Question:** Existence of a model \( M \) and a run \( \sigma \) such that \( M, \sigma, 0 \models \varphi \)?

**Theorem**

The satisfiability problem for $\text{CTL}^*$ is 2-EXPTIME-complete.
Outline

Introduction

Models

Temporal Specifications

Satisfiability and Model Checking

More on Temporal Specifications

- Expressivity
- Ehrenfeucht-Fraïssé games
- Separation
**Definition: Equivalence**

Let \( C \) be a class of time flows.

Two formulae \( \varphi, \psi \in \text{TL}(\text{AP}, \text{SU}, \text{SS}) \) are equivalent over \( C \) if for all temporal structures \( w = (T, <, \lambda) \) over \( C \) and all time points \( t \in T \) we have

\[
 w, t \models \varphi \iff w, t \models \psi
\]

Two formulae \( \varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS}) \) and \( \psi(x) \in \text{FO}_{\text{AP}}(<) \) are equivalent over \( C \) if for all temporal structures \( w = (T, <, \lambda) \) over \( C \) and all time points \( t \in T \) we have

\[
 w, t \models \varphi \iff w, x \mapsto t \models \psi
\]

We also write \( w \models \psi(t) \).

**Remark:** \( \text{TL}(\text{AP}, \text{SU}, \text{SS}) \subseteq \text{FO}^3_{\text{AP}}(<) \subseteq \text{FO}_{\text{AP}}(<) \)

\( \forall \varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS}), \exists \psi(x) \in \text{FO}^3_{\text{AP}}(<) \) such that \( \varphi \) and \( \psi(x) \) are equivalent.

**Expressivity problem:** \( \text{LTL} = \text{FO} \)?
**Expressivity**

**Definition: complete linear time flows**

A time flow \((\mathbb{T}, <)\) is **linear** if \(<\) is a **total** strict order.

A linear time flow \((\mathbb{T}, <)\) is **complete** if every nonempty and bounded subset of \(\mathbb{T}\) has a **least upper bound** and a **greatest lower bound**.

\((\mathbb{N}, <), (\mathbb{Z}, <)\) and \((\mathbb{R}, <)\) are complete.

\((\mathbb{Q}, <)\) and \((\mathbb{R} \setminus \{0\}, <)\) are not complete.

**Theorem: Expressive completeness [12, Kamp 68]**

For **complete** linear time flows, \(\text{TL}(\text{AP}, \text{SU}, \text{SS}) = \text{FO}_{\text{AP}}(<)\)

Elegant algebraic proof of \(\text{TL}(\text{AP}, \text{SU}) = \text{FO}_{\text{AP}}(<)\) over \((\mathbb{N}, <)\) due to Wilke 98.

See also Diekert-Gastin [18]: \(\text{TL} = \text{FO} = \text{SF} = \text{AP} = \text{CFBA} = \text{VWAA}\).

**Example: Translate in \(\text{TL}(\text{AP}, \text{SU}, \text{SS})\)**

\(\psi(x) = \neg P_a(x) \land \neg P_b(x) \land \forall y \forall z \ (P_a(y) \land P_b(z) \land y < z) \rightarrow \exists v \ y < v < z \land \left( \begin{array}{ll} P_c(v) \land x < y \land \lor P_d(v) \land z < x \land \lor P_e(v) \land y < x < z \end{array} \right)\)
Stavi connectives: Time flows with gaps

**Definition: Stavi Until:** \( \overline{U} \)

Let \( w = (\mathbb{T}, <, \lambda) \) be a temporal structure and \( i \in \mathbb{T} \). Then, \( w, i \models \varphi \overline{U} \psi \) if

\[
\exists k \ i < k \\
\wedge \exists j \ (i < j < k \wedge w, j \models \neg \varphi) \\
\wedge \exists j \ (i < j < k \wedge \forall \ell \ (i < \ell < j \rightarrow w, \ell \models \varphi)) \\
\wedge \forall j \left[ i < j < k \rightarrow \left[ \exists k' \ [j < k' \wedge \forall j' \ (i < j' < k' \rightarrow w, j' \models \varphi)] \\
\wedge \exists \ell \ (i < \ell < j \wedge w, \ell \models \neg \varphi) \right] \right] \\
\right]
\]

Similar definition for the Stavi Since \( \overline{S} \).

**Example:**

1. Let \( w = (\mathbb{R} \setminus \{0\}, <, h) \) with \( h(p) = \mathbb{R}_- \) and \( h(q) = \mathbb{R}_+ \).
   Then, \( w, -1 \not\models p \ SU \ q \) but \( w, -1 \models p \overline{U} \ q \).

2. Let \( w' = (\mathbb{R} \setminus \{0\}, <, h') \) with \( h'(p) = \mathbb{R} \setminus \{1, \frac{1}{2}, \frac{1}{4}, \ldots, 0\} \) and \( h'(q) = \mathbb{R}_+ \).
   Then, \( w', -1 \models p \overline{U} \ q \).
Stavi connectives: Time flows with gaps

Theorem: [14, Gabbay, Hodkinson, Reynolds]

$\text{TL}(\text{AP}, \text{SU}, \text{SS}, \overline{S}, \overline{U})$ is expressively complete for $\text{FO}_{\text{AP}}(<)$ over the class of all linear time flows.

Exercise: Isolated gaps (3)

Let $\varphi_p = p \text{ SU } p \land \overline{\text{SF}} \neg p \land \neg(p \text{ SU } \neg p) \land \neg(p \text{ SU } \neg(p \text{ SU } \top))$.

Let $w = (T, <, \lambda)$ with $T \subseteq \mathbb{R}$ and $t \in T$.

Show that if $w, t \models \varphi_p$ then $T$ has a gap.

Let $\psi_{p,q} = \varphi_p \land (q \lor \varphi_p) \text{ SU } (q \land \neg p)$.

Show that $\psi_{p,q}$ is equivalent to $p \overline{U} q$ over the time flow $(\mathbb{R} \setminus \{0\}, <)$.

Show that $\text{TL}(\text{AP}, \text{SU}, \text{SS})$ is $\text{FO}_{\text{AP}}(<)$-complete over the time flow $(\mathbb{R} \setminus \mathbb{Z}, <)$. 
Temporal depth

Definition: Temporal depth of $\varphi \in TL(AP, SU, SS)$

\[
\begin{align*}
\text{td}(p) &= 0 &\text{if } p \in AP \\
\text{td}(\neg \varphi) &= \text{td}(\varphi) \\
\text{td}(\varphi \lor \psi) &= \max(\text{td}(\varphi), \text{td}(\psi)) \\
\text{td}(\varphi \text{ SS } \psi) &= \max(\text{td}(\varphi), \text{td}(\psi)) + 1 \\
\text{td}(\varphi \text{ SU } \psi) &= \max(\text{td}(\varphi), \text{td}(\psi)) + 1
\end{align*}
\]

Lemma:
Let $B \subseteq AP$ be finite and $k \in \mathbb{N}$. There are (up to equivalence) finitely many formulae in $TL(B, SU, SS)$ of temporal depth at most $k$. 
\[ k \]-equivalence

**Definition:**

Let \( w_0 = (T_0, <, h_0) \) and \( w_1 = (T_1, <, h_1) \) be two temporal structures. Let \( i_0 \in T_0 \) and \( i_1 \in T_1 \). Let \( k \in \mathbb{N} \).

We say that \( (w_0, i_0) \) and \( (w_1, i_1) \) are \( k \)-equivalent, denoted \( (w_0, i_0) \equiv_k (w_1, i_1) \), if they satisfy the same formulae in TL(AP, SU, SS) of temporal depth at most \( k \).

**Lemma:** \( \equiv_k \) is an equivalence relation of finite index.

**Example:**

Let \( a = \{p\} \) and \( b = \{q\} \). Let \( w_0 = babaababaa \) and \( w_1 = baababaaba \).

\[
(w_0, 3) \equiv_0 (w_1, 4) \\
(w_0, 3) \equiv_1 (w_1, 4) \\
(w_0, 3) \equiv_1 (w_1, 6)
\]

Here, \( T_0 = T_1 = \{0, 1, 2, \ldots, 9\} \).
The EF-game has two players: **Spoiler (Player I)** and **Duplicator (Player II)**.

The **game board** consists of 2 temporal structures:
\[ w_0 = (T_0, <, h_0) \text{ and } w_1 = (T_1, <, h_1). \]

There are **two tokens**, one on each structure: \( i_0 \in T_0 \) and \( i_1 \in T_1 \).

A **configuration** is a tuple \( (w_0, i_0, w_1, i_1) \)
or simply \( (i_0, i_1) \) if the game board is understood.

Let \( k \in \mathbb{N} \).

The **\( k \)-round EF-game** from a configuration proceeds with (at most) \( k \) moves.

There are 2 available moves for **\( TL(AP, SU, SS) \)**: **SU-move** or **SS-move** (see below).

**Spoiler** chooses which move is played in each round.

**Spoiler wins if**

- Either **duplicator cannot answer** during a move (see below).
- Or a configuration such that \( (w_0, i_0) \not\equiv_0 (w_1, i_1) \) is reached.

Otherwise, **duplicator wins**.
Strict Until and Since moves

Definition: SU-move

- Spoiler chooses $\varepsilon \in \{0, 1\}$ and $k_\varepsilon \in T_\varepsilon$ such that $i_\varepsilon < k_\varepsilon$.
- Duplicator chooses $k_{1-\varepsilon} \in T_{1-\varepsilon}$ such that $i_{1-\varepsilon} < k_{1-\varepsilon}$.

Spoiler wins if there is no such $k_{1-\varepsilon}$.

Either spoiler chooses $(k_0, k_1)$ as next configuration of the EF-game, or the move continues as follows:

- Spoiler chooses $j_{1-\varepsilon} \in T_{1-\varepsilon}$ with $i_{1-\varepsilon} < j_{1-\varepsilon} < k_{1-\varepsilon}$.
- Duplicator chooses $j_\varepsilon \in T_\varepsilon$ with $i_\varepsilon < j_\varepsilon < k_\varepsilon$.

Spoiler wins if there is no such $j_\varepsilon$.

The next configuration is $(j_0, j_1)$.

Similar definition for the SS-move.
**Winning strategy**

**Definition: Winning strategy**

Duplicator has a winning strategy in the $k$-round EF-game starting from $(w_0, i_0, w_1, i_1)$ if he can win all plays starting from this configuration. This is denoted by $(w_0, i_0) \sim_k (w_1, i_1)$.

Spoiler has a winning strategy in the $k$-round EF-game starting from $(w_0, i_0, w_1, i_1)$ if she can win all plays starting from this configuration.

**Example:**

Let $a = \{p\}$, $b = \{q\}$, $c = \{r\}$. Let $w_0 = aaabbc$ and $w_1 = aababc$.

$$(w_0, 0) \sim_1 (w_1, 0)$$

$$ (w_0, 0) \not\sim_2 (w_1, 0)$$

Here, $T_0 = T_1 = \{0, 1, 2, \ldots, 5\}$. 
**EF-games for TL(AP, SU, SS)**

**Lemma: Determinacy**

The $k$-round EF-game for TL(AP, SU, SS) is determined:
For each initial configuration, either spoiler or duplicator has a winning strategy.

**Theorem: Soundness and completeness of EF-games**

For all $k \in \mathbb{N}$ and all configurations $(w_0, i_0, w_1, i_1)$, we have

$$(w_0, i_0) \sim_k (w_1, i_1) \text{ iff } (w_0, i_0) \equiv_k (w_1, i_1)$$

**Example:**

Let $a = \{p\}$, $b = \{q\}$, $c = \{r\}$.

Then, $aaaabbc, 0 \models p \text{ SU } (q \text{ SU } r)$ but $aaababc, 0 \not\models p \text{ SU } (q \text{ SU } r)$.

$p \text{ SU } (q \text{ SU } r)$ cannot be expressed with a formula of temporal depth at most 1.

$p \text{ SU } (q \wedge X q)$ cannot be expressed with a formula of temporal depth at most 1.

**Exercise:**

On finite linear time flows, “even length” cannot be expressed in TL(AP, SU, SS).
Moves for Strict Future and Past modalities

**Definition: SF-move**

- Spoiler chooses $\varepsilon \in \{0, 1\}$ and $j_\varepsilon \in T_\varepsilon$ such that $i_\varepsilon < j_\varepsilon$.
- Duplicator chooses $j_{1-\varepsilon} \in T_{1-\varepsilon}$ such that $i_{1-\varepsilon} < j_{1-\varepsilon}$.

Spoiler wins if there is no such $j_{1-\varepsilon}$.

The new configuration is $(j_0, j_1)$.

Similar definition for the SP-move.

**Example:**

$p \mathbf{SU} q$ is not expressible in $\mathbf{TL}(\mathbf{AP}, \mathbf{SP}, \mathbf{SF})$ over linear flows of time.

Let $a = \emptyset$, $b = \{p\}$ and $c = \{q\}$.

Let $w_0 = (abc)^n a (abc)^n$ and $w_1 = (abc)^n (abc)^n$.

If $n > k$ then, starting from $(w_0, 3n, w_1, 3n)$, duplicator has a winning strategy in the $k$-round EF-game using SF-moves and SP-moves.
Moves for Next and Yesterday modalities

Notation: \( i < j \overset{\text{def}}{=} i < j \land \neg \exists k \ (i < k < j) \).

Definition: X-move

- Spoiler chooses \( \varepsilon \in \{0, 1\} \) and \( j_\varepsilon \in \mathbb{T}_\varepsilon \) such that \( i_\varepsilon < j_\varepsilon \).
- Duplicator chooses \( j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon} \) such that \( i_{1-\varepsilon} < j_{1-\varepsilon} \).

Spoiler wins if there is no such \( j_{1-\varepsilon} \).

The new configuration is \((j_0, j_1)\).

Similar definition for the Y-move.

Exercise:

Show that \( p \ SU \ q \) is not expressible in \( \text{TL}(\text{AP}, Y, SP, X, SF) \) over linear time flows.
Definition: U-move

- Spoiler chooses \( \varepsilon \in \{0, 1\} \) and \( k_\varepsilon \in \mathbb{T}_\varepsilon \) such that \( i_\varepsilon \leq k_\varepsilon \).
- Duplicator chooses \( k_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon} \) such that \( i_{1-\varepsilon} \leq k_{1-\varepsilon} \).

Either spoiler chooses \((k_0, k_1)\) as new configuration of the EF-game, or the move continues as follows:

- Spoiler chooses \( j_{1-\varepsilon} \in \mathbb{T}_{1-\varepsilon} \) with \( i_{1-\varepsilon} \leq j_{1-\varepsilon} < k_{1-\varepsilon} \).
- Duplicator chooses \( j_\varepsilon \in \mathbb{T}_\varepsilon \) with \( i_\varepsilon \leq j_\varepsilon < k_\varepsilon \).

Spoiler wins if there is no such \( j_\varepsilon \).

The new configuration is \((j_0, j_1)\).

- If duplicator chooses \( k_{1-\varepsilon} = i_{1-\varepsilon} \) then the new configuration must be \((k_0, k_1)\).
- If spoiler chooses \( k_\varepsilon = i_\varepsilon \) then duplicator must choose \( k_{1-\varepsilon} = i_{1-\varepsilon} \), otherwise he loses.

Similar definition for the S-move.

Exercise:

1. Show that SU is not expressible in \( \text{TL}(\text{AP}, S, U) \) over \((\mathbb{R}, <)\).
2. Show that SU is not expressible in \( \text{TL}(\text{AP}, S, U) \) over \((\mathbb{N}, <)\).
Syntactic Separation

Definition: Syntactically pure formulae and separation

A formula $\varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS})$ is

- **syntactically pure present** if it is a boolean combination of formulae in $\text{AP}$,
- **syntactically pure future** if it is a boolean combination of formulae of the form $\alpha \text{ SU } \beta$ where $\alpha, \beta \in \text{TL}(\text{AP}, \text{SU})$,
- **syntactically pure past** if it is a boolean combination of formulae of the form $\alpha \text{ SS } \beta$ where $\alpha, \beta \in \text{TL}(\text{AP}, \text{SS})$.
- **syntactically separated** if it is a boolean combination of syntactically pure formulae.

A logic $\mathcal{L}$ is **syntactically separable** over a class $\mathcal{C}$ of time flows if each formula $\varphi \in \mathcal{L}$ is equivalent to some (finite) boolean combination of syntactically pure formulae.

Example: (5)
The formulae $\varphi_1 = \text{SF}(q \land \text{SP} \, p)$ and $\varphi_2 = \text{SF}(q \land \neg \text{SP} \, \neg p)$ are not separated but we can find equivalent syntactically separated formulae.
Separation

Theorem: [9, Gabbay, Pnueli, Shelah & Stavi 80]
TL(AP, SU, SS) is syntactically separable over discrete and complete linear orders.

Definition: Discrete linear order
A linear time flow $(\mathbb{T}, <)$ is discrete if every non-maximal element has an immediate successor and every non-minimal element has an immediate predecessor.

- $(\mathbb{N}, <)$ is the unique (up to isomorphism) discrete and complete linear order with a first point and no last point.
- $(\mathbb{Z}, <)$ is the unique (up to isomorphism) discrete and complete linear order with no first point and no last point.
- Any discrete and complete linear order is isomorphic to a sub-flow of $(\mathbb{Z}, <)$.

Theorem: Gabbay, Reynolds, see [8]
TL(AP, SU, SS) is syntactically separable over $(\mathbb{R}, <)$. 
Semantic Separation

Definition:
Let $w = (\mathbb{T}, <, \lambda)$ and $w' = (\mathbb{T}, <, \lambda')$ be temporal structures over the same time flow, and let $t \in \mathbb{T}$ be a time point.

- $w, w'$ agree on $t$ if $\lambda(t) = \lambda'(t)$
- $w, w'$ agree on the past of $t$ if $\lambda(s) = \lambda'(s)$ for all $s < t$
- $w, w'$ agree on the future of $t$ if $\lambda(s) = \lambda'(s)$ for all $s > t$

Definition: Pure formulae
Let $\mathcal{C}$ be a class of time flows. A formula $\varphi$ over some logic $\mathcal{L}$ is pure past (resp. pure present, pure future) over $\mathcal{C}$ if

$$w, t \models \varphi \iff w', t \models \varphi$$

for all temporal structures $w = (\mathbb{T}, <, \lambda)$ and $w' = (\mathbb{T}, <, \lambda')$ over $\mathcal{C}$ and all time points $t \in \mathbb{T}$ such that

$w, w'$ agree on the past of $t$ (resp. on $t$, on the future of $t$).
Remark: Syntax versus semantic

Every formula $\varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS})$ which is syntactically pure present (resp. future, past) is also semantically pure present (resp. future, past).

Definition: Separation

A logic $\mathcal{L}$ is separable over a class $\mathcal{C}$ of time flows if each formula $\varphi \in \mathcal{L}$ is equivalent to some (finite) boolean combination of pure formulae.

Theorem: [13, Gabbay 89] (already stated by Gabbay in 81)

Let $\mathcal{C}$ be a class of linear time flows.

Let $\mathcal{L}$ be a temporal logic able to express SF and SP.

Then, $\mathcal{L}$ is separable over $\mathcal{C}$ iff it is expressively complete for $\text{FO}_{\text{AP}}(<)$ over $\mathcal{C}$.

Exercise: Checking semantically pure

Is the following problem decidable? If yes, what is his complexity?

Input: A formula $\varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS})$

Question: Is the formula $\varphi$ semantically pure future?
Definition: Initial Equivalence

Let $\mathcal{C}$ be a class of time flows having a least element (denoted 0). Two formulae $\varphi, \psi \in \text{TL}(\text{AP}, \text{SU}, \text{SS})$ are initially equivalent over $\mathcal{C}$ if for all temporal structures $w = (T, <, \lambda)$ over $\mathcal{C}$ we have

$$w, 0 \models \varphi \iff w, 0 \models \psi$$

Two formulae $\varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS})$ and $\psi(x) \in \text{FO}_{\text{AP}}(<)$ are initially equivalent over $\mathcal{C}$ if for all temporal structures $w = (T, <, \lambda)$ over $\mathcal{C}$ we have

$$w, 0 \models \varphi \iff w \models \psi(0)$$

Corollary: of the separation theorem

For each $\varphi \in \text{TL}(\text{AP}, \text{SU}, \text{SS})$ there exists $\psi \in \text{TL}(\text{AP}, \text{SU})$ such that $\varphi$ and $\psi$ are initially equivalent over $(\mathbb{N}, <)$. 
**Initial equivalence**

**Example:** $\text{TL(} \text{AP, SU, SS)}$ versus $\text{TL(} \text{AP, SU)}$

$$G(\text{grant} \rightarrow (\neg \text{grant SS request}))$$

is initially equivalent to

$$(\text{request R} \neg \text{grant}) \land G(\text{grant} \rightarrow (\text{request} \lor (\text{request SR} \neg \text{grant})))$$

**Theorem:** (Laroussinie & Markey & Schnoebelen 2002)

$\text{TL(} \text{AP, SU, SS)}$ may be exponentially more succinct than $\text{TL(} \text{AP, SU)}$ over $(\mathbb{N}, <)$. 
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